LV2 Inertial Navigation Coordinate System

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This discussion is meant to serve as an orientation for our coordinate systems and reference frames. In addition, some mathematics has been included so that those not familiar with rotations in R^3 can get a feel for what's going on. For a more in depth presentation, I highly recommend reading Jack B. Kuipers' book, Quaternions and Rotation Sequences. This book is the best and most comprehensive text on the subject matter and should serve as a *bible* for this research.

Navigation Frame

First let us define our reference frame, which will be called the *navigation* frame from here forward. The navigation frame will have its origin at the point of launch and will form a local tangent plane made up of the XY-axes at that point. The positive X-axis points in the direction of North and the positive Y-axis points in the direction of East. The Z-axis points "up" or perpendicular to the local tangent plane as if directed into outer space, i.e. away from Earth's origin. While "hand" rules are not "rules" of any kind, but merely ways in which we remember certain relationships, we will define navigation frame to be left-handed in nature. That is, a CCW rotation to the observer located at the origin looking in the positive axis direction is a positive rotation. A positive rotation about the Z-axis will rotate the positive X-axis into the positive Y-axis. Positive rotation about the X-axis will rotate the positive Y-axis into the positive Z-axis. Finally, positive rotation about the Y-axis will rotate the positive Z-axis into the positive X-axis. Figure 1 below depicts this as best as possible. Rotation about the Z-axis is labeled θ, rotation about the X-axis is labeled ϕ and rotation about the Y-axis is labeled ψ.

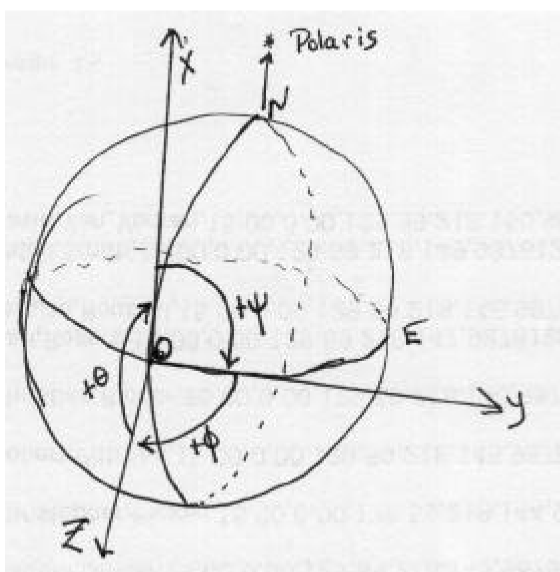

Figure 1 Navigation Frame

This frame defines a North, East, Up or NEU frame. As mentioned, we are really defining a left-handed coordinate frame because this makes the most sense from what you already have, and from what makes sense in terms of real-world orientation.

Body Frame

Furthermore, we define the rocket to have the following *body* frame: The positive Z-axis points out the nose cone; the positive X-axis arbitrarily points out a side; the positive Y-axis arbitrarily points out the perpendicular side. Figure 2 below depicts the rocket's body frame.

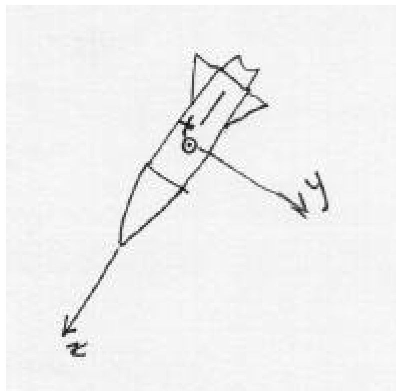

Figure 2 Body Frame (more or less aligned with Figure 1)

We define the following:

- Counter-clockwise rotations about an axis are positive
- Roll: Rotation about the Z-axis
- Pitch: Rotation about the X-axis
- Yaw: Rotation about the Y-axis
- Heading: $\mathrm{Proj}_{xy}(Z)$

For our frame, the following rotation matrices apply:

$$R^{n}_{b\,Z}(q) = \begin{bmatrix} \cos(q) & -\sin(q) & 0 \\ \sin(q) & \cos(q) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R^{n}_{b\,X}(f) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(f) & -\sin(f) \\ 0 & \sin(f) & \cos(f) \end{bmatrix}$$

$$R^{n}_{b\,Y}(y) = \begin{bmatrix} \cos(y) & 0 & \sin(y) \\ 0 & 1 & 0 \\ -\sin(y) & 0 & \cos(y) \end{bmatrix}$$

To follow the proper rotation sequence namely, Reference θ → ψ → φ Body, we would do the following:

$$C_n^b = R_{nY}^b(\psi)\left(R_{nX}^b(\phi)\, R_{nZ}^b(\theta)\right)$$

(Note: These rotations are given by Kuipers, but our axes are different; this is the equivalent rotation seq.)

However, from the properties of the Rotation Matrix,

$$C_b^n = C_n^{b\,T} = \left(R_{nY}^b(\psi)\left(R_{nX}^b(\phi)\, R_{nZ}^b(\theta)\right)\right)^T$$

$$= \left(R_{nX}^b(\phi)\, R_{nZ}^b(\theta)\right)^T \left(R_{nY}^b(\psi)\right)^T$$

$$= \left(R_{nZ}^b(\theta)\right)^T \left(R_{nX}^b(\phi)\right)^T \left(R_{nY}^b(\psi)\right)^T$$

$$= \left(R_{bZ}^n(\theta)\right)\left(R_{bX}^n(\phi)\right)\left(R_{bY}^n(\psi)\right)$$

(Note: This is what we have above)

$$= \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\psi) & 0 & \sin(\psi) \\ \sin(\phi)\sin(\psi) & \cos(\phi) & -\sin(\phi)\cos(\psi) \\ -\sin(\psi)\cos(\phi) & \sin(\phi) & \cos(\phi)\cos(\psi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta)\cos(\psi) - \sin(\theta)\sin(\phi)\sin(\psi) & -\sin(\theta)\cos(\phi) & \cos(\theta)\sin(\psi) + \sin(\theta)\sin(\phi)\cos(\psi) \\ \sin(\theta)\cos(\psi) + \cos(\theta)\sin(\phi)\sin(\psi) & \cos(\theta)\cos(\phi) & \sin(\theta)\sin(\psi) - \cos(\theta)\sin(\phi)\cos(\psi) \\ -\sin(\psi)\cos(\phi) & \sin(\phi) & \cos(\phi)\cos(\psi) \end{bmatrix}$$