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Volume IX - General Perturbations Theory

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by B. Kampos

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for George C. Marshall Space Flight Center

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Volume IX - General Perturbations Theory

By B. Kampos

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FOREWORD

This report was prepared under contract NAS 8-11495 and is one of a series intended to illustrate analytical methods used in the fields of Guidance, Flight Mechanics, and Trajectory Optimization. Derivations, mechanizations and recommended procedures are given. Below is a complete list of the reports in the series.

Volume I	Coordinate Systems and Time Measure
Volume II	Observation Theory and Sensors
Volume III	The Two Body Problem
Volume IV	The Calculus of Variations and Modern Applications
Volume V	State Determination and/or Estimation
Volume VI	The N-Body Problem and Special Perturbation Techniques
Volume VII	The Pontryagin Maximum Principle
Volume VIII	Boost Guidance Equations
Volume IX	General Perturbations Theory
Volume X	Dynamic Programming
Volume XI	Guidance Equations for Orbital Operations
Volume XII	Relative Motion, Guidance Equations for Terminal Rendezvous
Volume XIII	Numerical Optimization Methods
Volume XIV	Entry Guidance Equations
Volume XV	Application of Optimization Techniques
Volume XVI	Mission Constraints and Trajectory Interfaces
Volume XVII	Guidance System Performance Analysis

The work was conducted under the direction of C. D. Baker, J. W. Winch, and D. P. Chandler, Aero-Astro Dynamics Laboratory, George C. Marshall Space Flight Center. The North American program was conducted under the direction of H. A. McCarty and G. E. Townsend.

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1.0 STATEMENT OF THE PROBLEM

The equations of motion of a satellite, in the true force environment of the earth, are nonlinear differential equations which are analytically intractable. Historically, two approaches have been employed to obtain estimates of the trajectories which can be attained. The first, discussed in a previous monograph, is numerical integration. In this first approach (called special perturbations), series expansions formed about the most recent estimate of position and velocity are utilized to numerically estimate the next point on the curve (other equivalent techniques can be formulated). However, no simplifications need be made in the equations of motion. The second approach involves the simplification of the mathematical structure of the problem by the use of truncated series expansions substituted directly into the defining equations under the assumption that the coupling effects of the perturbations are negligible. The simplified differential equations are then integrated analytically to obtain an approximate solution. This solution process is called general perturbations.

The purpose of this monograph is to establish the nature of the solutions available by general perturbations techniques and to provide insight as to how these solutions can be profitably employed in the mission analysis and inflight phases of most space programs. To accomplish this objective, a critical review of the available literature will be presented for the dominating perturbative influences: the oblateness perturbation, the atmospheric drag perturbation, the extra-terrestrial gravitation perturbations, and the solar radiation pressure perturbation. Following each of these reviews, the development adjudged to be most outstanding will be analyzed in detail.

The presentation is initiated with a discussion of the dominant perturbation for most earth satellites, that derived from the earth's oblateness. This discussion presents two basic approaches to the definition of changes in the motion relative to that produced by a central force field. The first is based upon an assumed form for the spatial curve and is correct to the order of the second coefficient of the earth's potential. (This type of solution is typified by the works of King-Hele, Reference 1.11, and Struble, References 1.12, and 1.13). The second approach is based upon the method of variation of parameters as first applied to problems in celestial mechanics. This latter method can be applied to any order without excessive revision to the method or without excessively complicating the solution. (This type of solution is typified by the works of Kozai, Reference 1.5, Brouwer, Reference 1.7, Garfinkel, Reference 1.6, and others.) These developments are intended to demonstrate the assumptions implicit in the derivations and problems of conditioning in a numerical solution since both of these factors are extremely important in the application of the material.

The discussion continues with the development of the atmospheric perturbation to the motion of a satellite. In contrast to the oblateness derivations, only one basic approach is considered (that of Sterne, T. E., Reference 2.2. However, an extension reported by Kalil, F., Reference 2.3, is also presented.) This restriction in the presentation arises from a clear-cut superiority of the theory (relative to others available) resulting from the generality implicit in the formulation. This generality allows many factors which affect the atmospheric perturbation by altering the atmospheric density (solar activity, effects, diurnal effects, latitudinal effects, ... etc.) to be introduced (in an approximate sense) and allows the resultant displacement to be computed. Emphasis in the discussion of this material is placed on the simplifications to the structure of the problem necessary to provide an analytic solution. This emphasis allows a qualitative interpretation of the accuracy available and assures that the limitations of the formulation are understood.

The discussion of perturbative influences concludes with a presentation of the effects of extra-terrestrial gravitation (solar, lunar, ...) and of solar radiation pressure. These effects are normally negligible; however, many analyses require their inclusion to provide the necessary accuracy. Emphasis in these discussions is placed on the development of the perturbations themselves. The special cases where resonances can occur are not considered; rather, the existence of such cases is noted, and reference to some of the applicable literature is made.

The monograph concludes with the presentation of a scheme for approximating the net result of all of these perturbing influences and a mechanization to effect the solution. This mechanization is believed to reflect the optimum formulations of each phase of the analysis the date of publication.

2.0 STATE-OF-THE-ART

2.1 THE PERTURBATIVE EFFECTS OF EARTH OBLATENESS ON THE ORBIT OF AN ARTIFICIAL EARTH SATELLITE

2.1.1 Basic Review of the Problem

2.1.1.1 Definition of the Perturbing Force

If the earth were an ideal homogeneous sphere, the motion along any great circle would be periodic or harmonic. The true shape of the earth, however, is more closely that of a geoid; that is, the center of mass does not lie on the spin axis and neither the meridian nor the latitudinal contours are circles. The net result of the irregular mass distribution of the earth is to produce a variation in the gravitational acceleration relative to that predicted using a point mass description for the earth. Due to the asphericity of the central body, a perturbing component of force (transverse component) is produced which acts along the tangent to the instantaneous meridian and always points toward the equator. The magnitude of this transverse component depends upon the equatorial mass accumulation or oblateness. It reaches its maximum value at the 45° latitude and approaches zero at latitudes of 0° and 90°. The motion about a geoid can be visualized best by resolving it into individual motions along the meridian and latitudinal contours. The motion along a meridian can be thought of as consisting of a number of periodic (harmonic) motions, called zonal harmonics, of different frequency and amplitude. Similarly, the motion along a latitudinal contour can be visualized as consisting of a number of periodic (harmonic) motions called tesseral harmonics, of different frequency and amplitude. The zonal harmonics describe the deviations of a meridian from a great circle, whereas the tesseral harmonics describe the deviations of a latitudinal contour from a circle. The larger the number of these harmonics, the better the description of every detail of the true contour of the earth.

Since, at this stage of scientific progress, the tesseral harmonics are not sufficiently known, it is assumed by most investigators that the shape of the earth is an ellipsoid of revolution and, consequently, that all tesseral harmonics are zero.

The analytical representation of the zonal harmonic motions for an oblate earth, taken as an ellipsoid of revolution whose center coincides with the center of mass, is given by the simplified Vinti's potential, which was adopted in 1961 by the IAU,

$$U = \frac{\mu}{r}\left[1 - \sum_{n=2}^{\infty} J_n \left(\frac{R_{EQ}}{r}\right)^n P_n\ (\sin\delta)\right]$$

where μ is the earth's gravitational constant, r is the distance from the center of the earth to the satellite, R_{EQ} is the equatorial radius of the earth, J_n are Vinti's zonal harmonics and $P_n(\sin\delta)$ are Legendre polynomials of order n defined as follows,

$$P_n(\sin\delta) = \frac{(2n)!}{2^n(n!)^2}\left[(\sin\delta)^n - \frac{1}{2n-1}\binom{n}{2}(\sin\delta)^{n-2} + \frac{1\times 3}{(2n-1)(2n-3)}\binom{n}{4}(\sin\delta)^{n-4}\right]$$

and where δ is the geographic latitude. For,

$$n=0 \;.\;.\;.\;.\;.\; P_0 = 1$$

$$n=1 \;.\;.\;.\;.\;.\; P_1 = \sin\delta$$

$$n=2 \;.\;.\;.\;.\;.\; P_2 = \frac{1}{2}(3\sin^2\delta - 1)$$

$$n=3 \;.\;.\;.\;.\;.\; P_3 = \frac{1}{2}(5\sin^3\delta - 3\sin\delta)$$

$$n=4 \;.\;.\;.\;.\;.\; P_4 = \frac{1}{8}(35\sin^4\delta - 30\sin^2\delta + 3)$$

A derivation of this potential was presented in an earlier monograph Ref. (1.0).

However, since the two major works, selected for detailed analytical development in this monograph (King-Hele and Kozai), are based on Jeffreys potential, it is necessary to discuss the form of this potential. Jeffreys potential is defined as follows,

$$U = \frac{\mu}{r}\sum_{n=0}^{\infty}\left(\frac{B_M}{M}\right)\frac{P_n(\sin\delta)}{r^n}$$

where the Legendre polynomials $P_n(\sin\delta)$ have the same definitions as before, and where the $(\frac{B_n}{M})$ coefficients are constants, chosen to agree with observations, and are determined by the relation,

$$B_n \doteq \frac{1}{M}\int_0^n d^n P_n(\cos\delta)\,dm$$

in which M is the mass of the earth, d is the distance from the center of the earth to a particle inside the earth whose mass is m. For,

$$n = 0 \ldots\ldots \frac{B_0}{M} = 1$$

$$n = 1 \ldots\ldots \frac{B_1}{M} = 0$$

$$n = 2 \ldots\ldots \frac{B_2}{M} = -\frac{2}{3} J R_{EQ}^2$$

$$n = 3 \ldots\ldots \frac{B_3}{M} = -\frac{2}{5} H R_{EQ}^3$$

$$n = 4 \ldots\ldots \frac{B_4}{M} = \frac{8}{35} D R_{EQ}^4$$

If, in addition to the assumption that the center of the ellipsoid of revolution (representing the earth) is coincident with the center of mass, it is also assumed that the earth is symmetrical with respect to its equatorial plane, all of the odd harmonics must vanish.

The substitution of the expressions for the coefficients (B_n/M) and the polynomials P_n in the definition of Jeffreys potential yields

$$U = \frac{\mu}{r}\left[1 + \frac{J}{3}\left(\frac{R_{EQ}}{r}\right)^2 (1 - 3\sin^2\delta) - \frac{H}{5}\left(\frac{R_{EQ}}{r}\right)^3 (5\sin^3\delta - 3\sin\delta) + \frac{D}{35}\left(\frac{R_{EQ}}{r}\right)^4 (35\sin^4\delta - 30\sin^2\delta + 3)\right]$$

where the quantities J and H,D (second, third, and fourth harmonics) were introduced by Jeffreys.

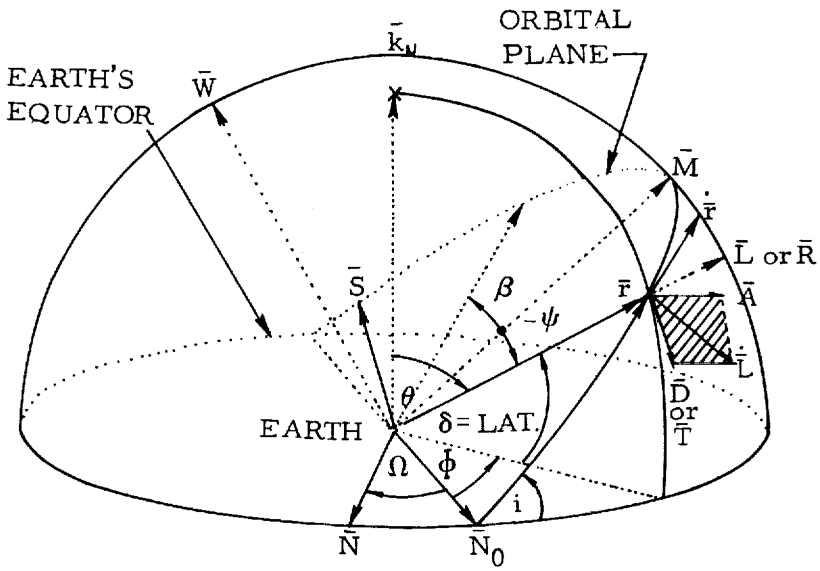
For the earth, with the center of the ellipsoid of revolution taken as the center of mass, the values of the zonal harmonics J, H, D, etc., can be found in many papers. Since the review and assessment of the material covered in this type of reference is not within the scope of this monograph, however, it is considered adequate at this point to provide a list of references which present such data. Accordingly, the reader is referred to References 1.1, 1.2, 1.3, and 1.4. Then the relationship between Vinti's and Jeffreys' zonal-harmonic coefficients is:

$$J_2 = \frac{2}{3} J$$

$$J_3 = \frac{2}{5} H$$

$$J_4 = -\frac{8}{35} D$$

The perturbing transverse or non-radial component of gravitational acceleration, T, produced by the equatorial bulge, can be resolved into two parts (Figure 1) as follows,



1. A horizontal component S in the orbital plane at right angle to the radial component R, and such that $\bar{S} \cdot \bar{V} \geq 0$.

2. An orthogonal component W normal to the orbital plane, and such that $\bar{W} \cdot \bar{h} = +1$, this component causes the nodal line to rotate.

These components can now be derived from the gravitational potential of the earth by representing it by the sum of the central field and the perturbing potentials,

$$U = \frac{\mu}{r} + Q$$

where Q is the perturbing potential,

$$Q = \frac{1}{3}\mu J \frac{R_{EQ}^2}{r^3}(1 - 3\sin^2\delta) - \frac{1}{5}\mu H \frac{R_{EQ}^3}{r^4}(5\sin^3\delta - 3\sin\delta) +$$

$$+ \frac{1}{35}\mu D \frac{R_{EQ}^4}{r^5}(35\sin^4\delta - 30\sin^2\delta + 3)$$

$$= \frac{1}{3}\mu J \frac{R_{EQ}^2}{r^3}(1 - 3\sin^2 i \sin^2 u^*) - \frac{1}{5}\mu H \frac{R_{EQ}^3}{r^4}(5\sin^3 i \sin^3 u^* - 3\sin i \sin u^*)$$

$$+ \frac{1}{35} D \frac{R_{EQ}^4}{r^5}(35\sin^4 i \sin^4 u^* - 30\sin^2 i \sin^2 u^* + 3)$$

i is the orbital inclination and u^* is the argument of latitude.

This perturbing potential can now be divided into secular and periodic parts to facilitate future efforts and to reveal the nature of the perturbation. This step is accomplished as follows,

$$Q = Q_{SEC} + Q_{PERIODIC}$$

The components of the perturbing force can now be derived from the perturbing potential Q: the radial R, the transverse T (tangent to the instantaneous meridian), the horizontal S, and the orthogonal W as follows,

$$R = \frac{\partial Q}{\partial r} = -\mu J \frac{R_{EQ}^2}{r^4} (1 - 3 \sin^2 \delta) - \frac{1}{7} \mu D \frac{R_{EQ}^4}{r^6} (35 \sin^4 \delta - 30 \sin^2 \delta + 3)$$

$$T = \frac{1}{r} \frac{\partial Q}{\partial \delta} = -\mu J \frac{R_{EQ}^2}{r^4} \sin 2\delta - \frac{2}{7} \mu D \frac{R_{EQ}^4}{r^6} (3 - 7 \sin^2 \delta) \sin 2\delta$$

$$S = \frac{1}{r} \frac{\partial Q}{\partial u^*} = -\mu J \frac{R_{EQ}^2}{r^4} \sin^2 i \sin 2u^* - \frac{2}{7} \mu D \frac{R_{EQ}^4}{r^6} (3 - 7 \sin^2 i \sin^2 u) \sin^2 i \sin 2u^*$$

$$W = \frac{1}{r \sin u^*} \frac{\partial Q}{\partial i} = -\mu J \frac{R_{EQ}^2}{r^4} \sin u^* \sin 2i - \frac{2}{7} \mu D \frac{R_{EQ}^4}{r^6} (3 - 7 \sin^2 i \sin^2 u^*) \sin u^* \sin 2i$$

(The terms in H have been dropped for the purpose of this illustration.)

Using the identities,

$$\sin^2 u^* = \frac{1}{2} - \frac{1}{2} \cos 2u^*$$

$$\sin^4 u^* = \frac{3}{8} - \frac{1}{2} \cos 2u^* + \frac{1}{8} \cos 4u^*$$

and the relationship,

$$dM = \frac{r^2}{a^2 \sqrt{1-e^2}} \, d\eta$$

** η is the true anomaly

will yield

$$Q_{SEC} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{3}\mu J \frac{R_{EQ}^2}{a^3}(1-e^2)^{-3/2}\left(\frac{3\cos^2 - 1}{2}\right)\left(1 + e\cos\eta\right) d\eta$$

$$+\frac{1}{2\pi}\int_0^{2\pi} \frac{3}{35}\mu D \frac{R_{EQ}^4}{a^5}(1-e^2)^{-7/2}\left(\frac{35\cos^4 i - 30\cos^2 i + 3}{8}\right)(1 + e\cos\eta)^3 d\eta$$

$$Q_{SEC} = \mu\left[\frac{1}{3}J\frac{R_{EQ}^2}{a^3}(1-e^2)^{-3/2}\left(\frac{3\cos^2 i - 1}{2}\right)\right.$$

$$\left. + \frac{3}{35}D\frac{R_{EQ}^4}{a^5}(1-e^2)^{-7/2}\left(1 + \frac{3}{2}e^2\right)\left(\frac{35\cos^4 i - 30\cos^2 i + 3}{8}\right)\right]$$

Since Q_{sec} is a function of a, e, i, these elements have no secular perturbations. (The same result would have been obtained for the secular part of Q if terms in H were present since integration over a complete period of the coefficient of H is zero.)

2.1.1.2 Summary of the Effects of the Perturbing Force on the Orbit

The gravitational perturbations due to earth oblateness produce secular and/or periodic changes in the orbits of all artificial satellites. These perturbations will be developed later; however, the effects will be summarized here to serve as motivation for the analysis.

1. Regression of the Node

   The orbital plane rotates about the earth's spin axis in the opposite direction of the satellite motion, resulting in monotonic regression (for posigrade orbits) of the ascending node along the earth's equator. For retrograde orbits the node moves counterclockwise.

2. Rotation of the Apsidal Line

   The major axis rotates in the orbital plane in the direction of the satellite motion for orbital inclinations $i < 63.4°$, and in the opposite direction for inclinations $i > 63.4°$. The rate of rotation is zero when $i = 63.4°$.

3. Change in Radial Distance

   As a result of the gravitational perturbations, the radial distance from the center of the earth is subject to periodic changes during the motion in inclined orbits. More exactly, the periodic change in radial distance is superimposed on a constant part displacement which is independent of the satellite position in the orbit. The superimposed oscillatory part has a period = 1/2 revolution and an amplitude = $\frac{1}{6}J r_{av}\left(\frac{R_{EQ}}{p}\right)^2 \sin^2 i$.

There is also a higher order oscillatory change, for elliptic orbits only, with a period = 1/3 revolution and an amplitude

$$= \frac{3}{24} J e\, r_{av} \left(\frac{R_{eq}}{p}\right)^2 \sin^2 i$$

4. Change in the Orbital Period

Because of a non-constant angular momentum and the change in radial distance, there is necessarily a change in the orbital period P. Two types of orbital periods are of interest in the satellite lifetime analysis: the nodal period, P_Ω, from one ascending node to the next, and the anomalistic, P_A, from one perigee to the next. Since the perigee is moving, the anomalistic period is longer than the nodal period for orbital inclinations $i < 63.4°$. Thus, by the time the perigee has rotated 360°, the number of anomalistic periods will be smaller by one than the number of nodal periods. For orbital inclinations $i > 63.4°$, the opposite is true. On the other hand, the period for inclined orbits, whether nodal or anomalistic, is always slightly greater than for an equatorial orbit.

The changes in the nodal longitude, $\Delta\Omega$, and in the argument of perigee, $\Delta\omega$, are both secular and periodic. These secular changes are produced by the unchanging direction of the perturbing equatorial bulge with respect to the orbital plane (that is, by the non-radial or transverse component of force).

The changes in the semi-major axis, Δa, the eccentricity, Δe, and the orbital inclination, Δi, are periodic; that is, oscillating with the cyclic change of the satellite position with respect to the perturbing equatorial bulge. More exactly, the periodic changes Δa and Δe are superimposed on a constant perturbation (relative to a conic solution). These effects are produced by the distortion of the central force field.

2.1.2 Review of the Available Literature

2.1.2.1 General Comments on the Papers Reviewed

There is an abundance of technical literature that is concerned with the perturbative effects of earth oblateness on the motion of artificial satellites. There are, however, only few with original theories. Thus, most of the papers duplicate each other and differ only in the degree of sophistication and the order to which the solutions are extended. Some of these papers consider only circular orbits, or place severe restrictions on either the eccentricity, or the inclination, or both. Unfortunately, comparison of the solutions presented in these papers is very difficult due to differences in the nomenclatures, the lack of inter-relationships between the various parameters, and the lack of general discussion and assessment of the results.

One group of authors concerns itself with the application of the principles of classical astronomy. They use basic variables, which are not always convenient for interpretation during the development of the theory, and employ arguments which are, at times, obscure. For this reason, such theories are more involved and are difficult to interpret during the various stages of the development.

There is also a second class of papers concerned with inference of earth oblateness from observations of orbital motion. Such analyses are usually complicated by choice of variables and the cumbersome form of the results. In most cases, a circular rather than an elliptic generating solution is used.

The order of the theory is defined, in all the papers, in relation to the highest harmonic of the perturbing potential which is employed. Theories based only on the second harmonic are referred to as the first order. Generally, the second order theories do not impose limitations on either the eccentricity or the orbital inclination. All qualitative aspects of the problem arise in the first order terms; the higher order terms function primarily to modify the numerical values.

2.1.2.2 Methods and Techniques

Two basic methods are employed in the literature. They are based upon the following considerations.

The orbit of an artificial satellite is a twisted space curve wound about the earth in a complicated wave pattern. The complicated waves of this curve are removed by introducing a rotating orbital plane, upon which the orbit itself may be represented as a plane curve. Since this plane curve is not a closed circuit, some artifice must be introduced to properly define the orbital motion. The choice of the nature of such an artifice defines the method of osculating ellipse or the method of basic coordinates.

2.1.2.2.1 The Method of Osculating Ellipse

This method has its origin in classical astronomy. The method introduces a precessing orbital plane and an osculating ellipse in this plane, which varies in size and shape throughout the satellite motion. The advantages of this method are that the variations of the osculating elements are small, the differential equations describing these variations are relatively easy to deal with, and the result of the analysis is the time history of a set of parameters which reflect the trend of the perturbation.

The principal disadvantage of this method is that the osculating ellipse does not represent a succession of satellite positions and hence, it does not in itself reveal the trend of the motion. Another disadvantage is the failure in case of very small eccentricities, resulting from the fact that the eccentricity appears as divisor in the expressions for the periodic perturbations in the osculating elements ω (argument of perigee) and M (mean anomaly). This difficulty is, admittedly, of an artificial nature and can be removed by using the combination of $e \sin \omega$ and $e \cos \omega$, and also $(M + \omega)$ instead of M. Some authors calculate a, e, ω, M separately and combine them in the radius vector and in the argument of latitude. The small divisor then cancels out.

This method is used by Kozai (Ref. 1.5), Garfinkel(Ref. 1.6), Brouwer (Ref. 1.7), Krause (Ref. 1.8), Anthony and Fosdick (Ref. 1.9), and others. Izsak's work (Ref. 1.10) is generally considered as belonging to this group, but Izsak considers himself as a proponent of the method of basic variables, (such a claim is not fully substantiated by his type of analytical treatment).

2.1.2.2.2 The Method of Basic Coordinates

In this method the motion of the orbital plane as a rigid body is introduced. The motion is represented by an instantaneous rotation about the position vector $\bar{r}$, reflecting the rotation of the velocity vector about $\bar{r}$ which, in turn, causes the change in Ω (nodal longitude) and i (orbital inclinations). Such a method possesses the desirable possibility of directly representing a succession of satellite positions and reveals the actual motion of the satellite. Basic rectangular, spherical, or oblate-spheroidal coordinates are used in the analysis, and the effect of the perturbation are expressed by differential equations of motion in terms of such coordinates and the perturbing accelerations. Struble (Ref. 1.12) and King-Hele (Ref. 1.11) are the principal proponents of this method.

2.1.2.3 Integration Procedures

The integration procedures employed in various papers depend upon the specific method used in the analytical treatment of the problem. In certain cases, even though two papers can be categorized under the same method, the respective analytical treatments of the basic philosophy (which is characteristic of the method itself) may not follow the same line and, consequently, different integration procedures will necessarily apply.

Kozai, Krause, Anthony, and Fosdick, whose works were categorized under the method of osculating ellipse, replace the time argument, in the definitions of the rates of the osculating elements, by the true anomaly and proceed to integrate these rates directly over a revolution. However, Garfinkel and Brouwer, who were also categorized under this method, do not define the rates of the osculating elements directly. Instead, they derive the perturbations in the osculating elements by Von Zeipel's modification of the method of Delaunay and arrive at closed form solutions in terms of elliptic integrals. In his lunar theory, Delaunay uses a succession of transformations to remove, one by one, the periodic terms of the determining Hamiltonian, whereas in Von Zeipel's modification, a single transformation accomplishes the same purpose. The secular terms, however, are derived directly from the Hamiltonian.

The integration procedure used in connection with the method of basic coordinates consists of the following steps: first, the equations of satellite motion are defined in terms of some basic coordinates (rectangular, spherical, oblate-spheroidal, etc.) and then the integration is performed by seeking a particular solution in one of the following forms,

$$u = \frac{1}{r} = \frac{1}{p}\left[1 + e\cos(\Psi - \beta) + J\nu + Jew\right]$$

or

$$u = \frac{1}{r} = \frac{1}{r_*}\left[1 + e\cos(\bar{\beta} - \omega) - Jc + J^2 d\right]$$

where ψ is the central angle referred to the argument of latitude u* by the relation, $\psi = u^* - 90°$; β is referred to the argument of perigee ω by the relation, $\beta = \omega - 90°$; $\bar{\beta}$ is the perturbed argument of latitude; r* is the harmonic mean value of the radius, resulting from the distortion of the central force field; (v, w) and (c, d) are unknown functions to be determined by integration.

The integration process involves a number of artifices and the replacement of the time argument by one of the coordinates, usually by ψ or $\bar{\beta}$. The choice of ψ or $\bar{\beta}$ has the geometrical advantage that these coordinates are less subject to perturbational variations than is the true anomaly η when the eccentricity is small.

2.1.2.4 Critical Evaluation of the Papers Reviewed

2.1.2.4.1 The Method of Osculating Ellipse

This is a classical astronomy concept based on a precessing orbital plane and an osculating ellipse (in the rotating plane) which is defined by the instantaneous position and velocity vectors. The osculating elements for each revolution are obtained either by direct integration of their rates of change, using Lagrange's planetary equations, or the perturbations in the osculating elements are derived by the principles of Hamiltonian mechanics, or by some other classical astronomy artifice (as in the case of Anthony and Fosdick, who employ Lindstedt technique to obtain approximate solutions for the differential equations of satellite motion).

2.1.2.4.1.1 The Work of Y. Kozai (Reference 1.5)

Assumptions: The earth is an oblate spheroid with axial symmetry only; the density distribution of the earth is symmetrical about its axis of rotation; the gravitational field is represented by the standard potential with spherical harmonics from the second through the fourth present; no limitations are imposed as to the order of magnitudes of the eccentricity and orbital inclination.

Completeness: This is a complete first-order theory which includes secular and both short- and long-periodic perturbations. A complete set of workable osculating elements is furnished including expressions for the perturbed radius and the argument of latitude. The special cases of e = 0 and i = 0 are also covered. The secular terms contain the J_2, J_4, and J_2^2 harmonics, the short-period terms are limited to J_2, and the long-period terms are expressed in J_2, J_3/J_2, and J_4/J_2.

Evaluation: The perturbations in the osculating elements are expressed as functions of the mean orbital elements, the perturbing acceleration, and time by making use of Lagrangian definitions of the time rates of change. The time argument is replaced by the true anomaly and the first-order secular Q_1, second-order secular Q_2, long periodic Q_3, and the short periodic Q_4 parts of the perturbing potential are derived. The analytical treatment that follows is simple and provides a clear geometrical interpretation of the problem. To remove, for instance, the short-period perturbations, one has only to replace the perturbing potential Q by Q_4 in the Lagrangian definition for the variations of the osculating elements. Kozai's is a complete first-order theory, very rigorous, meaningful, and easy to follow. It provides a clear insight into the geometrical aspects of the problem and appeals to an engineer with its straightforward analytical treatment. The solutions are simple, elegant, and meaningful.

2.1.2.4.1.2 The Work of B. Garfinkel (Reference 1.6)

Assumptions: The earth is a non-uniform spheroid with axial and equatorial symmetry; the gravitational field of the earth is independent of longitude and is represented by a special potential function (Garfinkel's potential) which does not fit exactly the standard earth's potential (with the second and fourth harmonics present), but approximates it closely enough to make the Hamilton-Jacobi equation separable.

Completeness: The theory is complete in position coordinates only. The perturbations in the osculating elements are derived which, in turn, define the perturbed spherical position coordinates. The periodic changes are of the first order and the secular changes are of the second order.

Evaluation: Garfinkel's technique involves the preliminary determination of a non-Keplerian intermediary orbit based on an approximation of the standard potential. The approximating potential incorporates a major portion of the second-spherical harmonic and preserves all of the basic features of the standard potential. This unique potential affords separability of the Hamilton-Jacobi equation in spherical coordinates and leads to closed-form solution in terms of elliptic functions with no secular variations of the first order. The non-Keplerian intermediary orbit is then taken as the unperturbed orbit in Delaunay theory. The secular terms are obtained directly from the determining Hamiltonian. To remove the periodic terms from the determining Hamiltonian, Garfinkel makes use of Von Zeipel's modification of Delaunay's method. Garfinkel's analytical treatment of the problem is focused primarily on the satisfaction of the principles of classical astronomy. There is no discussion of the geometrical and physical aspects of the problem and no assessment of the solutions derived.

2.1.2.4.1.3 The Work of D. Brouwer (Reference 1.7)

Assumptions: The earth is a non-uniform spheroid with axial symmetry only; the gravitational field of the earth is independent of longitude and may be represented by Struble's potential which is a slightly modified Jeffrey's potential; there is no limitation on the eccentricity or orbital inclination.

Completeness: A complete set of workable osculating elements is presented. This is a complete second-order theory in both position and velocity coordinates. The periodic terms, both of short and long period, are developed to first order and the secular terms are developed to second order.

Evaluation: Brouwer defines the problem in terms of Delaunay variables by using Hamiltonian mechanics. He then applies Von Zeipel's modification of Delaunay's method to remove the periodic terms from the determining Hamiltonian, whereas he obtains the secular terms directly from the Hamiltonian. It appears from the analytical treatment of the problem that Brouwer's primary intention was to present a solution satisfying the basic principles of classical astronomy. From this point of view, Brouwer's technique is perhaps one of the most remarkable. Unfortunately, the author presents neither a comprehensive discussion, other than of a purely mathematical nature, nor an assessment of the results.

2.1.2.4.1.4 The Work of H. G. L. Krause (Reference 1.8)

Assumptions: The earth is an oblate spheroid with axial and equatorial symmetry; the gravitational field of the earth is independent of longitude and is represented by the standard potential with the second and fourth harmonics present.

Completeness: The solution is an approximate first-order theory, since long-period terms are not derived. Short period and secular terms are limited to those containing eccentricity up to the third power. Periodic perturbations are of first order and the secular are of second order. A complete set of workable osculating elements is furnished. The solutions are of closed form.

Evaluation: Lagrange's definitions for the time rates of change of the osculating elements are used, in which the time argument is replaced by the true anomaly, and the rates are then integrated in closed form over a revolution. Only short period and secular terms are obtained. Since long-period perturbations are neglected, this is an approximate first-order theory. The analytical treatment is simple and straightforward.

2.1.2.4.1.5 The Work of M. L. Anthony and G. E. Fosdick (Reference 1.9)

Assumptions: The earth is an oblate spheroid with axial and equatorial symmetry; the potential field of the earth is independent of longitude and may be represented by the standard potential limited to the principal term and the second harmonic; the initial position of the satellite is at an apsis; no restriction is placed on either the eccentricity or orbital inclination.

Completeness: This is an incomplete and approximate first-order theory in the second harmonic J. Solutions are derived for the perturbations in radial distance r, speed V, and angular momentum P, deviation from the initial plane of motion Θ, and the rate of apsidal advance $\Delta\omega$.

Evaluation: The equations of satellite motion are defined in terms of the spherical coordinates r, Θ (deviation from the initial plane of motion), ϕ (argument of latitude). Approximate solutions of the differential equations are found by the method of Lindstedt, by assuming power series expansions in the second harmonic J for all variables and, then, truncating the series beyond the first power of J. The truncated series expansions are of the form

$$u = u_o(\xi) + J u_1(\xi)$$

$$P = P_o(\xi) + J P_1(\xi)$$

$$\theta = \frac{\pi}{2} + J\theta_1(\xi)$$

where $u = 1/r$, P is the angular momentum, and ξ is a new independent variable defined by

$$\phi = \xi(1 + J\phi_1)$$

The quantities with the "o" subscript apply to the two-body problem (for which $J = 0$), whereas the quantities with the subscripts "1" reflect perturbations due to oblateness and are determined by integration of the differential equations of motion.* The analog of the eccentricity is expressed by the parameter

$$\eta = c^2 - 1 = \left(\frac{V_o}{V}\right)^2 - 1$$

where V_o/V_c is the ratio of the initial orbital speed (at an apsis) and the corresponding circular speed.

It is not very clear how the independent variable ξ_o compares with the classical ω_o (initial argument of perigee), nor how the eccentricity analog η may depend on the classical eccentricity e. The transformations are far from obvious. The constant Φ_1 is so chosen as to eliminate secular terms in the solution for $u = 1/r$ expressed as a function of ξ. The new variables employed in the analytical treatment do not have a simple geometrical interpretation, and the nature of the periodic perturbations, embodied in the pseudo-argument of latitude ξ, is not defined. It is not clear whether the periodic perturbations are short-term periodic, or perhaps a combination of both short and long periodic terms. It is also obscure as to how the secular variation in Ω (the nodal longitude) is obtained.

2.1.2.4.2 The Method of Basic Coordinates

The orbital plane is considered to be a rigid body rotating about the instantaneous position vector, and the motion of the satellite in the orbital plane is along a non-closed plane curve representing a succession of satellite positions. Basic coordinates are used instead of osculating elements, and the equations of satellite motion, expressed in terms of such coordinates, are integrated by seeking a particular solution of the form

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* except for Φ_1 which is a constant.

$$u = \frac{1}{r} = \frac{1}{p}\left[1 + e \cos(\nu - \beta) + J v + J e w\right]$$

or

$$u = \frac{1}{r} = \frac{1}{r_*}\left[1 + e \cos(\bar{\beta} - \omega) - J c + J^2 d\right]$$

2.1.2.4.2.1 The Work of D. G. King-Hele (Reference 1.11)

Assumptions: The earth is a non-uniform oblate spheroid; the gravitational field of the earth is independent of longitude, symmetrical about the equatorial plane, and is defined by Jeffrey's potential function; the eccentricity $e \leq 0.05$.

Completeness: The theory in itself is self-sufficient to describe the problem completely, but unfortunately the author does not extend it to its full capacity and does not derive expressions for the periodic changes in a (semi-major axis), e (eccentricity), and i (orbital inclination). However, the fact that King-Hele gives an incomplete set of workable elements does not weaken the power and the originality of his approach, as the analysis can easily be extended to also cover the periodic changes in a, e, and i.

Evaluation: King-Hele has developed a novel and powerful method for the solution of the earth oblateness perturbation problem which is completely divorced from the classical astronomy concept of osculating ellipse. His analysis is very rigorous and easy to follow. The method assumes that the actual equation of the plane curve, representing a succession of satellite positions, is of the form $u = 1/r = [1 + e \cos(\nu - \beta) + Jv + Jew]1/p$ and the equations of satellite motion are integrated by imposing this particular solution.

2.1.2.4.2.2 The Work of R. A. Struble (Reference 1.12)

Assumption: The earth is a non-uniform oblate spheroid symmetrical about the equatorial plane; the gravitational field of the earth is independent of longitude and is defined by the standard potential function; there is no limitation whatsoever as to the eccentricity of orbital inclination.

Completeness: The analysis is not completely self-sufficient since no procedure is presented for the integration of $d\Omega/dt$ and for the $dt/d\bar{\beta}$ (where $\bar{\beta}$ is the perturbed argument of latitude). Instead, Struble suggests the introduction of these quantities into the solution via the method of averaging which he developed in a separate paper (Reference 1.13).

Evaluation: Struble's approach follows basically the concept and the principles employed by King-Hele and may have been influenced by him, although King-Hele is not reported in Struble's list of references. The analysis, however, is extended to yield second-order solutions. Equations are presented with the burden of proof on the reader; and the logic of successive steps in the analytical treatment is indicated in an intricate and confusing manner of cross-referencing to various sets of equations. Similarly to King-Hele's approach, Struble assumes that the actual equation of the plane curve, representing a succession of satellite positions, is of the form

$$u = \frac{1}{r} = \frac{1}{r_*}\left[1 + e\cos(\bar{\beta} - \omega) - Jc + J^2 d\right]$$

where $\bar{\beta}$ is the perturbed argument of latitude, and proceeds to integrate the satellite equations of motion by imposing this solution. The short-period perturbations are isolated in the c and d functions, which are determined by integration; the mean radius r^*, the mean eccentricity e, and the mean argument of perigee ω exhibit only long-period oscillations (with a secular variation in ω). Unlike King-Hele, who employs basic coordinates as variables, Struble introduces the perturbed argument of latitude $\bar{\beta}$ as the independent variable**, related to the unperturbed argument of latitude β by the identity $d\beta/d\bar{\beta} = k/(k\cos i)\, r^2 (d\beta)/dt$, where the parameter k is not known a priori, but is determined later in such a manner as to make β and $\bar{\beta}$ exhibit the same secular behavior. The solutions are expressed in terms of the mean inclination i_o, the perturbed argument of latitude $\bar{\beta}$, the argument of perigee ω , the eccentricity e, the constants of integration i_{oo}, ω_o, e_o, and the functions η_1, η_2, η_3, η_4, η_5, η_6, v_1 v_2. These functions represent lengthy collections of trigonometric terms; some of them are one-page long expressions. Hence, Struble's analytical treatment appears to be extremely lengthy, intricate, and cumbersome. However, this is not a reflection on the method employed, but rather on the mathematical application of the method.

2.1.2.5 Selection of Papers for Detailed Development

The two papers selected for detailed analytical development are:

King-Hele, D. G. "The Effect of the Earth Oblateness on the Orbit of a Near Satellite," Proceedings of the Royal Society. Series A, No. 1248 (September 1959).

Kozai, Y. "The Motion of a Close Earth Satellite," Astronomical Journal. (October 1959).

These two authors were selected because their works appear to be the most outstanding representatives of the two methods of approach. Kozai's work is based on the concept of osculating ellipse, whereas the work of King-Hele follows the method of basic coordinates. The analytical treatment in both papers is rigorous and easy to follow. Further, both papers provide clear geometrical interpretation of the problem and display original and interesting characteristics in the results and in the approach which are unique. From the point of view of engineering applicability, these two papers excel with their straightforward presentation of this complex problem.

** The new independent variable $\bar{\beta}$, according to Struble, preserves some of the mathematical simplicity of the system which would be lost if the unperturbed β were used as independent variable.

2.1.3 Analytical Development of King-Hele's Technique

2.1.3.1 Derivation of the Equations of Motion

The coordinate frame of reference is the $\bar{i}_N\ \bar{j}_N\ \bar{k}_N$ earth-equatorial frame centered at the earth; $\bar{i}_N$ pointing in the direction of the ascending node $\bar{N}$ of the orbital plane at the earth's equator, $\bar{k}_N$ along the earth's spin axis, and $\bar{j}_N$ completes the right-hand system. A vector in this system is defined by the complement of the geocentric latitude Θ, and by the angle Φ, measured counterclockwise from the instantaneous node $\bar{N}$ to the projection of the vector onto the earth's equatorial plane.

Let $\bar{r}$ and $\dot{\bar{r}}$ be the position and velocity vectors of the spacecraft at some time t and $\bar{L}^*$ a unit vector in the direction of $\bar{r}$,

$$\bar{r} = r\bar{L}^* \tag{1.1}$$

The acceleration vector in rotating coordinates is

$$\ddot{\bar{r}} = \ddot{r}\bar{L}^* + 2\dot{r}\dot{\bar{L}}^* + r\ddot{\bar{L}}^* \tag{1.2}$$

The acceleration vector $\ddot{\bar{r}}$ has a component α_r in the direction of $\bar{L}^*$, a component α_Θ in the direction in which Θ increases, and a component α_Φ in the direction in which Φ increases. The direction in which Θ increases is represented by the tangent to the meridian passing through the position vector $\bar{r}$, and is denoted by $\bar{D}^*$, and that in which Φ increases is represented by the normal to the meridian of $\bar{r}$ and is denoted by $\bar{A}^*$. (See Figure 1.)

All three reference pointings, $\bar{L}^*$, $\bar{D}^*$, $\bar{A}^*$, can be defined in terms of Θ and Φ as follows:

$$\begin{aligned}
\bar{L}^* &= i_N \sin\theta\cos\Phi + j_N \sin\theta\sin\Phi + k_N\cos\theta \\
\bar{D}^* &= i_N \cos\theta\cos\Phi + j_N\cos\theta\sin\Phi - k_N\sin\theta \\
\bar{A}^* &= -i_N\sin\Phi + j_N\cos\Phi + k_N(0) = \bar{L}\times\bar{D}
\end{aligned} \tag{1.3}$$

Thus, the derivatives with respect to time are:

$$\begin{aligned}
\dot{\bar{L}}^* &= (i_N\cos\theta\cos\Phi + j_N\cos\theta\sin\Phi - k_N\sin\theta)\dot{\theta} \\
&\quad + (-i_N\sin\Phi + j_N\cos\Phi)\dot{\Phi}\sin\theta = \bar{D}\dot{\theta} + \bar{A}\dot{\Phi}\sin\theta \\
\dot{\bar{D}}^* &= -(i_N\sin\theta\cos\Phi + j_N\sin\theta\sin\Phi + k_N\cos\theta)\dot{\theta} \\
&\quad + (-i_N\sin\Phi + j_N\cos\Phi)\dot{\Phi}\cos\theta = -\bar{L}\dot{\theta} + \bar{A}\dot{\Phi}\cos\theta \\
\dot{\bar{A}}^* &= \dot{\bar{L}}^*\times\bar{D}^* + \bar{L}^*\times\dot{\bar{D}}^* = (\bar{D}^*\dot{\theta} + A^*\dot{\Phi}\sin\theta)\times\bar{D}^* \\
&\quad + \bar{L}^*\times(-\bar{L}^*\dot{\theta} + \bar{A}^*\dot{\Phi}\cos\theta) = -\bar{L}^*\dot{\Phi}\sin\theta - \bar{D}^*\dot{\Phi}\cos\theta
\end{aligned} \tag{1.4}$$

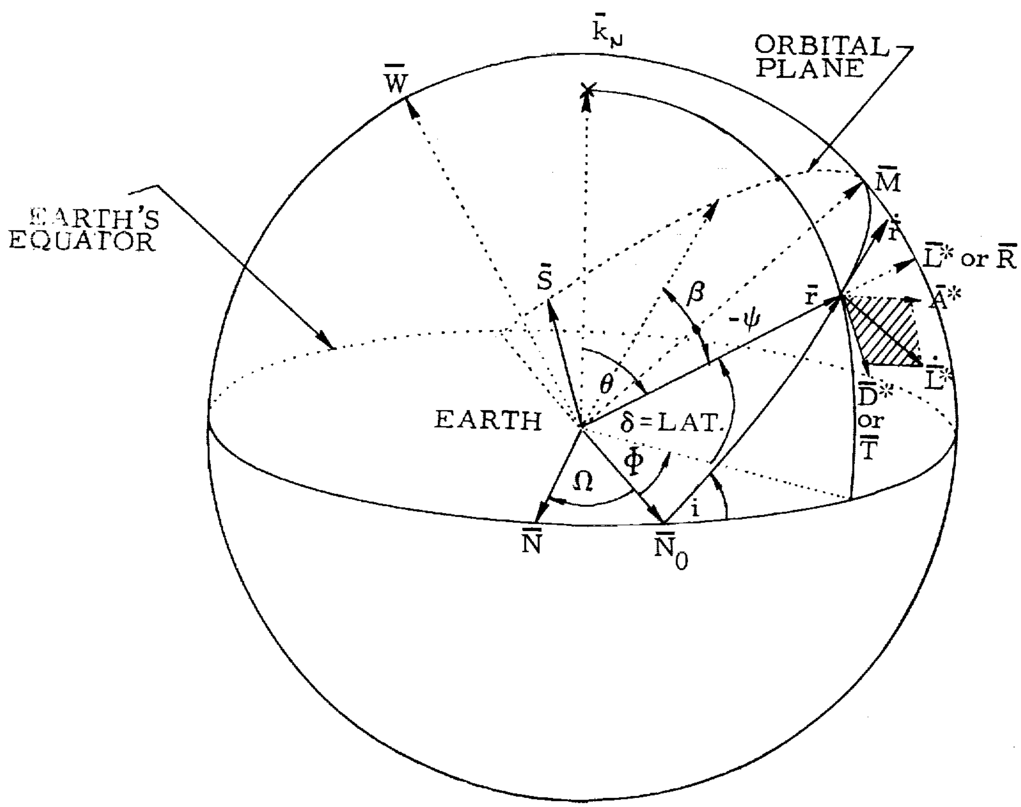

FIGURE 1. ORBITAL GEOMETRY

and

$$\ddot{\bar{L}} = \bar{D}^*\ddot{\theta} + \dot{\bar{D}}^*\dot{\theta} + \dot{\bar{A}}^*\dot{\Phi}\sin\theta + \bar{A}^*(\ddot{\Phi}\sin\theta + \dot{\Phi}\dot{\theta}\cos\theta)$$

$$= \bar{D}^*\ddot{\theta} + (-\bar{L}^*\dot{\theta} + \bar{A}^*\dot{\Phi}\cos\theta)\dot{\theta} - (\bar{L}^*\sin\theta + \bar{D}^*\cos\theta)\dot{\Phi}^2\sin\theta$$

$$+ \bar{A}^*(\ddot{\Phi}\sin\theta + \dot{\Phi}\dot{\theta}\cos\theta) \qquad (1.5)$$

$$= -\bar{L}^*(\dot{\theta}^2 + \dot{\Phi}^2\sin^2\theta) + \bar{D}^*(\ddot{\theta} - \dot{\Phi}^2\sin\theta\cos\theta)$$

$$+ \bar{A}^*(\ddot{\Phi}\sin\theta + 2\dot{\Phi}\dot{\theta}\cos\theta)$$

Substitution of $\dot{\bar{L}}^*$ and $\ddot{\bar{L}}^*$ from Equations (1.4) and (1.5) into Equation (1.2) yields the acceleration vector in rotating coordinates,

$$\ddot{\bar{r}} = \bar{L}^*\left[\ddot{r} - r(\dot{\theta}^2 + \dot{\Phi}^2\sin^2\theta)\right] + \bar{D}^*\left[2\dot{r}\dot{\theta} + r(\ddot{\theta} - \dot{\Phi}^2\sin\theta\cos\theta)\right] \qquad (1.6)$$

$$+ \bar{A}^*\left[2\dot{r}\dot{\Phi}\sin\theta + r(\ddot{\Phi}\sin\theta + 2\dot{\Phi}\dot{\theta}\cos\theta)\right]$$

The gravitation acceleration vector for the earth's spheroid is defined as follows:

$$\nabla \bar{U} = \bar{L}^* \frac{\partial U}{\partial r} + \bar{D}^* \frac{1}{r}\frac{\partial U}{\partial \theta} + \bar{A}^* \frac{1}{r \sin\theta}\frac{\partial U}{\partial \Phi} \tag{1.7}$$

Since, in King-Hele's theory, the earth is assumed to be symmetrical with respect to the equatorial plane, the third harmonic H is necessarily equal to zero and, therefore, U, the gravitational potential function, is given by

$$U = \mu\left[\frac{1}{r} + \frac{J}{3}\frac{R^2}{r^3}(1-3\cos^2\theta) + \frac{D}{35}\frac{R^4}{r^5}(35\cos^4\theta - 30\cos^2\theta + 3)\right]^{**} \tag{1.8}$$

Truncating this expansion at J and substituting the resulting expression for U into (1.7) yields

$$\nabla \bar{U} = \bar{L}^*\left[-\frac{\mu}{r^2} - \mu J\frac{R^2}{r^4}(1-3\cos^2\theta)\right] + \bar{D}^*\left[2\mu J\frac{R^2}{r^4}\sin\theta\cos\theta\right] + A^*(0)^{**} \tag{1.9}$$

Finally, term-by-term comparison of the respective components of (1.6) and (1.9) gives

$$\ddot{r} = r(\dot{\theta}^2 + \dot{\Phi}^2\sin^2\theta) = -\frac{\mu}{r^2} - \mu J\frac{R^2}{r^4}(1-3\cos^2\theta) \tag{1.10}$$

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) - r\dot{\Phi}^2\sin\theta\cos\theta = 2\mu J\frac{R^2}{r^4}\sin\theta\cos\theta \tag{1.11}$$

$$\frac{1}{r\sin\theta}\frac{d}{dt}(r^2\dot{\Phi}\sin^2\theta) = 0 \tag{1.12}$$

2.1.3.2 Preparatory Steps for the Integration of the Equations of Motion

The angles θ and Φ were defined in the preceding section. Now, two additional angles ψ and β will be defined; both of these angles are measured counterclockwise in the orbital plane from the direction $\bar{M}$, which is 90° ahead of the Node in the direction of motion. ψ defines the position of the spacecraft in the orbital plane and β that of the perigee point. The inclination of the orbital plane to the earth's equator plane is denoted by i, and the instantaneous angle of regression of the node N along the earth's equator by

** The fourth zonal harmonic D is neglected in this analysis.

Ω. It should not be confused with the nodal longitude Ω which defines the node relative to the vernal equinox.

With these conventions, the following orbital relationships exist:

$$\cot\theta = \tan i \sin(\Phi + \Omega) \tag{1.13}$$

$$\cos\theta = \sin i \cos\Psi \tag{1.14}$$

$$\sin\Psi = -\sin\theta \cos(\Phi + \Omega) \tag{1.15}$$

Differentiation of (1.13) and (1.14) with respect to Φ yields

$$\frac{d}{d\Phi}(\cot\theta) = \tan i \cos(\Phi + \Omega)\left(1 + \frac{d\Omega}{d\Psi}\ \frac{d\Psi}{d\Phi}\right) \tag{1.16}$$

$$-\sin\theta\ \frac{d\theta}{d\Phi} = -\sin i \sin\Psi\ \frac{d\Psi}{d\Phi} \tag{1.17}$$

Now substituting for sin ψ from (1.15), relation (1.17) becomes

$$-\frac{d\theta}{d\Phi} = \sin i \cos(\Phi + \Omega)\ \frac{d\Psi}{d\Phi} \tag{1.18}$$

or

$$\sin^2\theta\ \frac{d}{d\Phi}(\cot\theta) = \sin i \cos(\Phi + \Omega)\ \frac{d\Psi}{d\Phi} \tag{1.19}$$

Finally, after substitution for $d/d\Phi(\cot\theta)$ from (1.16), relation (1.19) becomes

$$\sec i \sin^2\theta\left(1 + \frac{d\Omega}{d\Psi}\ \frac{d\Psi}{d\Phi}\right) = \frac{d\Psi}{d\Phi} \tag{1.20}$$

which, upon solving for $d\psi/d\Phi$, yields

$$\frac{d\Psi}{d\Phi} = \frac{\sec i \sin^2\theta}{1 - \sec i \sin^2\theta\ \frac{d\Omega}{d\Psi}} = \sec i \sin^2\theta\left(1 + \sec i \sin^2\theta\ \frac{d\Omega}{d\Psi}\right) \tag{1.21}$$

But, from (1.13) and (1.15)

$$\tan i \sin(\Phi + \Omega) = \cot\theta$$

$$\sin\theta \cos(\Phi + \Omega) = -\sin\Psi$$

Thus, substitution of these expressions into (1.26) yields

$$\frac{d^2}{d\Phi^2}(\cot\theta) + \cot\theta = -2\left(\frac{d\Omega}{d\Psi}\frac{d\Psi}{d\Phi}\right)\sin\theta\cos\theta\left(\frac{1}{\sin^2\theta} + \tan^2 i - \frac{\cos^2\theta}{\sin^2\theta}\right) - \sec^3 i \sin^3\theta \sin i \sin\Psi \frac{d^2\Omega}{d\Psi^2} \qquad (1.27)$$

But, from Equation (1.14)

$$\sin i = \frac{\cos\theta}{\cos\Psi}$$

and

$$\frac{1}{\sin^2\theta} + \tan^2 i - \frac{\cos^2\theta}{\sin^2\theta} = \sec^2 i$$

by identity, so that substitution of these expressions and (1.22) into (1.27) yields

$$\frac{d^2}{d\Phi^2}(\cot\theta) + \cot\theta = -2\sec^3 i \sin^3\theta \cos\theta\left(\frac{d\Omega}{d\Psi} + \frac{1}{2}\tan\Psi\frac{d^2\Omega}{d\Psi^2}\right) \qquad (1.28)$$

2.1.3.3 Integration of the Equations of Motion

Equation (1.12) can be integrated in a straightforward manner:

$$r^2\dot{\Phi}\sin^2\theta = h\cos i \qquad (1.29)$$

where (h cos i) is the component of the angular momentum normal to the earth's equator.

Equation (1.11) cannot, however, be integrated in a direct manner. Instead, it will be rewritten in a different form suitable for the purpose of this analysis. First, the time derivatives will be transformed to derivatives with respect to Φ , Terms having J^2 or Je^2 as a factor are neglected in this analysis and since $d\Omega/d\psi$ has J as a factor**, $(d\Omega/d\psi)^2$ is consequently neglected.

Therefore, the following approximations are introduced:

$$\frac{d\Omega}{d\Psi}\;\frac{d\Psi}{d\Phi} \sim \sec i \sin^2\theta\;\frac{d\Omega}{d\Psi} \tag{1.22}$$

$$\left(1+\frac{d\Omega}{d\Psi}\;\frac{d\Psi}{d\Phi}\right)^2 \sim 1+2\sec i\sin^2\theta\;\frac{d\Omega}{d\Psi} \tag{1.23}$$

$$\frac{d^2\Omega}{d\Psi^2}\;\frac{d\Psi}{d\Phi} \sim \sec i\sin^2\theta\;\frac{d^2\Omega}{d\Psi^2} \tag{1.24}$$

Differentiating (1.22) with respect to Φ

$$\frac{d}{d\Phi}\left(\frac{d\Omega}{d\Psi}\;\frac{d\Psi}{d\Phi}\right) \sim 2\sec i\sin\theta\cos\theta\;\frac{d\theta}{d\Phi}\;\frac{d\Omega}{d\Psi} + \sec i\sin^2\theta\;\frac{d^2\Omega}{d\Psi^2}\;\frac{d\Psi}{d\Phi}$$

which, after substituting for $d\theta/d\Phi$ from (1.18) and for $\frac{d^2\Omega}{d\Psi^2}\;\frac{d\Psi}{d\Phi}$ from (1.24), becomes

$$\frac{d}{d\Phi}\left(\frac{d\Omega}{d\Psi}\;\frac{d\Psi}{d\Phi}\right) \sim -2\tan i\cos(\Phi+\Omega)\sin\theta\cos\theta\left(\frac{d\Omega}{d\Psi}\;\frac{d\Psi}{d\Phi}\right) + \sec^2 i\sin^4\theta\;\frac{d^2\Omega}{d\Psi^2} \tag{1.25}$$

Similarly, differentiation of relation (1.16) with respect to Φ yields

$$\frac{d^2}{d\Phi^2}(\cot\theta) = -\tan i\sin(\Phi+\Omega)\left(1+\frac{d\Omega}{d\Psi}\;\frac{d\Psi}{d\Phi}\right)^2 + \tan i\cos(\Phi+\Omega)\frac{d}{d\Phi}\left(\frac{d\Omega}{d\Psi}\;\frac{d\Psi}{d\Phi}\right)$$

and making use of relations (1.23) and (1.25)

$$\begin{aligned}\frac{d^2}{d\Phi^2}(\cot\theta) = {}&\tan i\sin(\Phi+\Omega)\left(1+2\frac{d\Omega}{d\Psi}\;\frac{d\Psi}{d\Phi}\right)\\ &-2\tan^2 i\cos^2(\Phi+\Omega)\sin\theta\cos\theta\left(\frac{d\Omega}{d\Psi}\;\frac{d\Psi}{d\Phi}\right)\\ &+\tan i\cos(\Phi+\Omega)\sec^2 i\sin^4\theta\;\frac{d^2\Omega}{d\Psi^2}\end{aligned} \tag{1.26}$$

** See Equations (1.39) and (1.40).

$$\frac{1}{r}\dot{\Phi}\frac{d}{d\Phi}\left(r^2\dot{\Phi}\frac{d\theta}{d\Phi}\right) - r\dot{\Phi}^2\sin\theta\cos\theta = 2\mu J\frac{R^2}{r^4}\sin\theta\cos\theta \qquad (1.30)$$

Dividing by $r\sin^2\theta$, (1.30) becomes

$$\frac{\dot{\Phi}}{r^2\sin^2\theta}\frac{d}{d\Phi}\left(r^2\dot{\Phi}\frac{d\theta}{d\Phi}\right) - \dot{\Phi}^2\cot\theta = 2\mu J\frac{R^2}{r^4}u\cot\theta \qquad (1.31)$$

where $u = 1/r$. Thus, substitution from relation (1.29) for $r^2\sin^2\theta$, $r^2\dot{\Phi}$, and r^4 yields

$$\frac{\dot{\Phi}^2}{h\cos i}h\cos i\frac{d}{d\Phi}\left(\frac{1}{\sin^2\theta}\frac{d\theta}{d\Phi}\right) - \dot{\Phi}^2\cot\theta$$
$$= 2\mu J\dot{\Phi}^2\frac{R^2}{h^2}u\sec^2 i\sin^3\theta\cos\theta \qquad (1.32)$$

Finally, dividing through by $\dot{\Phi}^2$ and using the identity

$$\frac{d}{d\Phi}\left(\frac{1}{\sin^2\theta}\frac{d\theta}{d\Phi}\right) = -\frac{d^2}{d\Phi^2}(\cot\theta)$$

yields

$$\frac{d^2}{d\Phi^2}(\cot\theta) + \cot\theta = -2\mu J\frac{R^2}{h^2}u\sec^2 i\sin^3\theta\cos\theta \qquad (1.33)$$

At this point, define

$$L = \frac{1}{p} = \frac{1}{a(1-e^2)} \qquad (1.34)$$

and note that

$$\frac{\mu}{h^2} = \frac{1}{p} = L \qquad (1.35)$$

and

$$u = \frac{1}{r} = L\left[1 + e\cos(\Psi - \beta)\right] \qquad (1.36)$$

Thus, on substituting these expressions for μ/h^2 and u, Equation (1.33) becomes

$$\frac{d^2}{d\Phi^2}(\cot\theta) + \cot\theta = -2JL^2R^2\sec^2 i\,\sin^3\theta\,\cos\theta\left[1 + e\cos(\psi - \beta)\right] \qquad (1.37)$$

Comparison of the right-hand sides of Equations (1.28) and (1.37) yields

$$\frac{2\,d\Omega}{d\psi} + \tan\psi\,\frac{d^2\Omega}{d\psi^2} = 2E\left[1 + e\cos(\psi - \beta)\right] \qquad (1.38)$$

where

$$\mathcal{E} = JL^2R^2\cos i \qquad (1.39)$$

A first order solution for $d\Omega/d\psi$, which retains terms of $O(Je)$ and neglects terms of $O(J^2)$, can be assumed to be of the form

$$\frac{d\Omega}{d\psi} = \mathcal{E}\left[1 + e\Lambda + O(J^2)\right] \qquad (1.40)$$

Hence,

$$\frac{d^2\Omega}{d\psi^2} = \mathcal{E}\,e\,\frac{d\Lambda}{d\psi} \qquad (1.41)$$

Substituting these expressions in Equation (1.38) yields

$$\tan\psi\,\frac{d\Lambda}{d\psi} + 2\Lambda = 2\cos(\psi - \beta) \qquad (1.42)$$

Integration of (1.42) with respect to ψ yields

$$\begin{aligned}\Lambda &= -\frac{1}{6\sin^2\psi}\left[3\cos(\psi + \beta) + \cos(3\psi - \beta)\right] \\ &= -\frac{2}{3\sin^2\psi}\left[\cos^3\psi\cos\beta - \sin^3\psi\sin\beta\right]\end{aligned} \qquad (1.43)$$

On substituting this expression for Λ, Equation (1.40) becomes

$$\frac{d\Omega}{d\psi} = \mathcal{E}\left\{1 - \frac{2e}{3\sin^2\psi}\left[\cos^2\psi\cos\beta - \sin^3\psi\sin\beta\right]\right\} \qquad (1.44)$$

Since Λ becomes infinite for multiples of 2π, it will be redefined by the following substitution:

$$\psi = u^* - 90^\circ$$
$$\beta = \omega - 90^\circ \tag{1.45}$$

where both u^* and ω are measured in the orbital plane counterclockwise from its node $\bar{N}$ at the earth's equator; u^* defines the instantaneous position of the spacecraft, and ω (the argument of perigee) defines the perigee point.

As a result of substitutions (1.45), relation (1.44) assumes the form

$$\frac{d\Omega}{du^*} = \varepsilon\left\{1 - \frac{2u}{3\cos^2 u^*}\left[\sin^3 u^* \sin\omega - \cos^3 u^* \cos\omega\right]\right\} \tag{1.46}$$

$$\frac{\Delta\Omega}{REV} = 2\pi\varepsilon - \frac{2Ee}{3}\int_0^{2\pi+\Delta\omega}\left[\sin\omega\,\frac{\sin u^*}{\cos^2 u^*} - \cos(u^* - \omega)\right]du^* \tag{1.47}$$

The first term, $2\pi\epsilon$, represents the regression of the node due to the rotation of the orbital plane about the earth's spin axis as a result of the earth's oblateness. The second term, $\epsilon e \Lambda$, represents the effect of the rotation of the semi-major axis on the nodal displacement (regression or precession).

Integration of the second term in (1.47) with respect to u^* yields

$$\int_0^{2\pi+\Delta\omega}\left[\sin\omega\,\frac{\sin u^*}{\cos^2 u^*} - \cos(u^*-\omega)\right]du^* = \left.\frac{\sin\omega}{\cos u^*} - \sin(u^*-\omega)\right|_0^{2\pi+\Delta\omega}$$
$$= \frac{\sin\omega}{\cos\Delta\omega} + \sin(\omega - \Delta\omega) - 2\sin\omega$$

where $\Delta\omega$ is given by Equation (1.87).

Substitution reduces Equation (1.47) to

$$\frac{\Delta\Omega}{REV} = -\varepsilon\left[2\pi - \frac{2e}{3}\left(\frac{\sin\omega}{\cos\Delta\omega} + \sin(\omega-\Delta\omega) - 2\sin\omega\right)\right]$$
$$\varepsilon = JL^2R^2\cos i \tag{1.48}$$

King-Hele attaches here the (-) sign because he used $(\Phi + \Omega)$ rather than $(\Phi - \Omega)$ in the basic relations (1.13) and (1.15).

It now remains to solve the differential equation (1.10).

$$\ddot{r} - r(\dot{\theta}^2 + \dot{\Phi}^2 \sin^2\theta) = -\frac{\mu}{r^2} - \mu J \frac{R^2}{r^4}(1 - 3\cos^2\theta) \tag{1.49}$$

This step is accomplished by noting from relation (1.35) that

$$\mu = L h^2$$

and from (1.14) that

$$\cos\theta = \sin i \cos\Psi$$

With these substitutions, Equation (1.49) becomes

$$\ddot{r} - r(\dot{\theta}^2 + \dot{\Phi}^2 \sin^2\theta) = -L h^2 u^2 - J L h^2 R^2 u^4 (1 - 3\sin^2 i \cos^2\Psi) \tag{1.50}$$

But,

$$\dot{r} = \frac{d}{dt}\left(\frac{1}{u}\right) = \frac{d\Psi}{dt}\frac{d}{d\Psi}\left(\frac{1}{u}\right) = \dot{\Psi}\left(-\frac{1}{u^2}\ \frac{du}{d\Psi}\right) \tag{1.51}$$

and from Equation (1.29)

$$\dot{\Phi} = \frac{h u^2 \cos i}{\sin^2\theta} \tag{1.52}$$

At this point, Equation (1.21), on using definition (1.40) for $d\Omega/d\Psi$, reduces to

$$\frac{d\Psi}{d\Phi} = \sec i \sin^2\theta \left[1 + \varepsilon(1 + e\Lambda)\sec i \sin^2\theta\right] \tag{1.53}$$

so that

$$\dot{\Psi} = \frac{d\Psi}{d\Phi}\dot{\Phi} = h u^2\left[1 + \varepsilon(1 + e\Lambda)\sec i \sin^2\theta\right] + O(J^2) \tag{1.54}$$

Substitution of (1.54) into (1.51) thus yields

$$\dot{r} = -h \frac{du}{d\Psi}\left[1 + \varepsilon(1+e\Lambda)\sec i \sin^2\theta\right] \tag{1.55}$$

It was mentioned before that, in this analysis, terms containing J^2 and Je^2 are neglected. Since $du/d\Psi$, as derived from Equation (1.36), contains e as a factor, and $\epsilon = JL^2R^2 \cos i$, the product $(du/d\Psi) * \varepsilon * e\Lambda$ will have Je^2 as a factor and is neglected. Therefore,

$$\dot{r} = -h \frac{du}{d\Psi}(1 + \varepsilon \sec i \sin^2\theta) \tag{1.56}$$

Differentiating (1.56) with respect to time give $\ddot{r}$ as

$$\ddot{r} = \frac{d}{dt}(\dot{r}) = \dot{\Psi}\frac{d}{d\Psi}(\dot{r}) \tag{1.57}$$

$$\ddot{r} = -h\dot{\Psi}\left[\frac{d^2u}{d\Psi^2}(1+\varepsilon \sec i \sin^2\theta) + 2\frac{du}{d\Psi}\varepsilon \sec i \cos\theta \sin\theta \frac{d\theta}{d\Psi}\right] \tag{1.58}$$

But from relation (1.17)

$$\sin\theta \frac{d\theta}{d\Psi} = \sin i \sin\Psi \tag{1.59}$$

and from relation (1.54), after dropping the term $\epsilon e \Lambda$ which will yield products with Je^2 as a factor resulting from multiplication by $d^2u/d\Psi^2$ and $du/d\Psi$, $\dot{\Psi}$ reduces to

$$\dot{\Psi} = hu^2(1 + \varepsilon \sec i \sin^2\theta) \tag{1.60}$$

On substituting (1.59) and (1.60) into Equation (1.58), $\ddot{r}$ is obtained as

$$\ddot{r} = -h^2u^2\left[\frac{d^2u}{d\Psi^2}(1+\varepsilon \sec i \sin^2\theta)^2 \right.$$
$$\left. + 2\frac{du}{d\Psi}\varepsilon \tan i \cos\theta \sin\Psi(1+\varepsilon \sec i \sin^2\theta)\right] \tag{1.61}$$

Since $\epsilon^2 = (JL^2R^2 \cos i)^2$ has J^2 as a factor, it should be neglected; so that, to the first order, $\ddot{r}$ is

$$\ddot{r} = -h^2u^2\left[\frac{d^2u}{d\psi^2}(1+2\varepsilon \sec i \sin^2\theta) + 2\frac{du}{d\psi}\varepsilon \tan i \cos\theta \sin\psi\right] \tag{1.62}$$

and since from relation (1.14)

$$\cos\theta = \sin i \cos\psi$$

$$\ddot{r} = -h^2u^2\left\{\frac{d^2u}{d\psi^2} + 2\varepsilon \sec i\left[\frac{d^2u}{d\psi^2}(1-\sin^2 i \cos^2\psi) + \frac{du}{d\psi}\sin^2 i \sin\psi \cos\psi\right]\right\} + O(J^2) \tag{1.63}$$

The next step is to redefine the term $r(\dot{\theta}^2 + \dot{\Phi}^2 \sin^2\theta)$ of Equation (1.50). Since from (1.17)

$$\dot{\theta} = \frac{\sin i \sin\psi}{\sin\theta}\dot{\psi} \tag{1.64}$$

and from relation (1.29)

$$\dot{\Phi} = \frac{h u^2 \cos i}{\sin^2\theta} \tag{1.65}$$

it follows that

$$r(\dot{\theta}^2 + \dot{\Phi}^2\sin^2\theta) = \frac{1}{u}\left(\frac{\sin^2 i \sin^2\psi}{\sin^2\theta}\dot{\psi}^2 + \frac{h^2u^4\cos^2 i}{\sin^2\theta}\right) \tag{1.66}$$

Thus, substituting for $\dot{\psi}$ from (1.54) and neglecting ϵ^2 yields

$$r(\dot{\theta}^2 + \dot{\Phi}^2\sin^2\theta) = h^2u^3\left[\left(\frac{\sin^2 i \sin^2\psi}{\sin^2\theta} + \frac{\cos^2 i}{\sin^2\theta}\right) + 2\varepsilon(1+e\Lambda)\sec i \sin^2 i \sin^2\psi\right] \tag{1.67}$$

Since $\sin^2 i \cos^2\psi = \cos^2\theta$ by relation (1.14), the first term in the brackets of Equation (1.67) reduces to

$$\frac{\sin^2 i \sin^2\psi}{\sin^2\theta} + \frac{\cos^2 i}{\sin^2\theta} = \frac{\sin^2 i - \sin^2 i \cos^2\psi + \cos^2 i}{\sin^2\theta} = \frac{1-\sin^2 i \cos^2\psi}{\sin^2\theta} = 1$$

and, therefore, Equation (1.67) becomes

$$r(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) = h^2 u^3 \left[1 + 2\varepsilon(1 + e\Lambda) \sec i \sin^2 i \sin^2\psi\right] \tag{1.68}$$

Subtraction of (1.68) from (1.63) yields

$$\ddot{r} - r(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) = -h^2 u^2 \left\{ \left(\frac{d^2 u}{d\psi^2} + u\right) + 2\varepsilon \sec i \left[\frac{d^2 u}{d\psi^2}(1 - \sin^2 i \cos^2\psi) \right.\right.$$
$$\left.\left. + \frac{du}{d\psi} \sin^2 i \sin\psi \cos\psi + u(1 + e\Lambda) \sin^2 i \sin^2\psi\right]\right\} + O(J^2) \tag{1.69}$$

where

$$\varepsilon = JL^2 R^2 \cos i$$

Comparing (1.69) with the right-hand side of (1.50) and dividing through by $h^2 u^2$, one has that

$$\frac{d^2 u}{d\psi^2} + u = L + JLR^2 u^2 (1 - 3\sin^2 i \cos^2\psi) - 2JL^2 R^2 \left[\frac{d^2 u}{d\psi^2}(1 - \sin^2 i \cos^2\psi)\right.$$
$$\left. + \frac{du}{d\psi} \sin^2 i \sin\psi \cos\psi + u(1 + e\Lambda) \sin^2 i \sin^2\psi\right] + O(J^2) \tag{1.70}$$

To solve the differential Equation (1.70), a particular solution in the following form is assumed:

$$u = L\left[1 + e\cos(\psi - \beta) + Jv + Jew\right] = \frac{1}{r} \tag{1.71}$$

where v and w are functions to be determined.

Differentiating Equation (1.71) twice with respect to ψ, and denoting the derivatives of β, v, w by primes, yields

$$\frac{d^2 u}{d\psi^2} = L\left[-e\cos(\psi - \beta)(1 - 2\beta') + \beta'' e \sin(\psi - \beta) + Jv'' + Jew''\right] \tag{1.72}$$

where $(1 - \beta')^2$ was approximated by $(1 - 2\beta')$ because $(\beta')^2$ is of order J^2.

Addition of (1.71) and (1.72) yields

$$\frac{d^2u}{d\psi^2}+u=L\left[1+2\beta' e\cos(\psi-\beta)+\beta'' e\sin(\psi-\beta)+J(v''+v)+Je(w''+w)\right] \quad (1.73)$$

But, since u and its derivatives, on the right-hand side of Equation (1.70), are independent of v and w, it is given in the form

$$u=L\left[1+e\cos(\psi-\beta)\right] \quad (1.74)$$

Thus, substituting u and its derivatives from relation (1.74)** in the right-hand side of Equation (1.70) produces

$$\begin{aligned} L+JL^3R^2&\left[1+2e\cos(\psi-\beta)\right](1-3\sin^2 i\cos^2\psi) \\ &-2JL^3R^2\Big[-e\cos(\psi-\beta)(1-\sin^2 i\cos^2\psi) \\ &-\frac{e}{2}\sin(\psi-\beta)\sin^2 i\sin 2\psi \\ &+1+e\cos(\psi-\beta)\sin^2 i\sin^2\psi+e\Lambda\sin^2 i\sin^2\psi\Big] \end{aligned} \quad (1.75)$$

where the term $e^2\Lambda\cos(\psi-\beta)$ was neglected because it is of order Je^2.

The expression in (1.75) will be rewritten as follows,

$$\begin{aligned} L+JL^3R^2&\left[(1-3\sin^2 i\cos^2\psi)-2\sin^2 i\sin^2\psi\right] \\ &+2JL^3R^2e\cos(\psi-\beta)\left[1-3\sin^2 i\cos^2\psi+1-\sin^2 i\cos^2\psi-\sin^2 i\sin^2\psi\right] \\ &+JL^3R^2e\sin(\psi-\beta)\sin^2 i\sin 2\psi-2JL^3R^2e\Lambda\sin^2 i\sin^2\psi \end{aligned}$$

which can be further simplified

$$\begin{aligned} L+\frac{JL^3R^2}{2}&\left[(5\cos^2 i-3)-\sin^2 i\cos 2\psi\right] \\ &+JL^3R^2e\cos(\psi-\beta)\left[(5\cos^2 i-1)-3\sin^2 i\cos 2\psi\right] \\ &+JL^3R^2e\sin(\psi-\beta)\sin^2 i\sin 2\psi-2JL^3R^2e\Lambda\sin^2 i\sin^2\psi \end{aligned} \quad (1.76)$$

Equating (1.73) and (1.76) and dividing through by L, yields

$$\begin{aligned} 2\beta' e\cos(\psi-\beta)&+\beta'' e\sin(\psi-\beta)+J(v''+v)+Je(w''+w) \\ &=\frac{JL^2R^2}{2}\left[(5\cos^2 i-3)-\sin^2 i\cos 2\psi\right] \\ &+JL^2R^2e\Big[\cos(\psi-\beta)(5\cos^2 i-1-3\sin^2 i\cos 2\psi) \\ &+\sin(\psi-\beta)\sin^2 i\sin 2\psi-2\Lambda\sin^2 i\sin^2\psi\Big] \end{aligned} \quad (1.77)$$

** Holding β constant

Since v is the change in radial distance arising from the second-order solution in which terms having Je as a factor are not retained, isolation of $J(v'' + v)$ from Equation (1.77) is accomplished by dropping the Je terms **, so that

$$v'' + v = \frac{L^2R^2}{2}(5\cos^2 i - 3 - \sin^2 i \cos 2\psi) \tag{1.78}$$

Assuming a particular solution,

$$v = A + B\cos 2\psi \tag{1.79}$$

and differentiating twice

$$v'' = -4B\cos 2\psi$$

Substituting v'' and v into Equation (1.78) yields

$$A - 3B\cos 2\psi = \frac{L^2R^2}{2}(5\cos^2 i - 3) - \frac{L^2R^2}{2}\sin^2 i \cos 2\psi$$

whence

$$A = \frac{L^2R^2}{2}(5\cos^2 i - 3)$$

$$B = \frac{L^2R^2}{6}\sin^2 i$$

so that a particular solution of Equation (1.78) will be as follows:

$$v = L^2R^2\left[\frac{5\cos^2 i - 3}{2} + \frac{\sin^2 i}{6}\cos 2\psi\right] \tag{1.80}$$

Substitution for $(v'' + v)$ from (1.78) in Equation (1.77) yields

$$\begin{aligned} &2\beta'\cos(\psi-\beta) + \beta''\sin(\psi-\beta) + J(w''+w) = \\ &= JL^2R^2\Big[(5\cos^2 i - 1)\cos(\psi-\beta) - 3\sin^2 i\cos 2\psi\cos(\psi-\beta) \\ &\quad + \sin^2 i \sin 2\psi \sin(\psi-\beta) - 2\Lambda \sin^2 i \sin^2\psi\Big] \end{aligned} \tag{1.81}$$

** β' and β'' are functions of J.

where, from Equation (1.43),

$$\Lambda = -\frac{1}{6\sin^2\Psi}\left[3\cos(\Psi+\beta)+\cos(3\Psi-\beta)\right]$$

$$-2\Lambda\sin^2 i\sin^2\Psi = \sin^2 i\cos(\Psi+\beta)+\frac{1}{3}\sin^2 i\cos(3\Psi-\beta) \tag{1.82}$$

and

$$\begin{aligned}
-3\sin^2 i\cos 2\Psi\cos(\Psi-\beta)+\sin^2 i\sin 2\Psi\sin(\Psi-\beta) &= \\
&= -2\sin^2 i\cos 2\Psi\cos(\Psi-\beta) \\
&\quad -\sin^2 i\left[\cos 2\Psi\cos(\Psi-\beta)-\sin 2\Psi\sin(\Psi-\beta)\right] \\
&= -\sin^2 i\left[\cos(3\Psi-\beta)+\cos(\Psi+\beta)\right]-\sin^2 i\cos(3\Psi-\beta) \\
&= -2\sin^2 i\cos(3\Psi-\beta)-\sin^2 i\cos(\Psi+\beta)
\end{aligned} \tag{1.83}$$

Combining (1.82) and (1.83) gives

$$\begin{aligned}
&-3\sin^2 i\cos 2\Psi\cos(\Psi-\beta)+\sin^2 i\sin 2\Psi\sin(\Psi-\beta) \\
&\qquad -2\Lambda\sin^2 i\sin^2\Psi = -\frac{5}{3}\sin^2 i\cos(3\Psi-\beta)
\end{aligned}$$

so that Equation (1.81) will assume the form

$$\begin{aligned}
&2\beta'\cos(\Psi-\beta)+\beta''\sin(\Psi-\beta)+J(w''+w) \\
&\qquad = JL^2R^2\left[(5\cos^2 i-1)\cos(\Psi-\beta)-\frac{5}{3}\sin^2 i\cos(3\Psi-\beta)\right]
\end{aligned} \tag{1.84}$$

At this point, King-Hele notes that there is only one way to split (1.84) into two parts without producing divergent solutions for β' and w; that is,

$$2\beta'\cos(\Psi-\beta)+\beta''\sin(\Psi-\beta) = JL^2R^2(5\cos^2 i-1)\cos(\Psi-\beta) \tag{1.85}$$

$$w''+w = -\frac{5}{3}L^2R^2\sin^2 i\cos(3\Psi-\beta) \tag{1.86}$$

Equation (1.85) is linear and of the first order with respect to β'.

$$2\beta' \cos(\Psi-\beta) + \beta'' \sin(\Psi-\beta) = 0$$

$$\beta' = \frac{C}{\sin^2(\Psi-\beta)}$$

$$\beta'' = \frac{C'}{\sin^2(\Psi-\beta)} - \frac{2C}{\sin^3(\Psi-\beta)} \cos(\Psi-\beta)$$

$$\frac{C'}{\sin(\Psi-\beta)} = JL^2R^2(5\cos^2-1)\cos(\Psi-\beta)$$

$$C = JL^2R^2(5\cos^2 i-1)\int \sin(\Psi-\beta)\cos(\Psi-\beta)\,d\Psi + K$$

$$C = \frac{JL^2R^2}{2}(5\cos^2 i-1)\sin^2(\Psi-\beta) + K$$

$$\beta' = \frac{JL^2R^2}{2}(5\cos^2 i-1) + \frac{K}{\sin^2(\Psi-\beta)}$$

$$\frac{\Delta\beta}{R_{EV}} = \pi JL^2R^2(5\cos^2 i-1) = \frac{\Delta\omega}{R_{EV}}\ ^{**} \tag{1.87}$$

NOTE: The arbitrary constant K was set equal to zero to avoid infinite values for $\Delta\beta$ at the limits 0 and 2π.

In order to integrate Equation (1.86), a particular solution is assumed of the form

$$w = K\cos(3\Psi-\beta)$$

$$w'' = -9K\cos(3\Psi-\beta)$$

$$w''+w = -8K\cos(3\Psi-\beta) = -\frac{5}{3}L^2R^2\sin^2 i\cos(3\Psi-\beta)$$

** $\beta = \omega - 90°$ according to relation (1.45), $\beta' = \omega'$ and $\Delta\beta = \Delta\omega$.

$$K = \frac{5}{24} L^2 R^2 \sin^2 i$$

$$W = \frac{5}{24} L^2 R^2 \sin^2 i \cos(3\Psi - \beta) \tag{1.88}$$

$Je(w)$ is the third-order change in radial distance, whereas Jv is the second-order change. See Equation (1.71). $Je(w)$ is of oscillatory nature with a period equal to 1/3 orbital revolution and an amplitude proportional to ($5/24$ $Je \sin^2 i$). Jv consists of two parts: (1) linear constant change over a revolution; (2) oscillatory change with a period equal to 1/2 orbital revolution and an amplitude proportional to ($J/6 \sin^2 i$).

Equation (1.71) will now be rewritten as follows:

$$\frac{1}{r} = L\left[1 + e \cos(\Psi - \beta)\right] + JL\,(v + ew) \tag{1.89}$$

Remembering that the function v is independent of Je (see comment following Equation (1.77)), and that Je^2 terms are not retained in this analysis, Equation (1.89) can be written as

$$\frac{1}{r} = L\left[1 + e \cos(\Psi - \beta)\right]\left[1 + Jv + Jew\right] \tag{1.90}$$

But from Equation (1.74)

$$L\left[1 + e \cos(\Psi - \beta)\right] = \frac{1}{r_0} \tag{1.91}$$

Therefore,

$$\frac{1}{r} = \frac{1}{r_0}\left[1 + Jv + Jew\right] \tag{1.92}$$

and

$$r \sim r_0\left[1 - Jv - Jew\right] \tag{1.93}$$

where

$$v = L^2R^2\left[\frac{5\cos^2 i-3}{2} + \frac{\sin^2 i}{6}\cos 2\psi\right] \tag{1.94}$$

$$w = \frac{5}{24}L^2R^2\sin^2 i\cos(3\psi-\beta) \tag{1.95}$$

Substituting relations (1.94) and (1.95), Equation (1.93) can, therefore, be written in the following form:

$$r = r_0\left[1-JL^2R^2\frac{5\cos^2 i-3}{2}\right]\left[1-\frac{JL^2R^2}{6}\sin^2 i\cos 2\psi - \frac{5}{24}JL^2R^2 e\sin^2 i\cos(3\psi-\beta)\right] \tag{1.96}$$

where

$$\bar{r} = r_0\left(1-JL^2R^2\frac{5\cos^2 i-3}{2}\right)$$

represents the constant part due to the distortion of the central force field.

2.1.3.4 Summary of the Change in the Orbital Elements

2.1.3.4.1 Argument of Perigee and Node

The change in the argument of perigee, $\Delta\omega$, is given by Equation (1.87),

$$\frac{\Delta\omega}{R_{EV}} = \pi JL^2R^2(5\cos^2 i-1) \qquad \text{RAD.} \tag{1.97}$$

The regression of the node, $\Delta\Omega$ is given by Equation (1.48),

$$\frac{\Delta\Omega}{R_{EV}} = -\varepsilon\left[2\pi - \frac{2e}{3}\left(\frac{\sin\omega_0}{\cos\Delta\omega} + \sin(\omega_0-\Delta\omega) - 2\sin\omega_0\right)\right] \quad \text{RAD.} \tag{1.98}$$

where,

$$\varepsilon = JL^2R^2\cos i \tag{1.99}$$

and

$$L = \frac{1}{p} = \frac{1}{a_0(1-e_0^2)} \qquad FT^{-1}$$

2.1.3.4.2 Radial Distance

Using relations (1.45)

$$\psi = u^* - 90^\circ$$
$$\beta = \omega - 90^\circ$$

allows Equations (1.80) and (1.88)** to be written as functions of the argument of latitude u*.

$$v = L^2R^2\left[\frac{5\cos^2 i - 3}{2} - \frac{\sin^2 i}{6}\cos 2u^*\right] \text{(non-dimensional)} \quad (1.100)$$

$$w = -\frac{5}{24}L^2R^2\sin^2 i\cos(3u^* - \omega) \quad \text{(non-dimensional)} \quad (1.101)$$

The magnitude of the perturbed radius vector is obtained from Equation (1.96). Defining

$$\bar{r} = r_0\left[1 - JL^2R^2\,\frac{5\cos^2 i - 3}{2}\right] \quad (1.102)$$

where

$$r_0 = \frac{1/L}{1 + e\cos(u^* - \omega)}$$

yields for r,

$$r = \bar{r}\left[1 + \frac{JL^2R^2}{6}\sin^2 i\cos 2u^* + \frac{5}{24}L^2R^2 e\sin^2 i\cos(3u^* - \omega)\right] \quad (1.103)$$

so that

$$\Delta r = r - \bar{r} = JL^2R^2\bar{r}\left[\frac{\sin^2 i}{6}\cos 2u^* + \frac{5}{24}e\sin^2 i\cos(3u^* - \omega)\right] \quad (1.104)$$

where $\bar{r}$, as given by Equation (1.102) represents the constant change in r_0 due to the distortion of the central force field, and the terms within the brackets represent the periodic oscillatory change with respect to the constant part.

** or Equations (1.94) and (1.95)

2.1.3.4.3 Inclination

King-Hele does not derive an expression for the periodic change in the orbital inclination i. However, his theory is easily extended to cover this change.

From Equation (1.11), the integral (1.29) and relations (1.13), (1.14), and (1.15), it follows that

$$\frac{d\Phi}{dt} = \frac{\cos i}{h \sin\theta}\left[2\mu J \frac{R^2}{r^3}\sin\theta\cos\theta\right] \tag{1.105}$$

Now, from the geometry alone

$$\frac{di}{dt} = -\cos u^* \frac{d\Phi}{dt} = -2\mu J \frac{R^2}{h r^3}\cos i \cos\theta \cos u^* \tag{1.106}$$

From relation (1.14) and the fact that $\Psi = u^* - 90°$

$$\cos\theta = \sin i \cos\Psi = \sin i \sin u^*$$

Therefore, Equation (1.106) can be rewritten as

$$\frac{di}{dt} = -\mu J \frac{R^2}{h r^3}\sin i \cos i \sin 2u \tag{1.107}$$

The time argument is now replaced by the argument of latitude u* by means of Equation (1.60), where $\epsilon = J(R/p)^2 \cos i$

$$dt = \frac{r^2}{h}\left[1 - J\left(\frac{R}{p}\right)^2 \sin^2\theta\right] du^* \tag{1.108}$$

Substituting (1.108) into (1.107) and neglecting terms in J^2 yields

$$\frac{di}{du^*} = -\frac{1}{2} J \frac{R^2}{rp}\sin 2i \sin 2u \tag{1.109}$$

and since by Equation (1.90)

$$\frac{1}{r} = \frac{1}{p}\left[1 + e\cos(u^* - \omega)\right]\left[1 + Jv + Jew\right]$$

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** Since $\Psi = u^* - 90°$, it follows that $d\Psi = du^*$

Equation (1.109) reduces to

$$\frac{di}{du^*} = -\frac{1}{2} J\left(\frac{R}{p}\right)^2 \sin 2i \left[\sin 2u^* + e \sin 2u^* \cos(u^* - \omega)\right] + O(J^2) \tag{1.110}$$

$$= -\frac{1}{4} J\left(\frac{R}{p}\right)^2 \sin 2i \left[2 \sin 2u^* + e \sin(3u^* - \omega) + e \sin(u^* + \omega)\right] + O(J^2)$$

Finally, integration with respect to u^* yields

$$i = i_o + \frac{1}{4} J\left(\frac{R}{p}\right)^2 \sin 2i_o \left[\cos 2u^* + e \cos(u^* + \omega) + \frac{e}{3} \cos(3u^* - \omega)\right] \tag{1.111}$$

2.1.3.4.4 Nodal Period

King-Hele's approach for the determination of the nodal period, P_Ω, is the following

$$P_\Omega = \int_0^{2\pi} \frac{d\Psi}{\dot{\Psi}} \tag{1.112}$$

But $\dot{\Psi}$ is given by Equation (1.60), where $\epsilon = JL^2R^2 \cos i$

$$\dot{\Psi} = \frac{h}{r^2}\left[1 + JL^2R^2 \sin^2\theta\right] \overset{**}{=} \frac{h}{r^2}\left[1 + JL^2R^2(1 - \sin^2 i \cos^2\Psi)\right]$$

$$= \frac{h}{r^2}\left[1 + JL^2R^2\left(\frac{1+\cos^2 i}{2} - \frac{\sin^2 i}{2} \cos 2\Psi\right)\right] \tag{1.113}$$

and r is given by Equation (1.89)

$$r = p_o\left[1 + e \cos(\Psi - \beta) + Jv + Jew\right]^{-1} \tag{1.114}$$

Thus,

$$P_\Omega = \frac{p_o^2}{h} \int_0^{2\pi} \left[1 + e \cos(\Psi - \beta) + Jv + Jew\right]^{-2} \left[1 - JL^2R^2\left(\frac{1+\cos^2 i}{2} - \frac{\sin^2 i}{2} \cos 2\Psi\right)\right] d\Psi \tag{1.115}$$

Using the definitions (1.94) and (1.95) for v and w, and integrating from 0 to 2π yields

---

** $\cos\Theta = \sin i \cos\Psi$ by Equation (1.14)

$$P_\Omega = 2\pi \frac{p_0^{3/2}}{\sqrt{\mu}}\left[1 + \frac{3}{2}e^2 - JL^2R^2 \; \frac{11\cos^2 i - 5}{2}\right] \tag{1.116}$$

where the angular momentum h was assumed constant and set $= \sqrt{\mu p_0}$

Next, the average p is derived as the harmonic mean value of r,

$$\frac{1}{\bar{p}} = \frac{1}{2\pi}\int_0^{2\pi}\frac{d\psi}{r} = \frac{1}{p_0}\left[1 + JL^2R^2 \; \frac{5\cos^2 i - 3}{2}\right] \tag{1.117}$$

whence

$$p_0 = \bar{p}\left[1 + JL^2 \; \frac{5\cos^2 i - 3}{2}\right] \tag{1.118}$$

Considering the momentum as non-constant, p_o in Equation (1.116) is replaced by relation (1.118)

$$P_\Omega = 2\pi \frac{\bar{p}^{3/2}}{\sqrt{\mu}}\left[1 + \frac{3}{2}e^2 - J\left(\frac{R}{\bar{p}}\right)^2 \frac{7\cos^2 i - 1}{4}\right] \tag{1.119}$$

or in terms of $\bar{a}$

$$P_\Omega = 2\pi \frac{\bar{a}^{3/2}}{\sqrt{\mu}}\left[1 - J\,\frac{R^2}{\bar{a}^2} \; \frac{7\cos^2 i - 1}{4}\right] \quad \text{SEC.} \tag{1.120}$$

where $\bar{a}$ is the harmonic mean average value of the semi-major axis given by

$$\bar{a} = a_0\left[1 - JL^2R^2 \; \frac{5\cos^2 i - 3}{2}\right] \tag{1.121}$$

The change in the nodal period, ΔP_Ω , is given by

$$\Delta P_\Omega = P_\Omega - (P_\Omega)_0 = (P_\Omega)_0\left[\frac{P_\Omega}{(P_\Omega)_0} - 1\right] \quad \text{SEC.} \tag{1.122}$$

in which $(P_\Omega)_o$ is the nodal period of the preceding revolution.

2.1.4 Analytical Development of Kozai's Approach

2.1.4.1 The Composite Parts of the Perturbing Potential

Kozai uses the standard potential function of Jeffreys and includes the third harmonic, since he assumes asymetry of the gravitational field with respect to the earth's equatorial plane. Thus, the perturbing potential Q in Kozai's work is given by,

$$Q = U - \frac{\mu}{r} = \mu\left[\frac{A_2}{3r^3}(1 - 3\sin^2\delta) + \frac{A_3}{2r^4}(5\sin^2\delta - 3\sin\delta) + \frac{A_4}{35r^5}(35\sin^4\delta - 30\sin^2\delta + 3)\right] \quad (1.123)$$

where,

$$A_2 = JR_{EQ}^2$$
$$A_3 = -\frac{2}{5}HR_{EQ}^3 \quad (1.124)$$
$$A_4 = DR_{EQ}^4$$

A_2 is of the first order; A_3 and A_4 are of the second order. Kozai derives the periodic perturbations to the first order and the secular perturbations up to the second order. This potential is transformed using the relation,

$$\sin\delta = \sin i \sin u^* \quad (1.125)$$

(where $u^* = \eta + \omega$ is the argument of latitude), and the trigonometric identities

$$\sin^2 u^* = \frac{1}{2} - \frac{1}{2}\cos 2u^*$$
$$\sin^3 u^* = \frac{3}{4}\sin u^* - \frac{1}{4}\sin 3u^* \quad (1.126)$$
$$\sin^4 u^* = \frac{3}{8} - \frac{1}{2}\cos 2u^* + \frac{1}{8}\cos 4u^*$$

The perturbing potential Q then assumes the form,

$$
\begin{aligned}
Q = \mu \Big\{ & \frac{A_2}{a^3}\left(\frac{a}{r}\right)^3 \left[\left(\frac{1}{3}-\frac{1}{2}\sin^2 i\right)+\frac{1}{2}\sin^2 i \cos 2u^*\right] \\
& + \frac{A_3}{a^4}\left(\frac{a}{r}\right)^4 \left[\left(\frac{15}{8}\sin^2 i - \frac{3}{2}\right)\sin u^* - \frac{5}{8}\sin^2 i \sin 3u^*\right]\sin i \\
& + \frac{A_4}{a^5}\left(\frac{a}{r}\right)^5 \Big[3\left(\frac{1}{8}\sin^4 i - \frac{1}{7}\sin^2 i + \frac{1}{35}\right) \\
& + \left(\frac{3}{7}-\frac{1}{2}\sin^2 i\right)\sin^2 i \cos 2u^* + \frac{1}{8}\sin^4 i \cos 4u^*\Big]\Big\}
\end{aligned}
\tag{1.127}
$$

Since $u^* = \eta + \omega$, where η is the true anomaly and ω is the argument of perigee,

$$
\begin{aligned}
\cos 2u^* &= \cos 2\eta \cos 2\omega - \sin 2\eta \sin 2\omega \\
\sin u^* &= \sin \eta \cos \omega + \cos \eta \sin \omega \\
\sin 3u^* &= \sin 3\eta \cos 3\omega + \cos 3\eta \sin 3\omega \\
\cos 4u^* &= \cos 4\eta \cos 4\omega - \sin 4\eta \sin 4\omega
\end{aligned}
\tag{1.128}
$$

In Equation (1.127), the factors in front of the brackets, $(a/r)^3$, $(a/r)^4$, $(a/r)^5$, are multiplied by terms (free from trigonometric functions in η and ω) and by trigonometric terms expounded in relations (1.128). The former products yield the secular contribution of the perturbing potential; the latter products, involving trigonometric functions in η and ω, yield the periodic contribution of the perturbing potential. Terms depending on ω only, and not on η , are long periodic; terms depending on η are short periodic.

In order to separate the first-order secular, second-order secular, the long periodic, and the short-periodic parts of the disturbing potential, the harmonic-mean values of the following terms are first evaluated, employing the relations,

$$r = \frac{p}{1 + e \cos \eta} \tag{1.129}$$

$$dM = \frac{r^2}{a^2 \sqrt{1-e^2}}\, d\eta = (1-e^2)^{3/2} \frac{d\eta}{(1+e\cos\eta)^2} \tag{1.130}$$

$$\left.\begin{aligned}
&\overline{\left(\frac{a}{r}\right)^3} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{a}{r}\right)^3 dM = (1-e^2)^{-3/2} \\
&\overline{\left(\frac{a}{r}\right)^4} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{a}{r}\right)^4 dM = (1-e^2)^{-5/2}\left(1+\frac{1}{2}e^2\right) \\
&\overline{\left(\frac{a}{r}\right)^5} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{a}{r}\right)^5 dM = (1-e^2)^{-7/2}\left(1+\frac{3}{2}e^2\right) \\
&\overline{\left(\frac{a}{r}\right)^3 \sin 2\eta} = \overline{\left(\frac{a}{r}\right)^3 \cos 2\eta} = 0 \\
&\overline{\left(\frac{a}{r}\right)^4 \sin \eta} = \overline{\left(\frac{a}{r}\right)^4 \sin 3\eta} = \overline{\left(\frac{a}{r}\right)^4 \cos 3\eta} = 0 \\
&\overline{\left(\frac{a}{r}\right)^4 \cos \eta} = e(1-e^2)^{-5/2} \\
&\overline{\left(\frac{a}{r}\right)^5 \sin 2\eta} = \overline{\left(\frac{a}{r}\right)^5 \sin 4\eta} = \overline{\left(\frac{a}{r}\right)^5 \sin 4\eta} = 0 \\
&\overline{\left(\frac{a}{r}\right)^5 \cos 2\eta} = \frac{3}{4} e^2 (1-e^2)^{-7/2}
\end{aligned}\right\} \tag{1.131}$$

Introducing identities (1.128) in the perturbing potential (1.127), and multiplying the terms within the brackets by the respective $(a/r)^n$, yields [upon substitution of the mean-harmonic values (1.131)] , the four parts of the disturbing potential,

$$Q_1 = \mu \frac{A_2}{a^3} (1-e^2)^{-3/2} \left(\frac{1}{3} - \frac{1}{2} \sin^2 i\right)$$

$$Q_2 = 3\mu \frac{A_4}{a^5} (1-e^2)^{-7/2} \left(1 + \frac{3}{2} e^2\right)\left(\frac{1}{8} \sin^4 i - \frac{1}{7} \sin^2 i + \frac{1}{35}\right)$$

$$Q_3 = \mu\left[\frac{3}{2} \frac{A_3}{a^4} e(1-e^2)^{-5/2} \left(\frac{5}{4} \sin^2 i - 1\right) \sin i \sin \omega \right. \tag{1.132}$$

$$\left. + \frac{3}{4} \frac{A_4}{a^5} e^2 (1-e^2)^{-7/2} \left(\frac{3}{7} - \frac{1}{2} \sin^2 i\right) \sin^2 i \cos 2\omega\right]$$

$$Q_4 = \mu \frac{A_2}{a^3}\left[\left(\frac{a^3}{r^3} - (1-e^2)^{-3/2}\right)\left(\frac{1}{3} - \frac{1}{2} \sin^2 i\right)\right.$$

$$\left. + \frac{1}{2}\left(\frac{a}{r}\right)^3 \sin^2 i \cos 2(\eta + \omega)\right]$$

where Q_1 is the first-order secular part of the disturbing potential, Q_2 is the second-order secular, Q_3 is the long periodic, and Q_4 is the short-periodic part of the disturbing potential. Note that Q_4 was obtained by subtracting Q_1 from the portion of the disturbing potential (1.127) which has A_2 as a factor.

2.1.4.2 Lagrangian Definitions for the Variations of the Orbital Elements

Kozai uses Lagrange's definitions for the rates of change in the orbital elements and replaces the time argument by the true anomaly by means of the relation,

$$dt = \frac{r^2}{h} d\eta$$

where h is the angular momentum per unit mass.

The Lagrangian definitions for the variations of the orbital elements are:

$$\frac{da}{dt} = \frac{2}{na} \frac{\partial Q}{\partial M} \tag{1.133}$$

$$\frac{de}{dt} = \frac{1-e^2}{he}\left(\sqrt{1-e^2} \frac{\partial Q}{\partial M} - \frac{\partial Q}{\partial \omega}\right) \tag{1.134}$$

$$\frac{d\omega}{dt} = \frac{1}{k}\left(-\cot i \ \frac{\partial Q}{\partial i} + \frac{1-e^2}{e} \ \frac{\partial Q}{\partial e}\right) = \frac{1-e^2}{ke} \ \frac{\partial Q}{\partial e} - \cos i \ \frac{d\Omega}{dt} \quad (1.135)$$

$$\frac{di}{dt} = \frac{\cot i}{k} \ \frac{\partial Q}{\partial \omega} \quad (1.136)$$

$$\frac{d\Omega}{dt} = \frac{1}{k \sin i} \ \frac{\partial Q}{\partial i} \quad (1.137)$$

$$\frac{dM}{dt} = n - \frac{\sqrt{1-e^2}}{k}\left(\frac{1-e^2}{e} \ \frac{\partial Q}{\partial e} + 2a \ \frac{\partial Q}{\partial a}\right) \quad (1.138)$$

$$\frac{d\sigma}{dt} = \frac{dM}{dt} - n \quad (1.139)$$

NOTE: a, n, e, i, and ω may be regarded as constant on the right-hand sides of Equations (1.33) through (1.139) except for n in (1.138) and (1.139). In those later equations, n must be considered a variable. However, it is a known function of time after obtaining the expression for a (semi-major axis).

2.1.4.3 The Secular Perturbations of the First Order

These perturbations are obtained by replacing Q by Q_1 in Lagrangian definitions for d ω /dt, d Ω /dt, dM/dt, as given by relations (1.135), (1.137), and (1.138). To accomplish this objective, the partial derivatives of Q_1 with respect to the elements i, e, a, must be derived.

$$\frac{\partial Q_1}{\partial i} = -\mu \frac{A_2}{a^3} (1-e^2)^{-3/2} \sin i \cos i \quad (1.140)$$

$$\frac{\partial Q_1}{\partial e} = \mu \frac{A_2}{a^3} e(1-e^2)^{-5/2}\left(1-\frac{3}{2}\sin^2 i\right) \quad (1.141)$$

$$\frac{\partial Q_1}{\partial a} = -\mu \frac{A_2}{a^4} (1-e^2)^{-3/2} \left(1 - \frac{3}{2} \sin^2 i\right) \qquad (1.142)$$

Substitution of these partial derivatives into Equations (1.135), (1.137), and (1.138) yields,

$$\overline{\frac{d\omega}{dt}} = \frac{1}{h}\left(-\cot i \frac{\partial Q_1}{\partial i} + \frac{1-e^2}{e} \frac{\partial Q_1}{\partial e}\right) = \frac{A_2}{p^2} \bar{n} \left(2 - \frac{5}{2}\sin^2 i\right) \qquad (1.143)$$

$$\overline{\frac{d\Omega}{dt}} = \frac{1}{h \sin i} \frac{\partial Q_1}{\partial i} = -\frac{A_2}{p^2} \bar{n} \cos i \qquad (1.144)$$

$$\overline{\frac{dM}{dt}} = \bar{n} = n_0\left[1 + \frac{A_2}{p^2}\left(1 - \frac{3}{2}\sin^2 i\right)\sqrt{1-e^2}\right] \qquad (1.145)$$

First Order Secular Perturbations

where,

$$p = a(1-e^2)$$

2.1.4.4 The Short-Period Perturbations

These perturbations are obtained by replacing Q by Q_4 in Lagrange's definition for the variations in the orbital elements as given by relations (1.133) through (1.138). However, the partial derivatives of Q_4 with respect to: M, ω, i, e, a, must be derived first.

$$\frac{\partial Q_4}{\partial M} = \frac{1}{2} A_2 n \frac{d}{dt}\left[\frac{2}{3}\left(\frac{a^3}{r^3} - (1-e^2)^{-3/2}\right)\left(1 - \frac{3}{2}\sin^2 i\right) + \left(\frac{a}{r}\right)^3 \sin^2 i \cos 2(\eta + \omega)\right] \qquad (1.146)$$

$$\frac{\partial Q_4}{\partial \omega} = -\mu A_2 \left(\frac{1}{r}\right)^3 \sin^2 i \sin 2(\eta + \omega) \qquad (1.147)$$

$$\frac{\partial Q_4}{\partial i} = -\mu\frac{A_2}{a^3}\sin i\cos i\left[\left(\frac{a}{r}\right)^3-\left(1-e^2\right)^{-3/2}-\left(\frac{a}{r}\right)^3\cos 2(\eta+\omega)\right]$$
$$= -\mu\frac{A_2}{r^2}\sin i\cos i\left[\frac{1}{r}-\frac{1}{p}\left(\frac{r^2}{a^2\sqrt{1-e^2}}\right)-\frac{\cos 2(\eta+\omega)}{r}\right] \quad (1.148)$$

$$\frac{\partial Q_4}{\partial e} = \frac{\mu}{p}\frac{A_2}{r^2}\left(\frac{e}{1-e^2}\right)\left\{\frac{1}{e}\left(\frac{p}{r}\right)^2\cos\eta\left[\left(1-\frac{3}{2}\sin^2 i\right)+\frac{3}{2}\sin^2 i\cos 2(\eta+\omega)\right]\right.$$
$$\left.-\frac{1}{e}\left(\frac{p}{r}\right)^2\left(1+\frac{r}{p}\right)\sin^2 i\sin\eta\sin 2(\eta+\omega)-\left(1-\frac{3}{2}\sin^2 i\right)\left(\frac{r^2}{a^2\sqrt{1-e^3}}\right)\right\} \quad (1.149)$$

$$\frac{\partial Q_4}{\partial a} = -\mu\frac{A_2}{a^4}\left[\left(\frac{a^3}{r^3}-\left(1-e^2\right)^{-3/2}\right)\left(1-\frac{3}{2}\sin^2 i\right)+\frac{3}{2}\left(\frac{a}{r}\right)^3\sin^2 i\cos 2(\eta+\omega)\right] \quad (1.150)$$

NOTE: The term $\frac{r^2}{a^2\sqrt{1-e^2}}$ is equal to $dM/d\eta$. Hence, the integral of this term with respect to the true anomaly η will be M.

Substitution of these partial derivatives in Equations (1.133) through (1.138) and replacing the time argument by the true anomaly, $dt = \frac{r^2}{h}d\eta$, yields the following integrated solutions in which the argument must be replaced by the limits of integration η_0 and η_1.

$$da_{SHORT} = \frac{2}{na}\frac{\partial Q_4}{\partial M}dt = \frac{A_2}{a}\left[\frac{2}{3}\left(\frac{a^3}{r^3}-\left(1-e^2\right)^{-3/2}\right)\left(1-\frac{3}{2}\sin^2 i\right)\right.$$
$$\left.+\left(\frac{a}{r}\right)^3\sin^2 i\cos 2(\eta+\omega)\right] \quad (1.151)$$

$$de_{SHORT} = \frac{1-e^2}{\hbar e}\left(\sqrt{1-e^2}\,\frac{\partial Q_4}{\partial M} - \frac{\partial Q_4}{\partial \omega}\right)dt = \left(\frac{1-e^2}{e}\right)\left(\frac{1}{2}\,\frac{da_{SHORT}}{a} - \frac{\sin i}{\cos i}\,di_{SHORT}\right) \quad (1.52)$$

$$= \frac{1-e^2}{e}\,\frac{A_2}{a^2}\left\{\frac{1}{3}\left(\frac{a^3}{r^3} - (1-e^2)^{-3/2}\right)\left(1 - \frac{3}{2}\sin^2 i\right) + \frac{1}{2}\left(\frac{a}{r}\right)^3 \sin^2 i \cos 2(\eta+\omega)\right.$$

$$\left. - \frac{\sin^2 i}{2(1-e^2)^2}\left[\cos 2(\eta+\omega) + e\cos(\eta+2\omega) + \frac{e}{3}\cos(3\eta+2\omega)\right]\right\}$$

$$di_{SHORT} = \frac{\cot i}{\hbar}\,\frac{\partial Q_4}{\partial \omega}\,dt = \frac{1}{4}\,\frac{A_2}{p^2}\,\sin 2i\left[\cos 2(\eta+\omega) + e\cos(\eta+2\omega)\right.$$

$$\left. + \frac{e}{3}\cos(3\eta+2\omega)\right] \quad (1.153)$$

$$d\Omega_{SHORT} = \frac{1}{\hbar \sin i}\,\frac{\partial Q_4}{\partial i}\,dt = -\frac{A_2}{p^2}\cos i\left[(\eta - M + e\sin\eta) - \frac{1}{2}\sin 2(\eta+\omega)\right. \quad (1.154)$$

$$\left. - \frac{e}{2}\sin(\eta+2\omega) - \frac{e}{6}\sin(3\eta+2\omega)\right]$$

$$d\omega_{SHORT} = \frac{1}{\hbar}\left(-\cot i\,\frac{\partial Q_4}{\partial i} + \frac{1-e^2}{e}\,\frac{\partial Q_4}{\partial e}\right)dt = \frac{1}{\hbar}\left(\frac{1-e^2}{e}\right)\frac{\partial Q_4}{\partial e}\,dt - \cos i\,d\Omega_{SHORT}$$

$$= \frac{A_2}{p^2}\left\{\left(2 - \frac{5}{2}\sin^2 i\right)\left(\eta - M + e\sin\eta\right)\right.$$

$$+ \left(1 - \frac{3}{2}\sin^2 i\right)\left[\frac{1}{e}\left(1 - \frac{e^2}{4}\right)\sin\eta + \frac{1}{2}\sin 2\eta + \frac{e}{12}\sin 3\eta\right]$$

$$- \frac{1}{e}\left[\frac{1}{4}\sin^2 i + \left(\frac{1}{2} - \frac{15}{16}\sin^2 i\right)e^2\right]\sin(\eta+2\omega)$$

$$+ \frac{1}{e}\left[\frac{7}{12}\sin^2 i - \frac{1}{6}\left(1 - \frac{19}{8}\sin^2 i\right)e^2\right]\sin(3\eta+2\omega)$$

$$- \frac{1}{2}\left(1 - \frac{5}{2}\sin^2 i\right)\sin 2(\eta+\omega) + \frac{e}{16}\sin^2 i\,\sin(\eta-2\omega) \quad (1.155)$$

$$\left. + \frac{3}{8}\sin^2 i\,\sin(4\eta+2\omega) + \frac{e}{16}\sin^2 i\,\sin(5\eta+2\omega)\right\}$$

$$dM_{SHORT} = n\,dt - \frac{\sqrt{1-e^2}}{\hbar}\left(\frac{1-e^2}{e}\,\frac{\partial Q_4}{\partial e} + 2a\,\frac{\partial Q_4}{\partial a}\right)dt$$

Here, n is a variable.

$$n = n_0\left(1 + \frac{dn}{n_0}\right) = n_0\left(1 - \frac{3}{2}\frac{da}{a}\right) = n_0 - \frac{3}{2} n_0 \frac{da}{a}$$

From (1.151) and (1.150), it follows that,

$$-\frac{3}{2}\frac{da}{a} = \frac{a^2}{\mu}\frac{\partial Q_4}{\partial a} = \left(\frac{\sqrt{1-e^2}}{k} a \frac{\partial Q_4}{\partial a}\right)\frac{a^{3/2}}{\sqrt{\mu}} = \frac{1}{n_0}\left(\frac{\sqrt{1-e^2}}{k} a \frac{\partial Q_4}{\partial a}\right)$$

Therefore,

$$n = n_0 + \left(\frac{\sqrt{1-e^2}}{k} a \frac{\partial Q_4}{\partial a}\right)$$

$$\begin{aligned}
dM_{SHORT} &= n_0\, dt - \frac{\sqrt{1-e^2}}{k}\left(\frac{1-e^2}{e}\frac{\partial Q_4}{\partial e} + a\frac{\partial Q_4}{\partial a}\right) dt \\
&= -\frac{A_2}{ep^2}\sqrt{1-e^2}\Big\{\left(1-\frac{3}{2}\sin^2 i\right)\Big[\left(1-\frac{e^2}{4}\right)\sin\eta + \frac{e}{2}\sin 2\eta \\
&\qquad + \frac{e^2}{12}\sin 3\eta\Big] - \frac{1}{4}\left(1+\frac{5}{4}e^2\right)\sin^2 i \sin(\eta + 2\omega) \\
&\qquad + \frac{7}{12}\left(1-\frac{e^2}{28}\right)\sin^2 i \sin(3\eta + 2\omega) \\
&\qquad + \frac{e^2}{16}\sin^2 i \sin(\eta - 2\omega) + \frac{3}{8} e \sin^2 i \sin(4\eta + 2\omega) \\
&\qquad + \frac{e^2}{16}\sin^2 i \sin(5\eta + 2\omega)\Big\}
\end{aligned}$$

(1.156)

2.1.4.5 The Mean Values of the Short-Periodic Perturbations

The mean values of the short-periodic perturbations with respect to the mean anomaly M do not vanish, except for that of the semi-major axis a. This fact can be shown by considering the point where r = a; that is, $\cos\eta = -e$ and $M = \pi/2 - e$.

$$\overline{de_{SHORT}} = \frac{A_2}{p^2}\sin^2 i\left(\frac{1-e^2}{6e}\right)\overline{\cos 2\eta}\cos 2\omega \tag{1.157}$$

$$\overline{di_{SHORT}} = -\frac{1}{12}\frac{A_2}{p^2}\sin 2i\,\overline{\cos 2\eta}\cos 2\omega \tag{1.158}$$

$$\overline{d\Omega_{SHORT}} = -\frac{1}{6}\frac{A_2}{p^2}\cos i\,\overline{\cos 2\eta}\sin 2\omega \tag{1.159}$$

$$\overline{d\omega_{SHORT}} = \frac{A_2}{p^2}\left[\sin^2 i\left(\frac{1}{8}+\frac{1-e^2}{6e^2}\overline{\cos 2\eta}\right)\sin 2\omega + \frac{1}{6}\cos^2 i\ \overline{\cos 2\eta}\sin 2\omega\right] \tag{1.160}$$

$$\overline{dM_{SHORT}} = -\frac{A_2}{p^2}\sqrt{1-e^2}\sin^2 i\left(\frac{1}{8}+\frac{1+\frac{1}{2}e^2}{6e^2}\overline{\cos 2\eta}\right)\sin 2\omega \tag{1.161}$$

where
$$\overline{\cos j\eta} = \left(\frac{-e}{1+\sqrt{1-e^2}}\right)^j\left(1+j\sqrt{1-e^2}\right)$$

$j = 1, 2, 3, \ldots, n$

The mean value $\bar{a}$ of a semi-major axis a is determined from the relation,

$$\overline{a^3}\ \overline{n^2} = \overline{\mu} = a_0^3 n_0^2\left[1-\frac{A_2}{p^2}\left(1-\frac{3}{2}\sin^2 i\right)\sqrt{1-e^2}\right].$$

and the known value of $\bar{n}$, which is given by Equation (1.145). Hence,

$$\bar{a} = a_0\left[1 - \frac{A_2}{p^2}\left(1 - \frac{3}{2}\sin^2 i\right)\sqrt{1-e^2}\right] \tag{1.162}$$

The mean values of the short-periodic perturbations, as given in Equations (1.157) through (1.161), were transcribed from Kozai's report. The derivation of these equations, however, yields results which are opposite in sign for $\overline{de_{short}}$, $\overline{di_{short}}$, and $\overline{d\Omega_{short}}$, and which are of a different form for $\overline{d\omega_{short}}$ and $\overline{dM_{short}}$. Thus, the derivation of these quantities, using Kozai's conditions,

$$\overline{\cos\eta} = -e$$

$$\overline{\sin j\eta} = 0$$

$$j = 1, 2, 3, \ldots, n$$

introduced into Equations (1.152) through (1.156), will be reported. Consider Equation (1.153)

$$di_{SHORT} = \frac{1}{4}\frac{A_2}{p^2}\sin 2i\left[\cos 2(\eta+\omega) + e\cos(\eta + 2\omega) + \frac{e}{3}\cos(3\eta+2\omega)\right]$$

The mean value $\overline{di_{short}}$ is obtained by replacing e by ($-\overline{\cos\eta}$) and the resulting expressions by,

$$-\overline{\cos(\eta+2\omega)}\ \overline{\cos\eta} = -\frac{1}{2}\left[\overline{\cos 2(\eta+\omega)} + \overline{\cos 2\omega}\right]$$

$$-\frac{1}{3}\overline{\cos(3\eta+2\omega)}\ \overline{\cos\eta} = -\frac{1}{6}\left[\overline{\cos 2(\eta+\omega)} + \overline{\cos(4\eta+2\omega)}\right]$$

The terms $\cos 2\omega$ (which is independent of η) and $\cos(4\eta + 2\omega)$ are neglected. Thus,

$$\overline{di_{SHORT}} = \frac{1}{12}\frac{A_2}{p^2}\sin 2i\ \overline{\cos 2\eta}\cos 2\omega \tag{1.158}$$

Next, consider Equation (1.152),

$$\overline{de_{SHORT}} = \left(\frac{1-e^2}{e}\right)\left(\frac{1}{2a}\overline{da_{SHORT}} - \frac{\sin i}{\cos i}\overline{di_{SHORT}}\right)$$

But, since,

$$\overline{da_{SHORT}} = 0$$

it follows that,

$$\overline{de_{SHORT}} = -\left(\frac{1-e^2}{e}\right)\frac{\sin i}{\cos i}\overline{di_{SHORT}} = -\frac{1}{6}\frac{A_2}{p^2}\sin^2 i\left(\frac{1-e^2}{e}\right)\overline{\cos 2\eta}\cos 2\omega \quad (1.157)$$

Similarly, consider Equation (1.154)

$$d\Omega_{SHORT} = -\frac{A_2}{p^2}\cos i\Big[(\eta - M + e\sin\eta) - \frac{1}{2}\sin 2(\eta+\omega)$$
$$- \frac{e}{2}\sin(\eta + 2\omega) - \frac{e}{6}\sin(3\eta + 2\omega)\Big]$$

But since,

$$\eta - M = 2e\sin\eta - \frac{3}{4}e^2\sin 2\eta + \dots$$

and

$$\overline{\sin j\eta} = 0$$

$j = 1, 2, 3, \dots, n$

It follows that,

$$(\overline{\eta - M}) + e\,\overline{\sin\eta}) = 0$$

Now, the mean value of $\overline{d\Omega_{short}}$ is obtained by replacing e by $(\overline{-\cos\eta})$ and the resulting expressions by,

$$\frac{1}{2}\,\overline{\sin(\eta+2\omega)}\;\overline{\cos\eta} = \frac{1}{4}\left[\overline{\sin 2(\eta+\omega)} + \overline{\sin 2\omega}\right]$$

$$\frac{1}{6}\,\overline{\sin(3\eta+2\omega)}\;\overline{\cos\eta} = \frac{1}{12}\left[\overline{\sin 2(\eta+\omega)} + \overline{\sin(4\eta+2\omega)}\right]$$

The terms $\overline{\sin 2\omega}$ (which is independent of η) and $\overline{\sin(4\eta + 2\omega)}$ are neglected. Thus,

$$\overline{d\Omega_{SHORT}} = \frac{1}{6}\,\frac{A_2}{p^2}\cos i\;\overline{\cos 2\eta}\,\sin 2\omega \tag{1.159}$$

The expression for $\overline{d\omega_{short}}$ is obtained by rewriting Equation (1.155) as follows:

$$\begin{aligned} d\omega_{SHORT} = \frac{A_2}{p^2}\Big\{ & \left(1-\frac{3}{2}\sin^2 i\right)\left[(\eta - M + e\sin\eta) + \frac{1}{e}\left(1-\frac{e^2}{4}\right)\sin\eta + \frac{1}{2}\sin 2\eta \right. \\ & \left. + \frac{e}{12}\sin 3\eta\right] + \frac{1}{e}\sin^2 i\left[-\frac{1}{4}\left(1-\frac{7}{4}e^2\right)\sin(\eta+2\omega) + \frac{1}{12}\left(7+\frac{11}{4}e^2\right)\sin(3\eta+2\omega)\right] \\ & + \frac{3}{4}\sin^2 i\,\sin 2(\eta+\omega) + \frac{e}{16}\sin^2 i\,\sin(\eta-2\omega) \\ & + \frac{3}{8}\sin^2 i\,\sin(4\eta+2\omega) + \frac{e}{16}\sin^2 i\,\sin(5\eta+2\omega)\Big\} - \cos i\, d\Omega_{SHORT} \end{aligned}$$

Here again,

$$\overline{\eta - M} + e\,\overline{\sin\eta} = 0$$

$$\overline{\sin j\eta} = 0$$

$$j = 1, 2, 3, \ldots, n$$

Thus, using the condition that $\overline{\cos\eta} = -e$, yields,

$$\overline{d\omega_{SHORT}} = \frac{A_2}{p^2}\Big\{-\frac{1}{e^2}\sin^2 i\Big[-\frac{1}{4}\Big(1-\frac{7}{4}e^2\Big)\overline{\sin(\eta+2\omega)\cos\eta}$$
$$+\frac{1}{12}\Big(7+\frac{11}{4}e^2\Big)\overline{\sin(3\eta+2\omega)\cos\eta}\Big]$$
$$+\frac{3}{4}\sin^2 i\,\overline{\sin 2(\eta+\omega)}+\frac{3}{8}\sin^2 i\,\overline{\sin(4\eta+2\omega)}$$
$$-\frac{1}{16}\sin^2 i\,\overline{\cos\eta}\Big[\overline{\sin(\eta-2\omega)}+\overline{\sin(5\eta+2\omega)}\Big]\Big\}-\cos i\,\overline{d\Omega_{SHORT}}$$

Now, considering the identities,

$$\overline{\sin(\eta+2\omega)}\;\overline{\cos\eta} = \frac{1}{2}\Big[\overline{\sin 2(\eta+\omega)}+\sin 2\omega\Big]$$
$$\overline{\sin(3\eta+2\omega)}\;\overline{\cos\eta} = \frac{1}{2}\Big[\overline{\sin 2(\eta+\omega)}+\overline{\sin(4\eta+2\omega)}\Big]$$
$$\overline{\sin(\eta-2\omega)}\;\overline{\cos\eta} = \frac{1}{2}\Big[\overline{\sin 2(\eta-\omega)}-\sin 2\omega\Big]$$
$$\overline{\sin(5\eta+2\omega)}\;\overline{\cos\eta} = \frac{1}{2}\Big[\overline{\sin(4\eta+2\omega)}+\overline{\sin(6\eta+2\omega)}\Big]$$

and neglecting the terms $\sin(4\eta+2\omega)$ and $\sin(6\eta+2\omega)$ yields

$$\overline{d\omega_{SHORT}} = \frac{A_2}{p^2}\Big\{\sin^2 i\Big[\Big(\frac{1}{8e^2}-\frac{3}{16}\Big)-\Big(\frac{1-e^2}{6e^2}-\frac{9}{32}\Big)\overline{\cos 2\eta}\Big]\sin 2\omega$$
$$-\frac{1}{6}\cos^2 i\,\overline{\cos 2\eta}\,\sin 2\omega\Big\} \tag{1.160}$$

Similarly, $\overline{dM_{short}}$ is obtained as,

$$\overline{dM_{SHORT}} = -\frac{A_2}{p^2}\Big\{\sin^2 i\Big[\Big(\frac{1}{8e^2}+\frac{3}{16}\Big)-\Big(\frac{1+\frac{1}{2}e^2}{6e^2}-\frac{9}{32}\Big)\overline{\cos 2\eta}\Big]\sin 2\omega\Big\}\sqrt{1-e^2} \tag{1.161}$$

2.1.4.6 The Long Period Perturbations

These perturbations are obtained by determining the respective deviations from the mean values of the short periodic perturbations, as defined by Eq. (1.157) through (1.161). There are no long periodic perturbations of the first order in the semi-major axis a. The long periodic perturbations of the remaining elements are defined as follows:

$$di_{LONG} = \overline{di_{SHORT}} + \delta_1 i + \delta_2 i \tag{1.163}$$

$$de_{LONG} = \overline{de_{SHORT}} + \delta_1 e + \delta_2 e \tag{1.164}$$

$$d\Omega_{LONG} = \overline{d\Omega_{SHORT}} + \delta_1 \Omega + \delta_2 \Omega \tag{1.165}$$

$$d\omega_{LONG} = \overline{d\omega_{SHORT}} + \delta_1 \omega + \delta_2 \omega \tag{1.166}$$

where the δi_i , δe_i ....., represent the deviations from the short periodic mean values.

Kozai does not present the technique he used to determine these deviations. However, through research of the literature on the subject, some useful clues were obtained pertaining to the way the $\delta_1 i$, $\delta_1 e$, $\delta_1 \Omega$, $\delta_1 \omega$ and $\delta_2 i$, $\delta_2 e$, $\delta_2 \Omega$, $\delta_2 \omega$ are to be derived.

The technique consists basically of replacing Q by Q_3 in relations (1.134) through (1.137), remembering that $\partial Q_3 / \partial M = 0$, and in adding certain terms to the basic definitions of d Ω /dt and d ω /dt.

$$\frac{d}{dt}\left(\delta_1 i\right) = \frac{1}{h} \cot i \, \frac{\partial Q_3}{\partial \omega} \tag{1.167}$$

$$\frac{d}{dt}\left(\delta_1 e\right) = -\frac{1}{\hbar}\left(\frac{1-e^2}{e}\right)\frac{\partial Q_3}{\partial \omega} \tag{1.168}$$

$$\frac{d}{dt}\left(\delta_1 \Omega\right) = \frac{1}{\hbar \sin i}\ \frac{\partial Q_3}{\partial i} + \frac{d}{di}\left(\dot{\Omega}\right)\delta_1 i + \frac{d}{de}\left(\dot{\Omega}\right)\delta_1 e \tag{1.169}$$

$$\frac{d}{dt}\left(\delta_1 \omega\right) = \frac{1}{\hbar}\left(-\cot i\,\frac{\partial Q_3}{\partial i} + \frac{1-e^2}{e}\ \frac{\partial Q_3}{\partial e}\right) + \frac{d}{di}\left(\dot{\omega}\right)\delta_1 i + \frac{d}{de}\left(\dot{\omega}\right)\delta_1 e \tag{1.170}$$

where,

$$\begin{aligned} \dot{\Omega} &= -\frac{A_2}{p^2}\, n \cos i \\ \dot{\omega} &= \frac{A_2}{p^2}\, n\left(2 - \frac{5}{2}\sin^2 i\right) \end{aligned} \tag{1.171}$$

$$\begin{aligned} \frac{d}{di}\left(\dot{\Omega}\right) &= \frac{A_2}{p^2}\, n \sin i = -\left(\tan i\right)\dot{\Omega} \\ \frac{d}{de}\left(\dot{\Omega}\right) &= -4\left(\frac{e}{1-e^2}\right)\frac{A_2}{p^2}\, n \cos i = 4\left(\frac{e}{1-e^2}\right)\dot{\Omega} \end{aligned} \tag{1.172}$$

$$\begin{aligned} \frac{d}{di}\left(\dot{\omega}\right) &= -5\,\frac{A_2}{p^2}\, n \sin i \cos i = 5\left(\sin i\right)\dot{\Omega} \\ \frac{d}{de}\left(\dot{\omega}\right) &= 4\left(\frac{e}{1-e^2}\right)\frac{A_2}{p^2}\, n\left(2 - \frac{5}{2}\sin^2 i\right) = 4\left(\frac{e}{1-e^2}\right)\dot{\omega} \end{aligned} \tag{1.173}$$

The function Q_3, given by Eq. (1.132), will be rewritten in a more convenient form for the purpose of the ensuing analytical development. As it was stated earlier, Q_3 represents the long-periodic part of the perturbing potential.

$$Q_3 = \frac{k}{\sqrt{p}}\left[-\left(\frac{3}{4}\frac{A_3}{a^{5/2}}\, e\,(1-e^2)^{-5/2} \sin i \sin\omega\right)\frac{\dot{\omega}}{\frac{A_2}{p^2}} + \frac{A_4}{a^{7/2}}\, e^2(1-e^2)^{-7/2}\sin^2 i\left(\frac{18-21\sin^2 i}{7}\right)\cos 2\omega\left(\frac{\dot{\omega}}{8\frac{A_2}{p^2}\left(2-\frac{5}{2}\sin^2 i\right)}\right)\right] \tag{1.174}$$

$$\frac{\partial Q_3}{\partial\omega} = k\frac{d}{dt}\left[-\frac{3}{4}\frac{A_3}{A_2 p}\, e \sin i \sin\omega + \frac{A_4}{A_2 p^2}\, e^2 \sin^2 i\left(\frac{18-21\sin^2 i}{7}\right)\frac{\cos 2\omega}{8\left(2-\frac{5}{2}\sin^2 i\right)}\right] \tag{1.175}$$

$$\frac{\partial Q_3}{\partial e} = k\left(\frac{e}{1-e^2}\right)\dot{\omega}\left[-\frac{3}{4}\frac{A_3}{A_2 p}\left(\frac{1}{e}+4e\right)\sin i \sin\omega + \frac{A_4}{A_2 p^2}\,(2+5e^2)\sin^2 i\left(\frac{18-21\sin^2 i}{7}\right)\frac{\cos 2\omega}{8\left(2-\frac{5}{2}\sin^2 i\right)}\right] \tag{1.176}$$

$$\frac{\partial Q_3}{\partial i} = k\left[-\left(\frac{3}{4}\frac{A_3}{A_2 p}\, e \cos i \sin\omega\right)\dot{\omega} - \left(\frac{15}{4}\frac{A_3}{A_2 p}\, e \sin^2 i \sin\omega\right)\dot{\Omega} + 4\frac{A_4}{A_2 p^2}\, e^2 \sin i \cos i\left(\frac{9-21\sin^2 i}{7}\right)\frac{\cos 2\omega}{8\left(2-\frac{5}{2}\sin^2 i\right)}\,\dot{\omega}\right] \tag{1.177}$$

From (1.167) and (1.175), after eliminating d/dt on both sides,

$$\delta_1 i = \left.-\frac{3}{4}\frac{A_3}{A_2 p}\, e \cos i \sin\omega + \frac{A_4}{A_2 p^2}\, e^2 \sin 2i\left(\frac{18-21\sin^2 i}{7}\right)\frac{\cos 2\omega}{8(4-5\sin^2 i)}\right|_{\omega_0}^{\omega_1} \tag{1.178}$$

Similarly, from (1.168) and (1.175), after eliminating d/dt on both sides,

$$\delta_1 e = \left.\frac{3}{4}(1-e^2)\frac{A_3}{A_2 p}\sin i \sin\omega - 2(1-e^2)\frac{A_4}{A_2 p^2}\, e \sin^2 i\left(\frac{18-21\sin^2 i}{7}\right)\frac{\cos 2\omega}{8(4-5\sin^2 i)}\right|_{\omega_0}^{\omega_1} \tag{1.179}$$

From (1.169) and (1.177), it follows that,

$$\frac{1}{h \sin i} \frac{\partial Q_3}{\partial i} = -\left(\frac{15}{4} \frac{A_3}{A_2 p} e \sin i \sin \omega\right) \dot{\Omega} - \left(\frac{3}{4} \frac{A_3}{A_2 p} e \frac{\cos i}{\sin i} \sin \omega\right) \dot{\omega} + 8 \frac{A_4}{A_2 p^2} e^2 \cos i \left(\frac{9 - 21 \sin^2 i}{7}\right) \frac{\cos 2\omega}{8(4 - 5 \sin^2 i)} \dot{\omega} \tag{1.180}$$

and from (1.172), (1.178), and (1.179), that

$$\frac{d}{di}(\dot{\Omega}) \delta_1 i = \left(\frac{3}{4} \frac{A_3}{A_2 p} e \sin i \sin \omega\right) \dot{\Omega} - 2 \frac{A_4}{A_2 p^2} e^2 \sin^2 i \left(\frac{18 - 21 \sin^2 i}{7}\right) * \left(\frac{\cos 2\omega}{8(4 - 5 \sin^2 i)}\right)\left(\frac{-\dot{\omega} \cos i}{2 - \frac{5}{2} \sin^2 i}\right) \tag{1.181}$$

$$\frac{d}{de}(\dot{\Omega}) \delta_1 e = \left(\frac{12}{4} \frac{A_3}{A_2 p} e \sin i \sin \omega\right) \dot{\Omega} - 8 \frac{A_4}{A_2 p^2} e^2 \sin^2 i \left(\frac{18 - 21 \sin^2 i}{7}\right) * \left(\frac{\cos 2\omega}{8(4 - 5 \sin^2 i)}\right)\left(\frac{-\dot{\omega} \cos i}{2 - \frac{5}{2} \sin^2 i}\right) \tag{1.182}$$

The addition of (1.180), (1.181), and (1.182) yields relation (1.169).

Integrating with respect to ω, yields,

$$\delta_1 \Omega = \frac{A_4}{A_2 p^2} e^2 \cos i \left(\frac{9 - 21 \sin^2 i}{7}\right) \frac{\sin 2\omega}{2(4 - 5 \sin^2 i)} + 5 \frac{A_4}{A_2 p^2} e^2 \sin^2 i \cos i \left(\frac{18 - 21 \sin^2 i}{7}\right) \frac{\sin 2\omega}{4(4 - 5 \sin^2 i)^2} + \frac{3}{4} \frac{A_3}{A_2 p} e \frac{\cos i}{\sin i} \cos \omega \tag{1.183}$$

in which the argument ω has to be replaced by the limits of integration ω_0 and ω_1.

In a similar manner, from (1.170) and (1.177) that,

$$-\frac{1}{k}\cot i\ \frac{\partial Q_3}{\partial i}=\left(\frac{15}{4}\frac{A_3}{A_2 p}\ e\sin i\cos i\sin\omega\right)\dot{\Omega}+\left(\frac{3}{4}\frac{A_3}{A_2 p}e\ \frac{\cos^2 i}{\sin i}\sin\omega\right)\dot{\omega} \qquad (1.184)$$

$$-\frac{8A_4}{A_2 p^2}\ e^2\cos^2 i\left(\frac{9-21\sin^2 i}{7}\right)\frac{\cos 2\omega}{8(4-5\sin^2 i)}\ \dot{\omega}$$

and from (1.170) and (1.176), it follows that,

$$\frac{1}{k}\left(\frac{1-e^2}{e}\right)\frac{\partial Q_3}{\partial e}=-\left(\frac{3}{4}\frac{A_3}{A_2 p}\left(\frac{1}{e}+4e\right)\sin i\ \sin\omega\right)\dot{\omega}$$

$$+2\frac{A_4}{A_2 p^2}\ (2+5e^2)\sin^2 i\left(\frac{18-21\sin^2 i}{7}\right)\frac{\cos 2\omega}{8(4-5\sin^2 i)}\ \dot{\omega} \qquad (1.185)$$

From (1.173), (1.178), and (1.179),

$$\frac{d}{di}(\dot{\omega})\delta_1 i=-\left(\frac{15}{4}\frac{A_3}{A_2 p}e\sin i\cos i\sin\omega\right)\dot{\Omega}$$

$$+5\frac{A_4}{A_2 p^2}e^2\sin^2 i\cos i\left(\frac{18-21\sin^2 i}{7}\right)\left(\frac{\cos 2\omega}{4(4-5\sin^2 i)}\right)\left(\frac{-\dot{\omega}\cos i}{2-\frac{5}{2}\sin^2 i}\right) \qquad (1.186)$$

$$\frac{d}{de}(\dot{\omega})\delta_1 e=\left(3\frac{A_3}{A_2 p}e\sin i\sin\omega\right)\dot{\omega}$$

$$-4\frac{A_4}{A_2 p^2}e^2\sin^2 i\left(\frac{18-21\sin^2 i}{7}\right)\frac{\cos 2\omega}{4(4-5\sin^2 i)}\ \dot{\omega} \qquad (1.187)$$

Addition of (1.184) through (1.187) yields relation (1.170). Integrating this result with respect to ω, yields,

$$\begin{aligned}\delta_1\omega=&\frac{A_4}{A_2 p^2}\sin^2 i\left(\frac{18-21\sin^2 i}{28}\right)\frac{\sin 2\omega}{4-5\sin^2 i}\\&-\frac{1}{2}\frac{A_4}{A_2 p^2}e^2\sin^2 i\left(\frac{18-21\sin^2 i}{28}\right)\left(\frac{\sin 2\omega}{4-5\sin^2 i}\right)\left[4+\frac{5\cos^2 i}{2-\frac{5}{2}\sin^2 i}\right]\\&-\frac{A_4}{A_2 p^2}e^2\left(\frac{\sin 2\omega}{4-5\sin^2 i}\right)\left[\frac{4\cos^2 i(9-21\sin^2 i)-5\sin^2 i(18-21\sin^2 i)}{56}\right]\\&+\frac{3}{4}\frac{A_3}{A_2 p}\left(\frac{\sin^2 i-e^2\cos^2 i}{e\sin i}\right)\cos\omega\end{aligned} \qquad (1.188)$$

in which the argument ω has to be replaced by the limits of integration ω_0 and ω_1.

The technique for the determination of $\delta_2 i$, $\delta_2 e$, $\delta_2\Omega$, $\delta_2\omega$ consists in the application of the following rule,

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_j \frac{\partial}{\partial f_j}\left(\frac{\partial f}{\partial t}\right)\delta f_j \tag{1.189}$$

where f represents any orbital element and the summation represents the sum of the partial derivatives of ($\partial f/\partial t$) with respect to the remaining elements.

First of all, the time argument in relation (1.189) will be eliminated in favor of the perturbed true anomaly η through the relationship,

$$dt = \frac{r^2}{h}\Big\{1 + \frac{1}{e}\frac{A_2}{r^2}\Big[\left(1-\frac{3}{2}\sin^2 i\right)\cos\eta + \frac{3}{2}\sin^2 i\cos\eta\cos 2(\eta+\omega) - \sin^2 i\left(1+\frac{r}{p}\right)\sin\eta\sin 2(\eta+\omega)\Big]\Big\}d\eta \tag{1.190}$$

Applying rule (1.189) to the orbital inclination,

$$\frac{di}{dt} = \frac{\partial i}{\partial t} + \frac{\partial}{\partial a}\left(\frac{\partial i}{\partial t}\right)\delta a + \frac{\partial}{\partial e}\left(\frac{\partial i}{\partial t}\right)\delta e + \frac{\partial}{\partial \omega}\left(\frac{\partial i}{\partial t}\right)\delta\omega \tag{1.191}$$

By Eq. (1.136) and (1.147),

$$\frac{\partial i}{\partial t} = \frac{1}{h}\cot i\,\frac{\partial Q_4}{\partial\omega} = -\frac{1}{2}\frac{\mu}{h}\frac{A_2}{r^3}\sin 2i\sin 2(\eta+\omega) \tag{1.192}$$

Replacement of the time argument by the perturbed true anomaly by means of Eq. (1.190) yields.

$$\left(\frac{\partial i}{\partial\eta}\right)_{TOTAL} = -\frac{1}{2}\frac{A_2}{rp}\sin 2i\sin 2(\eta+\omega) - \frac{1}{2e}\left(\frac{A_2}{p^2}\right)^2\sin 2i\sin 2(\eta+\omega)(1+e\cos\eta)^3\Big[\left(1-\frac{3}{2}\sin^2 i\right)\cos\eta + \frac{3}{2}\sin^2 i\cos\eta\cos 2(\eta+\omega) - \left(1+\frac{r}{p}\right)\sin^2 i\sin\eta\sin 2(\eta+\omega)\Big] \tag{1.193}$$

The first part of Eq. (1.193) represents the short periodic perturbation and, since the purpose here is the determination of the long periodic contribution arising from the perturbed part of the true anomaly, only the second term of Eq. (1.193) is to be considered. In fact, it represents the function $\partial i/\partial\eta$,

$$\frac{\partial i}{\partial \eta} = -\frac{1}{2e}\left(\frac{A_2}{p^2}\right)^2 (1+e\cos\eta)^3 \sin 2i \sin 2(\eta+\omega)\Big[(1-\frac{3}{2}\sin^2 i)\cos\eta + \frac{3}{2}\sin^2 i \cos\eta \cos 2(\eta+\omega) - \left(1+\frac{r}{p}\right)\sin^2 i \sin\eta \sin 2(\eta+\omega)\Big] \tag{1.194}$$

Since the effort here is concentrated on long periodic perturbations, only functions of ω are to be retained in relation (1.194). To obtain explicitly these functions of ω, one proceeds as follows,

$$(1+e\cos\eta)^3 = (1+\frac{3}{2}e^2) + 3e(1+\frac{e^2}{4})\cos\eta + \frac{3}{2}e^2\cos 2\eta + \frac{e^3}{4}\cos 3\eta$$

$$\cos\eta(1+e\cos\eta)^3 = \frac{3}{2}e\left(1+\frac{e^2}{4}\right) + \left(1+\frac{9}{4}e^2\right)\cos\eta + \frac{e}{2}(3+e^2)\cos 2\eta + \frac{3}{4}e^2\cos 3\eta + \frac{e^3}{8}\cos 4\eta$$

$$\sin 2(\eta+\omega)\cos\eta(1+e\cos\eta)^3 = \frac{e}{4}(3+e^2)\sin 2\omega \tag{1.195}$$

$$\sin\eta\sin^2 2(\eta+\omega) = \frac{1}{2}\sin\eta - \frac{1}{2}\sin\eta\cos 4(\eta+\omega) = \frac{1}{2}\sin\eta - \frac{1}{4}\sin(5\eta+4\omega) + \frac{1}{4}\sin(3\eta+4\omega)$$

$$\sin\eta\sin^2 2(\eta+\omega)(1+e\cos\eta)^3 = \frac{e^3}{32}\sin 4\omega \tag{1.196}$$

Introducing (1.195) and (1.196) in relation (1.194) one obtains,

$$\frac{\partial i}{\partial \eta} = -\frac{1}{8}\left(\frac{A_2}{p^2}\right)^2 \sin 2i\left[\left(1-\frac{3}{2}\sin^2 i\right)\left(3+e^2\right)\sin 2\omega - \frac{e^2}{8}\sin 4\omega\right] \tag{1.197}$$

Let $\delta_2 i$ be the long periodic perturbations in inclination arising from the perturbed part of the true anomaly. Then, the transformed relation (1.191) can be written as,

$$\frac{d}{d\eta}(\delta_2 i) = \frac{\partial i}{\partial \eta} + \frac{\partial}{\partial a}\left(\frac{\partial i}{\partial \eta}\right)\delta a + \frac{\partial}{\partial e}\left(\frac{\partial i}{\partial \eta}\right)\delta e + \frac{\partial}{\partial \omega}\left(\frac{\partial i}{\partial \eta}\right)\delta\omega \tag{1.198}$$

It now remains to evaluate the partial derivatives of relation (1.197),

$$\frac{\partial}{\partial a}\left(\frac{\partial i}{\partial \eta}\right)\delta a = \frac{1}{6}\left(\frac{A_2}{p^2}\right)^2\left(\frac{e^2}{1-e^2}\right)\sin 2i\Big[\left(1-\frac{3}{2}\sin^2 i\right)\left(6+e^2\right)\sin 2\omega - \frac{3}{32}e^2\sin^2 i\sin 4\omega\Big] \tag{1.199}$$

$$\frac{\partial}{\partial e}\left(\frac{\partial i}{\partial \eta}\right)\delta e = \frac{1}{24}\left(\frac{A_2}{p^2}\right)^2\left(\frac{1}{1-e^2}\right)\sin 2i\left(1-\frac{3}{2}\sin^2 i\right)(3+22e^2+3e^4)\sin 2\omega \tag{1.200}$$

$$\frac{\partial}{\partial \omega}\left(\frac{\partial i}{\partial \eta}\right)\delta\omega = \frac{1}{24}\left(\frac{A_2}{p^2}\right)^2\sin 2i\left[\left(1-\frac{3}{2}\sin^2 i\right)(12+11e^2)+3e^2\sin^2 i\right] \tag{1.201}$$

$$\frac{\partial}{\partial M}\left(\frac{\partial i}{\partial \eta}\right) = \frac{\partial}{\partial \Omega}\left(\frac{\partial i}{\partial \eta}\right) = 0 \tag{1.202}$$

Substitution of Eqs. (1.197) and (1.199) through (1.201) into relation (1.198), yields,

$$\begin{aligned}\frac{d}{d\eta}(\delta_2 i) &= -\frac{1}{24}\left(\frac{A_2}{p^2}\right)^2\sin 2i\left\{\left(1-\frac{3}{2}\sin^2 i\right)\left[3(3+e^2)-\frac{4e^2(6+e^2)}{1-e^2}\right.\right.\\ &\quad\left.\left.+\frac{3+22e^2+3e^4}{1-e^2}-(12+11e^2)\right]-3e^2\sin^2 i\right\}\sin 2\omega\\ &= \frac{1}{8}\left(\frac{A_2}{p^2}\right)^2 e^2\sin 2i\left(\frac{14-15\sin^2 i}{6}\right)\sin 2\omega\end{aligned} \tag{1.203}$$

Using the relation,

$$\left(\frac{d\omega}{d\eta}\right)_{SECULAR} = \frac{A_2}{p^2}\left(2-\frac{5}{2}\sin^2 i\right) \tag{1.204}$$

replacing $d\eta$ by,

$$d\eta = \frac{d\omega}{\left(\frac{d\omega}{d\eta}\right)} = \frac{2\,d\omega}{\frac{A_2}{p^2}(4-5\sin^2 i)} \tag{1.205}$$

and integrating Eq. (1.203) with respect to ω, yields,

$$\delta_2 i = -\frac{A_2}{p^2}e^2\sin 2i\left(\frac{14-15\sin^2 i}{6}\right)\left.\frac{\cos 2\omega}{8(4-5\sin^2 i)}\right|_{\omega_0}^{\omega_1} \tag{1.206}$$

$\delta_2 e$ is determined in a direct manner from the condition that,

$$\sqrt{a(1-e^2)}\cos i = \text{const.} \tag{1.207}$$

and the known fact that there are no long periodic perturbations in a.
Therefore,

$$\sqrt{1-e^2}\cos i = \frac{\text{const}}{a} \tag{1.208}$$

$$-\frac{e}{\sqrt{1-e^2}}\cos i\,\delta_2 e - \sqrt{1-e^2}\sin i\,\delta_2 e = 0 \tag{1.209}$$

whence,

$$\delta_2 e = -\left(\frac{1-e^2}{e}\right)\frac{\sin i}{\cos i}\,\delta_2 i \tag{1.210}$$

Substitution of (1.206) into (1.210), yields,

$$\delta_2 e = \frac{A_2}{p^2}\, e(1-e^2)\sin^2 i\left(\frac{14-15\sin^2 i}{6}\right)\frac{\cos 2\omega}{4(4-5\sin^2 i)}\Bigg|_{\omega_0}^{\omega_1} \tag{1.211}$$

$\delta_2\Omega$ and $\delta_2\omega$ are derived in exactly the same manner as $\delta_2 i$.

$$\begin{aligned}\delta_2\Omega = &-\frac{A_2}{p^2}\, e^2\cos i\left(\frac{7-15\sin^2 i}{6}\right)\frac{\sin 2\omega}{2(4-5\sin^2 i)}\\ &-5\frac{A_2}{p^2}\, e^2\sin^2 i\cos i\left(\frac{14-15\sin^2 i}{6}\right)\frac{\sin 2\omega}{4(4-5\sin^2 i)^2}\end{aligned} \tag{1.212}$$

$$\begin{aligned}\delta_2\omega = &-\frac{A_2}{p^2}\left[\sin^2 i\left(\frac{14-15\sin^2 i}{24}\right) - e^2\left(\frac{28-158\sin^2 i+135\sin^4 i}{48}\right)\frac{\sin 2\omega}{4-5\sin^2 i}\right]\\ &+\frac{1}{2}\frac{A_2}{p^2}\, e^2\sin^2 i\left(\frac{14-15\sin^2 i}{24}\right)\left(\frac{\sin 2\omega}{4-5\sin^2 i}\right)\left[4+\frac{5\cos^2 i}{2-\frac{5}{2}\sin^2 i}\right]\end{aligned} \tag{1.213}$$

where the argument ω has to be replaced by the limits of integration ω_0 and ω_1.

2.1.4.7 The Second Order Secular Perturbations

The secular perturbations of the second order with the A_4 harmonic are obtained from Eq. (1.135) and (1.137) by substituting Q_2 for Q. To these must be added the contribution of the square of the A_2 harmonic; that is, the terms in A_2^2.

$$\frac{d\Omega}{dt} = \frac{1}{h \sin i} \frac{\partial Q_2}{\partial i} + A_2^2 (TERM) \tag{1.214}$$

$$\frac{d\omega}{dt} = \frac{1}{h}\left(-\cot i \frac{\partial Q_2}{\partial i} + \frac{1-e^2}{e} \frac{\partial Q_2}{\partial e}\right) + A_2^2 \ (TERM) \tag{1.215}$$

Q_2, as given by Eq. (1.132), will be rewritten in the following form,

$$Q_2 = \frac{h}{\sqrt{p}} \frac{A_4}{a^{7/2}} n \left(1 + \frac{3}{2}e^2\right)\left(1 - e^2\right)^{-7/2}\left(\frac{3}{8}\sin^4 i - \frac{3}{7}\sin^2 i + \frac{3}{35}\right) \tag{1.216}$$

and the partial derivatives required in Eqs. (1.214) and (1.215) are,

$$\frac{\partial Q_2}{\partial i} = -h \frac{A_4}{p^4} n\left(1 + \frac{3}{2}e^2\right)\sin i \cos i\left(\frac{12 - 21\sin^2 i}{14}\right) \tag{1.217}$$

$$\frac{\partial Q_2}{\partial e} = h\left(\frac{e}{1-e^2}\right) \frac{A_4}{p^4} n\left(1 + \frac{3}{4}e^2\right)\left(\frac{15}{4}\sin^4 i - \frac{30}{7}\sin^2 i + \frac{6}{7}\right) \tag{1.218}$$

Substituting these partial derivatives into Eqs. (1.214) and (1.215), yields,

$$\left(\frac{d\Omega}{dt}\right)_{\substack{SECULAR\\2ND\ ORDER}} = -\frac{A_4}{p^4} n \cos i\left(1 + \frac{3}{2}e^2\right)\left(\frac{12 - 21\sin^2 i}{14}\right) + A_2^2 \ (TERM\ FOR\ \Omega) \tag{1.219}$$

$$\left(\frac{d\omega}{dt}\right)_{\substack{SECULAR\\2ND\ ORDER}} = \frac{A_4}{p^4} n\left[\frac{12}{7}\left(1+\frac{9}{8}e^2\right) - \frac{93}{14}\left(1+\frac{63}{62}e^2\right)\sin^2 i + \frac{21}{4}\left(1+\frac{27}{28}e^2\right)\sin^4 i\right] + A_2^2\,(\text{TERM FOR } \dot{\omega}) \tag{1.220}$$

Kozai does not present a derivation for the second order secular perturbations; not even a hint. His results for the A_2^2 (terms) do not agree with the results obtained by other theories. At this time, the apparent discrepancies have not been resolved. Kozai's A_2^2 (terms) are thus, transcribed varbatim.

$$A_2^2(\text{TERM FOR } \dot{\Omega}) = -\left(\frac{A_2}{\bar{p}^2}\bar{n}\cos\bar{i}\right)\times\frac{A_2}{p^2}\left[\frac{3}{2}\left(1+\frac{e^2}{9}\right) - \frac{5}{3}\left(1-\frac{e^2}{8}\right)\sin^2 i - 2\left(1-\frac{3}{2}\sin^2 i\right)\sqrt{1-e^2}\right] \tag{1.221}$$

$$A_2^2(\text{TERM FOR } \dot{\omega}) = \frac{A_2}{\bar{p}^2}\bar{n}\left(2-\frac{5}{2}\sin^2 i\right)* \frac{A_2}{p^2}\left[2\left(1+\frac{e^2}{4}\right) - \frac{43}{24}\left(1-\frac{e^2}{86}\right)\sin^2 i - 2\left(1-\frac{3}{2}\sin^2 i\right)\sqrt{1-e^2}\right] - \frac{5}{12}\left(\frac{A_2}{p^2}\right)^2 e^2 n \cos^4 i \tag{1.222}$$

where $\bar{a}$ is given by (1.162),

$$\bar{p} = \bar{a}\,(1-\bar{e}^2)$$

and

$$\bar{n} = n_0\left[1+\frac{A_2}{p^2}\left(1-\frac{3}{2}\sin^2 i\right)\sqrt{1-e^2}\right]$$

$\bar{i}$ and $\bar{e}$ are the mean values of the inclination and eccentricity over all periods with respect to M and ω.

2.1.4.8 The Sum of the Secular Perturbations of the First and Second Order

The rates of the secular perturbations of the first order are given by Eqs. (1.143) through (.1.45). The rates of the secular perturbations of the second order are given by Eqs. (1.219) and (1.220), with the A_2^2 (terms) defined by (1.221) and (1.222). Thus, if $\dot{\Omega}$ and $\dot{\omega}$ represent the sum of the rates of the secular perturbations of the first and second order,

$$\dot{\Omega} = \left(\frac{d\Omega}{dt}\right)_{1st\ ORDER} + \left(\frac{d\Omega}{dt}\right)_{2ND\ ORDER} \tag{1.223}$$

$$\dot{\omega} = \left(\frac{d\omega}{dt}\right)_{1st\ ORDER} + \left(\frac{d\omega}{dt}\right)_{2ND\ ORDER} \qquad (1.224)$$

the corresponding total secular perturbations during the time interval t are: $\dot{\Omega}t$ and $\dot{\omega}t$, respectively.

NOTE: In this analysis only the secular rate of the first order for the mean anomaly M is derived and is given by Eq. (1.145): $\dot{M} = \bar{n}$; $\dot{M}t = \bar{n}t$.

2.1.4.9 Perturbations in the Radius and Argument of Latitude

The perturbations in r and the argument of latitude u* are calculated for two reasons: (1) for the sake of completeness; (2) the expressions for short periodic perturbations in the mean anomaly and argument of perigee are, first of all, lengthy and complicated and, secondly, they fail in case of very small eccentricities. Therefore, it is very useful to combine (M, ω) together with (a, e) in the radius vector and in the argument of latitude.

Using the relation for the unperturbed increment in the true anomaly,

$$d\eta = \frac{a^2\sqrt{1-e^2}}{r^2}\,dM$$

and differentiating r with respect to η, a, e, yields,

$$\frac{dr}{a} = \frac{e}{\sqrt{1-e^2}}\sin\eta\,dM + \frac{r}{a}\frac{da}{a} - \cos\eta\,de \qquad (1.225)$$

Now, noting that the perturbed rate of the true anomaly is given by,

$$d\eta_{PERTURBED} = \frac{a^2\sqrt{1-e^2}}{r^2}\,dM + \frac{a}{r}\left(1+\frac{r}{p}\right)\sin\eta\,de \qquad (1.226)$$

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where,

$$
\begin{aligned}
da &= da_{SHORT} - \frac{A_2}{p^2} a_0 \left(1 - \frac{3}{2}\sin^2 i\right)\sqrt{1-e^2} \\
de &= de_{SHORT} \\
dM &= dM_{SHORT} + \frac{3}{8}\frac{A_2}{p^2}\sqrt{1-e^2}\,\sin^2 i\,\sin 2\omega \\
d\omega &= d\omega_{SHORT} - \frac{3}{8}\frac{A_2}{p^2}\sin^2 i\,\sin 2\omega
\end{aligned}
\qquad (1.227)
$$

and that the perturbed rate of the argument of latitude can be defined as $(d\eta + d\omega)$,

$$
du^* = \frac{a^2\sqrt{1-e^2}}{r^2}dM + \frac{a}{r}\left(1+\frac{r}{p}\right)\sin\eta\, de + d\omega \qquad (1.228)
$$

allows relations (1.225) and (1.228) to be written as,

$$
\frac{dr}{a} = -\frac{1}{3}\frac{A_2}{ap}\left(1-\frac{3}{2}\sin^2 i\right)\left[\left(\frac{1-\sqrt{1-e^2}}{e}\right)\cos\eta + \left(1-\frac{r}{a\sqrt{1-e^2}}\right)\right] + \frac{1}{6}\frac{A_2}{ap}\sin^2 i\cos 2(\eta+\omega) \qquad (1.229)
$$

$$
\begin{aligned}
du^* = \frac{A_2}{p^2}\Bigg[&\left(2-\frac{5}{2}\sin^2 i\right)\left(\eta - M + e\sin\eta\right) + \left(1-\frac{3}{2}\sin^2 i\right)\Bigg\{\frac{2}{3e}\left(1-\sqrt{1-e^2}-\frac{e^2}{2}\right)\sin\eta \\
&+\frac{1}{6}\left(1-\sqrt{1-e^2}\right)\sin 2\eta\Bigg\} - \frac{1}{2}\left(1-\frac{5}{3}\sin^2 i\right)e\sin(\eta+2\omega) \\
&-\frac{1}{2}\left(1-\frac{7}{6}\sin^2 i\right)\sin 2(\eta+\omega) - \frac{e}{6}\cos^2 i\,\sin(3\eta+2\omega)\Bigg]
\end{aligned}
\qquad (1.230)
$$

2.1.4.10 The Sum Total of All Perturbations

$$
a = \bar{a} + da_{SHORT}
$$

$$
e = \bar{e} + de_{SHORT} - \overline{de_{SHORT}} + de_{LONG}
$$

$$i = \bar{i} + di_{SHORT} - \overline{di_{SHORT}} + di_{LONG}$$

$$\omega = \omega_0 + \dot{\omega} t + d\omega_{SHORT} - \overline{d\omega_{SHORT}} + d\omega_{LONG}$$

$$\Omega = \Omega_0 + \dot{\Omega} t + d\Omega_{SHORT} - \overline{d\Omega_{SHORT}} + d\Omega_{LONG}$$

$$M = M_0 + \bar{n} t + dM_{SHORT}$$

where $\bar{a}$ is given by (1.162); $\bar{e}$ and $\bar{i}$ are the mean values with respect to M and ω; ω_0, Ω_0, M_0 are initial values from which periodic perturbations have been subtracted.

2.2 THE PERTURBATIVE EFFECTS OF ATMOSPHERIC DRAG ON THE ORBIT OF AN ARTIFICIAL SATELLITE

2.2.1 Basic Review of the Problem

2.2.1.1 Definition of the Perturbing Force

The atmospheric drag is directly dependent on the following factors:

A. The Drag Coefficient, C_D: The drag coefficient C_D is a function of the shape of the vehicle, its projected effective area, A, the accommodation coefficient α, and the orbital altitude, h.

B. The Projected Effective Area, A: The projected effective area is a function of attitude stabilization of the spacecraft.

C. The Mass Variation of the Spacecraft

D. The Relative Velocity, V_R, of the Spacecraft with Respect to the Atmosphere: Due to the fact that the atmosphere rotates, the velocity of the spacecraft relative to the rotating atmosphere differs from the inertial velocity of the spacecraft. Consequently, the drag force vector will not lie in the plane of unperturbed motion; and, therefore, all six orbital elements will be affected.

E. The Atmospheric Density: The atmospheric density is a rapidly decreasing function of altitude with superimposed effects of solar ultraviolet and corpuscular radiation in the upper atmosphere regions (above 200 KM). In other words, at altitudes above 200 KM, the atmospheric density is a function of both altitude and time; the dependence on time being implicit in the form of dependence on the position of the subsolar point and the amount of emitted solar energy (ultraviolet and corpuscular).

The drag acceleration is analytically defined in terms of these factors as follows,

$$\frac{\bar{D}}{m} = -B\rho V_R \bar{V}_R$$

where $B = C_D A/2m$ is the ballistic coefficient; ρ, is the instantaneous atmospheric mass density at altitude h above the oblate earth's surface; V_R, is the magnitude of $\bar{V}_R$.

The dependence of atmospheric density on orbital altitude is usually approximated by the exponential functional relationship,

$$\rho = \rho_p e^{-k(h-h_p)}$$

where K is the inverse of the density scale height $\left(K = -\frac{1}{\rho}\frac{d\rho}{dh}\right)$ and ρ_p is the density at the instantaneous perigee height. ρ and K are determined as a function of both perigee altitude and time from a preferred dynamic model atmosphere. In this manner, the integration of the perturbative effects is greatly simplified.

In addition to the direct dependence of atmospheric drag on the five factors listed heretofore, it also depends in an indirect manner on the attitude stabilization of the spacecraft, the rotation of the atmosphere, and the flattening of the atmosphere. This indirect dependence is implicit through the projected effective area, A, (and the drag coefficient, C_D), the relative velocity V_R and the term (h-hp) in the exponential definition of the density, respectively.

A. The attitude stabilization: The attitude stabilization affects the shape of the projected effective area, A, and through it, also the drag coefficient, C_D. The two extreme cases of attitude stabilization are: "nose-on," with the longitudinal axis of the vehicle along the instantaneous velocity vector (0° angle-of-attack), and "broadside," with a 90° angle-of-attack. All the other cases are contained between these two extremes. In absence of information on vehicle attitude stabilization, a locally fixed attitude geometry may be assumed. Such locally stabilized attitude yields a (nearly) constant effective drag area. Vice-versa, by assuming a constant effective area, a locally stabilized attitude is automatically imposed. Maximum lifetime is achieved by having the longitudinal axis of the vehicle locally stabilized in the direction of the instantaneous velocity vector to minimize the projected effective area.

B. The rotation of the atmosphere: The atmospheric drag in a stationary atmosphere causes the eccentricity, e, the semi-major axis, a, and hence the orbital period P to decrease secularly (per revolution), but causes no secular changes in the argument of perigee ω, the inclination i, and the longitude of the node Ω. In a rotating atmosphere, the drag acceleration vector is out of the plane of unperturbed motion, and the three orientational elements will also be affected. The effect of atmospheric rotation is: (1) to decrease the respective rates at which a, e, and P vary for $i<90°$, and to increase these rates for $i > 90°$; (2) to decrease the inclination i for all orbits; and (3) to produce secular regression of the node Ω of the argument of perigee ω.

C. The flattening of the atmosphere: The flattening of the atmosphere, assumed to be the same as that of the earth, affects the atmospheric density through the exponential term (h-hp). Since density varies rapidly with slight changes in altitude, the effect of the flattening of the atmosphere is rather significant.

2.2.1.2 The Effect of the Perturbing Force on Orbit Decay

The drag acceleration causes a distortion in the shape of the orbit and a continuous loss of kinetic energy of the satellite to the atmosphere. The net result of these periodically repeated effects is:

A. A cumulative variation of the orbital elements.

E. A drop in orbital altitude (increase in potential energy) to compensate for the loss in kinetic energy. Apogee altitude decays at much higher rates than does the perigee altitude. Thus, an initially circular orbit with uniform drag over its entire path will tend to remain nearly circular and an elliptic orbit will tend to become circular.

2.2.2 Review of the Available Literature

2.2.2.1 General Comments on the Papers Reviewed

The literature in the field of general perturbations, as applied to atmospheric drag effects on the orbit of an artificial earth satellite, is very extensive. Unfortunately, many of the papers duplicate one another and differ principally only in the manner in which the exponential density function is developed. Furthermore, most of the works do not include all of the factors which are pertinent to the problem, such as the rotation and the non-sphericity of the atmosphere; and some authors restrict the validity of the analysis by assuming that for elliptical orbits of eccentricity > 0.1, the perigee altitude may be considered constant, and that the uncertainties in the true variation of atmospheric density are greater than the differences between the results obtained by them and other authors. Practically, except for some superficial comments, no attempt is made by any author to discuss the variation of the drag coefficient C_D and the dependence of the projected effective area, A, on attitude stabilization. Rather, they assume these parameters to be constant. Further, the variability of the density scale height is completely ignored and assumed to be constant (except for King-Hele). Likewise, standard atmosphere models (mostly outdated) are considered for the determination of atmospheric density at perigee, ρ_p, and only as a function of altitude, completely ignoring the dynamic nature of the atmosphere.

2.2.2.2 Methods and Techniques

The method most commonly used by the majority of authors is that of general perturbations; that is, integration of the equations of motion by analytical methods. The time rates of change of the elements are defined in terms of the components of the perturbing acceleration in the radial ($\overline{R}$), local horizontal ($\overline{S}$), and orthogonal ($\overline{W}$) directions. Three alternative developments may then be used: expansion in series in terms of the true anomaly θ,* the eccentric anomaly E, and in the mean anomaly M. Most of the papers, however, use the expansion in terms of the eccentric anomaly E. Denoting by $\Delta\nu$ the secular changes in any of the six orbital elements, the series expansion in the eccentric anomaly E yields:

$$\Delta\nu = -(\text{CONSTANT})\int_0^{2\pi} \rho \sum_{n=0}^{n} D_n \cos^n E \, dE$$

* θ will be utilized as the true anomaly to be consistent with the notation of the papers to be reviewed.

where the coefficients D_n are functions of the eccentricity, e, and of the factor $\lambda = (\Omega_e \sqrt{1-e^2} \cos i)/n$, where Ω_e is the rate of rotation of the atmosphere and n is the mean motion. The density is generally approximated by the exponential function $\rho = \rho_p e^{-K(h-h_p)}$ and expanded in terms of one of the three anomalies. When the eccentric anomaly E is used, the expansion of the density exponential function yields:

$$\rho = \rho_p e^{-K(h-h_p)} = \rho_p e^{-c(1-\cos E)} e^{-Q(\sin^2 u^* - \sin^2 \omega)}$$

where $c = Kae$ and $Q = K R_{EQ} f \sin^2 i$. K is the inverse of the density scale height, f is the flattening of the earth, R_{EQ} is the equatorial radius, i is the orbital inclination, u^* is the equivalent $(\theta + \omega)$, and transforming the true anomaly θ in terms of E, the density becomes,

$$\rho = \rho_p e^{-c}\left[e^{c\cos E} + Q_1 e^{c\cos E}\left(1 + \sum_{n=1}^{n} A_n \cos^n E\right) + Q_2 e^{c\cos E}\left(1 + \sum_{n=1}^{n} B_n \cos^n E\right) + \ldots\right]$$

where,

$$Q_1 = (1-e^2)\left(-Q\cos 2\omega + \frac{Q^2}{2}\sin^2 2\omega\right)$$

$$Q_2 = (1-e^2)^2\left(\frac{Q^2}{2}\cos 4\omega\right)$$

Introducing this expansion for the density, ρ, in the foregoing definition for ΔV, and performing the respective series multiplication, yields:

$$\Delta V = -(\text{CONSTANT})\rho_p e^{-c}\int_0^{2\pi}\left[e^{c\cos E}\sum_{n=0}^{n} D_n \cos^n E + Q_1 e^{c\cos E}\sum_{n=0}^{n} A_n' \cos^n E + Q_2 e^{c\cos E}\sum_{n=0}^{n} B_n' \cos^n E\right] dE$$

2.2.2.3 Integration Procedures

The basic approach to the integration of the perturbative effects of atmospheric drag for orbits of relatively high eccentricity is to consider only the accumulated secular perturbations per perigee pass. This procedure leads to the assumption that the motion over the remainder of the orbital path is not significantly affected by atmospheric drag. In other words, the drag effect in the vicinity of perigee is assumed to be so much higher than elsewhere on the oribt, that it is nearly that of an impulse. For this case, which is assumed to occur when $c > 3$, the "asymptotic solutions" are used. The result is that, the larger c is, the more accurate are the results.

For nearly circular orbits, e is small and $c \leq 3$. In this case the osculating atmosphere remains a good approximation throughout the orbit, all parts of which contribute significantly to the integrals of the changes in the orbital elements, and the asymptotic solutions become useless. The integration is, therefore, performed over the entire orbital path between 0 and 2π. The resulting solutions are called the "General Solutions" and are applied to cases where $c \leq 3$.

The General Solutions for the Case when $c \leq 3$

The integrals in $e^{c \cos E} \cos^n E \, dE$, in the definitions of the changes in the orbital element $d\nu$ (as indicated in the preceding section), are expressed by most authors in a sequence of modified Bessel functions $I_n(c)$ of the first kind. This is usually done by first transforming the powers $\cos^n E$ into multiple angles $\cos nE$ and then, using the definition:

$$I_n(C) = \frac{1}{2\pi} \int_0^{2\pi} e^{c \cos E} \cos nE \, dE$$

where

$$I_n(C) = \sum_{j=0}^{\infty} \frac{\left(\frac{c}{2}\right)^{2j+n}}{j!(j+n)!} \qquad n = 1, 2, 3, \ldots n$$

Finally, the higher orders of the Bessel functions are expressed in terms of the zero and first orders using the following reduction formula:

$$I_{n+1} = I_{n-1} - 2\frac{n}{C} I_n$$

However, these two steps can be combined to express the integrals of $e^{c \cos E} \cos^n E \, dE$ directly in terms of the modified Bessel functions of the zero and first order by the application of the following table:

$$\frac{1}{2\pi} \int_0^{2\pi} e^{c \cos E} dE = I_0(C)$$

$$\frac{1}{2\pi} \int_0^{2\pi} e^{c \cos E} \cos E \, dE = I_1(C)$$

$$\frac{1}{2\pi} \int_0^{2\pi} e^{c \cos E} \cos^2 E \, dE = I_0(C) - \frac{I_1(C)}{C}$$

$$\frac{1}{2\pi}\int_0^{2\pi} e^{c\cos E}\cos^3 E\, dE = -\frac{I_0(C)}{C} + \left(1+\frac{2}{C^2}\right) I_1(C)$$

$$\frac{1}{2\pi}\int_0^{2\pi} e^{c\cos E}\cos^4 E\, dE = \left(1+\frac{3}{C^2}\right) I_0(C) - \frac{2}{C}\left(1+\frac{3}{C^2}\right) I_1(C)$$

$$\frac{1}{2\pi}\int_0^{2\pi} e^{c\cos E}\cos^5 E\, dE = -\frac{2}{C}\left(1+\frac{6}{C^2}\right) I_0(C) + \left(1+\frac{7}{C^2}+\frac{24}{C^4}\right) I_1(C)$$

The integrated form of the general solutions ($c \leq 3$) for the secular changes of the orbital elements will then assume the following form:

$$\Delta V = -(CONSTANT) \times 2\pi \rho_p e^{-c}\left[(a_0 + \frac{Q_1}{C} a_1 + 3\,\frac{Q_2}{C^2} a_2) I_0(C) + (b_0 + \frac{Q_1}{c} b_1 + 3\,\frac{Q_2}{C^2} b_2) I_1(c)\right]$$

The Asymptotic Solutions for the Case when $c > 3$

When $c > 3$, very accurate analytical solutions can be obtained by considering the accumulated secular changes per perigee pass; that is, by assuming that the drag effect in the vicinity of perigee is nearly that of an impulse. This assumption is made when $c > 3$. The larger c is, the more accurate are the results.

The integrated solutions for this case, called the "asymptotic solutions," can be directly derived from the "General Solutions" by replacing the modified Bessel functions $I_0(c)$ and $I_1(c)$ of the zero and first order with their equivalent asymptotic expansions $I_0^A(c)$ and $I_1^A(c)$, which are defined as follows:

$$I_0^A \sim \frac{e^c}{\sqrt{2\pi c}}\left(1+\frac{1}{8c}+\frac{9}{128c^2}+\frac{75}{1024c^3}+\dots\right)$$

$$I_1^A \sim \frac{e^c}{\sqrt{2\pi c}}\left(1-\frac{3}{8C}-\frac{15}{128c^2}-\frac{105}{1024c^3}+\dots\right)$$

or in a general form

$$I_n^A \sim \frac{e^c}{\sqrt{2\pi C}}\left[1+\sum_{j=1}^{\infty}(-1)^j\,\frac{(4n^2-1^2)(4n^2-3^2)\dots(4n^2-(2J-1)^2)}{j!\,(8c)^J}\right]$$

As before, the higher-order functions can be reduced to lower order by the use of the recurrent relation

$$I_{n+1}^{A} = I_{n-1} - \frac{2}{C} I_{n}^{A}$$

After substitution of the expressions for the asymptotic expansions of the modified Bessel function in the definitions of the "General Solutions," the exponentials e^{c} and e^{-c} will cancel out, and the "asymptotic solutions" will have the following form:

$$\Delta V = -(\text{CONSTANT})\rho_p \sqrt{\frac{2\pi}{c}} \left[\left(a_0 + \frac{Q_1}{c} a_1 + 3\frac{Q_2}{c^2} a_2\right) I_0^{A}(C) + \left(b_0 + \frac{Q_1}{c} b_1 + 3\frac{Q_2}{c^2} b_2\right) I_1^{A}(C) \right]$$

where the coefficients (a_0, a_1, a_2) and (b_0, b_1, b_2) have the same values as before.

2.2.2.4 Critical Evaluation of the Papers Reviewed

2.2.2.4.1 The Method of General Perturbations - Bessel Functions Introduced in the Solutions

This method is based on the principle of osculating ellipse with binomial series expansion in terms of the eccentric anomaly E, and the integrated secular changes in the orbital elements per revolution expressed in terms of modified Bessel functions of the first kind of the argument $c = Kae$:

The Work of T. E. Sterne (Reference 2.2).

Assumptions

The atmosphere is non-spherical and rotates with the angular velocity Ω_e of the Earth. The resultant aerodynamic force acts in the direction opposite to the relative velocity $\overline{V}_R$ of the satellite with respect to the rotating atmosphere. The atmospheric density (ρ) at any altitude (h) above the oblate Earth is approximated by the osculating exponential atmosphere $\rho = \rho_p e^{-K(h-h_p)}$, where ρ_p is the density at perigee, hp is the perigee height, and K is the inverse of the density scale height. The Earth's gravitational potential is taken as that of a point mass; the factor $Q = K R_{EQ} f \sin^2 i$ is assumed to be < 0.2 in the expansion of the exponential form of the atmosphere.

Completeness

Asymptotic solutions for the case of eccentric orbits are presented for the secular changes in all orbital elements. For nearly-circular orbits, however, only the secular change in the orbital period is presented ("General Solution" type).

Evaluation

All the factors influencing atmospheric drag are included. The analysis is more rigorous and accurate than the other analyses reported in the literature. However, in view of Sterne's assumption that $Q < 0.2$ and that he neglects powers of Q greater than 2, his results are somewhat less accurate for satellite altitudes < 200 n.mi. This limitation is not considered serious, however, since the formulation can be easily extended to include powers of Q higher than 2.

The Work Of F. Kalil (Reference 2.3)

Assumptions

The atmosphere is oblate, has the same flattening as the Earth, and varies exponentially with altitude. The atmosphere rotates with the same angular velocity as the Earth. The gravitational potential of the Earth is taken as that of a point mass. Sterne's assumptions, that $\frac{\Omega_e}{n} < \frac{1}{15}$ (ratio of the rate of the Earth's rotation and the mean motion of the satellite) and that $Q = K\, R_{EQ}\, f \sin^2 i$ is < 0.2 for orbital altitudes ≥ 200 n.mi., are retained; the eccentricity is contained within the boundaries of $0 \leq e \leq 0.01$.

Completeness

Only the "General Solutions" are presented for three of the orbital elements: the semi-major axis, the eccentricity, and the period of nearly circular orbits ($0 \leq e \leq 0.01$). The expansion of the exponential form of the atmosphere is extended to include powers of Q through Q^4, thus making the results fairly accurate for orbital altitudes lower than 200 n.mi.

Evaluation

Kalil uses Sterne's approach, his technique and assumptions. His work is primarily an extension of Sterne's analysis to the case of nearly-circular orbits. For this case, Sterne derives only the solution for the secular change in the orbital period, whereas Kalil proceeds to derive also the solutions for the semi-major axis and the eccentricity. In summary, Kalil's paper is limited to the special case of nearly-circular orbits and does not include solutions for the changes in all orbital elements. The solutions ("General Solutions") for this case could be derived from the "asymptotic solutions" for eccentric orbits, when expressed in terms of modified Bessel functions, by simply replacing the asymptotic expansions of the Bessel functions with their regular definitions.

The Work Of P. E. El'Yasberg (Reference 2.4)

Assumptions

The atmosphere is stationary and spherical. The atmospheric density is approximated by the exponential osculating atmosphere. The gravitational potential of the Earth is that of a point mass.

Completeness

Incomplete. The theory is limited to the hypothetical case of spherical non-rotating atmosphere.

Evaluation

It appears that this work was influenced by Sterne (that is, it generally follows his approach and technique). Failure to include all of the factors which affect atmospheric drag, however, restricts the analysis and limits the scope. In addition, further assumptions and approximations are made in deriving the "Asymptotic Solutions" for eccentric orbits.

The Work Of G. E. Cook and D. G. King-Hele (Reference 2.5)

Assumptions

The atmosphere is spherically symmetrical and rotates with the same angular velocity as the Earth. The air density is approximated by the exponential function $\rho = \rho_p [1 + b(r - r_p)^2] e^{-(r-r_p)/H_p}$, where H_p and b are taken constant over a revolution. The density scale height at perigee, H_p, varies linearly with perigee altitude, $H_p = H_{p_o} + k(r_p - r_{pe})$, in which k is a constant and < 0.2; the Earth's gravitational potential is that of a mass point. The orbital eccentricity is < 0.2. The parameter b is assumed to be related to the constant k, $b = k/2H_p^2$

Completeness

The non-sphericity of the atmosphere is neglected - incomplete set of orbital elements. Only expressions for the rates of change Δa and Δx of the semi-major axis, a, and the parameter $x = ae$ are presented. The solutions do not apply for eccentricities $e > 0.2$.

Evaluation

Solutions are given for the secular changes Δa, $\Delta x (x = ae)$, the perigee drop from its initial position $(r_{p_o} - r_p)$, the ratio of the current and initial periods, and for the current time t and total lifetime t_L in orbit for the cases of: $0 \leq e \leq 0.025$ and $0.025 < e < 0.2$. Also, the variations in perigee drop and orbital period as a function of (t/t_L), as well as the total lifetime t_L as a function of the initial period T_o, are given.

The analysis is rather cumbersome and difficult to follow. In the process, numerous assumptions are made and subsequently modified, so that the intricate inter-relationships in the development have to be mastered to follow the analysis. As an illustration, the original assumption of linear variation of the density scale height H with altitude is replaced by several intricate relationships in an effort to show that particular constant values of H for the entire lifetime may be used. To add to the confusion, subscripts are not sufficiently defined. In the case of eccentric orbits, the subscript "o" appears to refer to the zero-time conditions; this assumption is difficult to verify. In the case of nearly circular orbits, it appears that the subscript "1" is used to indicate zero-time conditions; but again, no clear definition is given. Numerous approximations are also made without apparent justification. For instance, in deriving a solution for the perigee drop for the case of eccentric orbits $(\beta \chi > 3)$, "a" was set equal to a_o. Finally, no reason is given why it is assumed that air density variation follows the law: $\rho = \rho_p[1 + b(r-r_p)^2]\, e^{-(r-r_p)/H_p}$, and no attempt is made to introduce the oblateness of the atmosphere into the analysis.

2.2.2.4.2 The Method of General Perturbations - Bessel Functions not Introduced in the Solutions

This method is based on the principle of osculating ellipse with Fourier series expansion in one of the three anomalies and the secular changes separated from the periodic changes in the integrated solutions.

The Work Of I. G. Izsak (Reference 2.6)

Assumptions

The atmosphere is spherical, rotates with the Earth, and is approximated by the empirical power function $\rho = \rho_p\left(\frac{h - 55}{h_p - 55}\right)^{-5}$. The Earth's gravitational potential is that of a point mass.

Completeness

Incomplete. The oblateness of the atmosphere is not included. There is no distinction between eccentric and nearly-circular orbits. The solutions for both cases are combined in a single set of solutions.

Evaluation

The assumption of a spherical atmosphere and the approximation of atmospheric density by an outdated empirical power function makes the analysis both incomplete and questionable. The basic equations for the rates of change of the orbital elements are taken "verbatim" from Sterne's paper. The expansion of the density power function by the method reported by Smart (1953) makes the coefficients of the series in the integrands rather cumbersome. Because of the use of a power representation for the atmospheric density variation, the integrals are not suitable for development in Bessel functions. Instead, Izsak uses indefinite integrals, integrates the rates of change of the orbital elements with respect to the eccentric anomaly, and obtains a term in E (free from trigonometric functions) and a series of terms in sin jE. Next, he replaces the "free" E by (nt + e sin E), and thus obtains the secular and the periodic changes. Normally, when development in Bessel functions is used, the respective solutions have the secular and periodic changes combined together.

The Work Of Y. V. Batrakov and V. F. Proscurin (Reference 2.7)

Assumptions

The atmosphere is stationary and spherically symmetrical. The air density is approximated by the function: $\rho = A\left(\log_{10} \frac{h}{10}\right)^{B}$, where A and B are constants. The Earth's gravitational potential is taken as that of a point mass.

Completeness

Incomplete. Two-dimensional analysis. The rotation and non-sphericity of the atmosphere are neglected. There is no distinction between eccentric and nearly-circular orbits. The solutions for both cases are combined in a single set of solutions.

Evaluation

The periodic and the secular terms are separated in the solutions. The second and higher-order secular terms are only suggested. The power of the author's approach is weakened by his neglect of the rotation and non-sphericity of the atmosphere, as well as through the representation of air density by an outdated and questionable model. The expression for the change in the longitude of perigee, π, has the eccentricity in the denominator which, for nearly circular orbits, would make the perturbative variation in the perigee direction approach infinity.

2.2.2.4.3 The Method of Canonical Variables

This method if based on a generalization of the method of variation of arbitrary constants. The equations of motion are defined in canonical variables, and the development of the drag acceleration in power of eccentricity and in multiples of the mean anomaly.

The Work Of D. Brouwer and H. Gen-Ichiro (Reference 2.8)

Assumptions

The atmosphere is stationary and spherical. The atmospheric density may be represented by a spherical exponential model from the perigee height upward. The density scale height is constant. The drag effects can be linearly superimposed upon the effects of Earth oblateness to the first order.

Completeness

Two-dimensional analysis, because the rotation of the atmosphere is not included. The oblateness of the atmosphere is also neglected. No distinction is made between nearly circular and eccentric orbits. Numerous simplifying assumptions are made. The drag perturbation effects are superimposed to the solutions of the drag-free problem.

Evaluation

Equations of motion for the canonical variables which are solutions of the drag-free problem are developed first. Next, the drag accelerations are introduced and expanded in powers of e and multiples of the mean anomaly. Finally, the integration is performed by the method of successive approximations. The 0th approximation corresponds to the solution of the drag-free problem. The solutions are very lengthly and extremely cumbersome. Because of the superimposition of the drag-free problem, the solutions fail at the critical inclination. The power of the author's approach is greatly weakened by the neglect of the rotation and non-sphericity of the atmosphere, by the spherical exponential approximation $\rho = \rho_{p} e^{-\alpha(h - h_p)}$ of the air density, and by assuming a constant value for α, the inverse of the density scale height. Furthermore, the effectiveness of the theory is greatly reduced by the unfavorable series convergence in the case of low perigee heights and also in the case of values of $(\alpha_{ae} \geq 1)$. The analytical treatment is more concerned with satisfying the classical astronomical principles than with actual satellite engineering needs.

2.2.2.4.4 The Method of Variation of Parameters

This method is based on the principles of general perturbations; the transformation of variables in the basic equations of motion, using either non-dimensional variables $(\xi = R_{EQ}/r, \eta = \mu R_{EQ}/h^2)$ or dimensional $(\xi = 1/r, \eta = h)$, where h is the angular momentum, and the application of the Krylov-Bogoliuboff averaging method over a full revolution.

The Work Of E. R. Roberson (Reference 2.9)

Assumptions

The atmosphere is stationary and spherical. The air density can be represented by the exponential function $\rho = \rho_{*} e^{-Kp^2(1/r - 1/p)}$, where ρ_* is the air density at distance $p = a(1 - e^2)$ from the Earth's center and K is the inverse of the density scale height taken as a constant. The radial component of the drag acceleration is small and may be neglected; the eccentricity is assumed to be small, and therefore, powers of $e > 1$ may be discarded.

Completeness

Two-dimensional analysis. The effects of the rotation and oblateness of the atmosphere are not included. The radial drag acceleration component is neglected. Solutions are derived only for the decay of eccentricity with the semi-latus rectum p, the decay of the semi-latus rectum p with the true anomaly, and for the "growth" of the true anomaly with time. The analysis is applicable only to nearly circular orbits.

Evaluation

The analysis is not rigorous. It does not include all of the factors affecting the drag forces which are assumed to act tangent to the path of motion. The angular momentum is assumed constant. The atmospheric model used is outdated (Kallman, 1952). Powers of the eccentricity higher than one are neglected in deriving the solutions; thus, the solutions apply only to nearly-circular orbits. The preliminary solutions for $dp/d\beta$ and $de/d\beta$ (where β is the true anomaly), which were obtained by the Krylov-Bogoliuboff averaging method, are subjected to intricate manipulation to derive expressions for e and p by an iterative process. The process requires the use of tabulated values for certain definite integral functions. It appears that there is an error in the solution for $de/d\beta$ (Equations 28 and 31). The averaging Krylov-Bogoliuboff method is questionable, as it leads to the invariance of the perigee.

The Work Of B. Billik (Reference 2.10)

Assumptions

The atmosphere is stationary and spherical. The air density may be approximated by the exponential function $\rho = \bar{\rho}_p \, e^{-K(r-R_{EQ})}$, where ρ_p and K are matching constants (K is the inverse of the density scale height). The perigee altitude remains invariant for eccentricities > 0.1; for $e > 0.1$, the modified Bessel functions of all orders are assumed to be equal in the definition of the asymptotic solutions.

Completeness

Two-dimensional analysis. The effects of the rotation and oblateness of the atmosphere are not included. Incomplete and obscure definition of the constant ρ_p in the exponential model of the atmosphere. The conclusion resulting from the application of the Krylov-Bogoliuboff averaging method for the invariance of perigee altitude when $e > 0.1$ is far from being true and weakens the power of the author's approach.

Evaluation

The author attempts a survey of 30 references listed but limits himself to a brief discussion of about one-third of the referenced papers. The main body of the discussion and the analysis are centered on the author's earlier report dated December 1960 and listed as his seventh reference. The survey is based entirely on a two-dimensional analysis of the drag problem; the effects of the rotation and non-sphericity of the atmosphere are ignored. Sterne's paper, "Effects of the Rotation of a Planetary Atmosphere upon the Orbit of Close Satellites," is listed among the references; but it is not discussed. In reporting Roberson's solutions for the case of nearly-circular orbits the author replaces Roberson's definition for air density by his own definition, $\rho = \bar{\rho}_p \, e^{-K(p - R_{EQ})}$. When deriving the asymptotic solutions for eccentric orbits with $e > 0.1$, the author assumes the modified Bessel functions of all orders to be equal. All the reviewed papers, according to Billik, may be used for adequate lifetime calculations and the differences between the results obtained by the various authors are smaller than the inherent uncertainties in the knowledge of the atmosphere, implying that a three-dimensional analysis is unwarranted.

2.2.2.5 Selection of Papers for Detailed Development

Two papers were selected for detailed, analytical development. They appear to be the most outstanding papers in the up-to-date literature for the following reasons:

A. They include all the factors which affect atmospheric drag.

B. The analysis is three-dimensional, very rigorous, and easy to follow.

C. The analysis applies to eccentric as well as to nearly-circular orbits (including circular orbits).

D. The only simplifying assumption is that $\frac{1}{2}\left(\frac{\Omega_e}{n}\right)$, half of the ratio of the rate of the Earth's rotation and the mean motion of the satellite is $<$ 1/30 for close Earth satellites; therefore, $\frac{1}{4}\left(\frac{\Omega_e}{n}\right)^2$ is <0.001 of the leading term in the definition of the relative velocity of the satellite with respect to the atmosphere and may be neglected.

E. The solutions are expressed in an elegant form convenient for computer development.

The two papers selected are: "Effect of the Rotation of a Planetary Atmosphere Upon the Orbit of a Close Satellite," by T. E. Sterne, ARS Journal. October 1959, Volume 29, No. 10; and "Effect of an Oblate Rotating Atmosphere on the Eccentricity, Semi-Major Axis and Period of a Close Earth Satellite," by F. Kalil, The Martin Company, Baltimore 3, Maryland.

2.2.3 Analytical Development of Sterne's Technique (Asymptotic Solutions)

2.2.3.1 The Acceleration Caused by the Perturbing Force Acting on the Spacecraft

Assuming that atmospheric drag is the only perturbation force acting on the spacecraft, the vector of the drag acceleration can be defined as follows:

$$\frac{\bar{D}}{m} = -B\rho V_R \bar{V}_R \tag{2.1}$$

where B is the ballistic coefficient $(C_D A/m)$ and $\bar{V}_R$ is the velocity vector of the spacecraft relative to the atmosphere,

$$\bar{V}_R = \bar{V} - \bar{V}_{ATM} \tag{2.2}$$

The inertial velocity vector $\bar{V}$ is given by,

$$\bar{V} = \bar{R}\dot{r} + \bar{S}r\dot{\theta} = \frac{an}{1 - e\cos E}\,(\bar{R}\,e\sin E + \bar{S}\sqrt{1-e^2}) \tag{2.3}$$

where $\bar{R}$ is a unit vector in the outward direction of the position vector $\bar{r}$, $\bar{S}$ is a unit vector perpendicular to $\bar{R}$ in the osculating orbital plane, $\bar{W}$ completes the right-hand frame.

The rotational velocity vector of the atmosphere, $\bar{V}_{ATM}$, is defined as

$$\bar{V}_{ATM} = \bar{\Omega}_e \times \bar{r} = \Omega_e\,(\bar{k}\times\bar{r}) = r\Omega_e\,(\bar{k}\times\bar{R}) \tag{2.4}$$

where Ω_e is the rotational rate of the atmosphere in rad/sec, and $\bar{k}$ is a unit vector in the direction of the Earth's spin axis

$$\bar{k} = \bar{R}\sin i\sin u + \bar{S}\sin i\cos u + \bar{W}\cos i \tag{2.5}$$

so that

$$\bar{V}_{ATM} = r\Omega_e\,(\bar{S}\cos i - \bar{W}\sin i\cos u) = a(1-e\cos E)\Omega_e\,(\bar{S}\cos i - \bar{W}\sin i\cos u) \tag{2.6}$$

Substitution of relations (2.3) and (2.6) in Equation (2.2) yields,

$$\bar{V}_R = \bar{R}\,\frac{ane\sin E}{1-e\cos E} + \bar{S}\left[\frac{an\sqrt{1-e^2}}{1-e\cos E} - a(1-e\cos E)\Omega_e\cos i\right] + \bar{W}a(1-e\cos E)\Omega_e\sin i\cos u \tag{2.7}$$

Next, the magnitude V_R is calculated,

$$V_R^2 = \left(\frac{an}{1-e\cos E}\right)^2 (e^2\sin^2 E + 1 - e^2) - 2a^2n^2\left(\frac{\Omega_e\sqrt{1-e^2}\cos i}{n}\right) + a^2(1-e\cos E)^2\Omega_e^2\,(1-\sin^2 i\sin^2 u) \tag{2.8}$$

Now, defining,

$$d = \frac{\Omega_e\sqrt{1-e^2}\cos i}{n} \tag{2.9}$$

and substituting into (2.8) yields,

$$V_R^2 = a^2 n^2 \left[\frac{1+e\cos E}{1-e\cos E} - 2d + \left(\frac{\Omega_e}{n}\right)^2 (1-e\cos E)^2 (1-\sin^2 i \sin^2 u) \right] \qquad (2.10)$$

$$V_R = an\sqrt{\frac{1+e\cos E}{1-e\cos E}} \sqrt{1-2d\ \frac{1-e\cos E}{1+e\cos E} + \left(\frac{\Omega_e}{n}\right)^2 \frac{(1-e\cos E)^3}{1+e\cos E}(1-\sin^2 i \sin^2 u)} \qquad (2.11)$$

Now, making use of Sterne's observation that $\frac{\Omega_e}{n}$ is always less than 1/15 for Earth orbits, the function under the radical can be expanded in a series. The third term will be smaller than $\frac{1}{2}(\frac{\Omega_e}{n})^2$; that is $<$ 1/450 times a number smaller than 1. Hence, the third term will average 0.001 of the leading term and can be neglected. For the same reason, $\frac{1}{2}\, d^2 < 0.001$. In the binomial expansion all terms after the second may be neglected so that Equation (2.11) is approximately

$$V_R = an\sqrt{\frac{1+e\cos E}{1-e\cos E}} \left(1 - d\ \frac{1-e\cos E}{1+e\cos E}\right) \qquad (2.12)$$

The vector $\bar{V}_R$ in Equation (2.7) will be now expressed in terms of the parameters,

$$d = \frac{\Omega_e \sqrt{1-e^2} \cos i}{n}$$

$$\dot{E} = \frac{n}{1-e\cos E} \qquad (2.13)$$

$$\bar{V}_R = a\dot{E}\left[\bar{R}\, e\sin E + \bar{S}\sqrt{1-e^2}\left(1-d\ \frac{(1-e\cos E)^2}{1-e^2}\right) + \bar{W}\ \frac{\Omega_e}{n}(1-e\cos E)^2 \sin i \cos u\right] \qquad (2.14)$$

Substitution of Equation (2.14) in Equation (2.1) yields,

$$\frac{\bar{D}}{m} = -B a \rho\, V_R\, \dot{E}\left[\bar{R}\, e\sin E + \bar{S}\sqrt{1-e^2}\left(1-d\ \frac{(1-e\cos E)^2}{1-e^2}\right) + \bar{W}\ \frac{\Omega_e}{n} \sin i \cos u\ (1-e\cos E)^2\right] \qquad (2.15)$$

where V_R is given by (2.12)

2.2.3.2 Rates of Change of the Orbital Elements Caused by the Perturbing Acceleration

In the previous section the acceleration of the perturbing force was derived:

$$\frac{\bar{D}}{m} = -Ba\rho\, V_R \dot{E}\left[\bar{R}\, e\, \mathrm{SIN}\, E + \sqrt{1-e^2}\left(1 - d\, \frac{(1-e\cos)^2}{1-e^2}\right)\bar{S} + \bar{W}\, \frac{\Omega_e}{n}\, \mathrm{SIN}\, i\, \cos u \left(1 - e \cos E\right)^2\right] \tag{2.16}$$

where V_R is the magnitude of the relative velocity of the vehicle with respect to the atmosphere, as defined by relation (2.12),

$$V_R = a n \sqrt{\frac{1+e\cos E}{1-e\cos E}}\left(1 - d\, \frac{1-e\cos E}{1+e\cos E}\right) \tag{2.17}$$

Thus, since the inertial velocity vector $\bar{V}$ in the osculating $\bar{R}\,\bar{S}\,\bar{W}$ frame, is given by Equation (2.3):

$$\bar{V} = \frac{a n}{1-e\cos E}\left(\bar{R}\, e\, \mathrm{SIN}\, E + \bar{S}\sqrt{1-e^2}\right) \tag{2.18}$$

the energy change, $d\epsilon/dt$, per unit mass may be found from the definition of work done on the vehicle by the perturbing force:

$$\frac{d\varepsilon}{dt} = \frac{\bar{D}}{m}\bullet\bar{V} = -Ba^2 n\rho\, V_R\, (1+e\cos E)\dot{E}\left(1 - d\, \frac{1-e\cos E}{1+e\cos E}\right) \tag{2.19}$$

But the total energy is,

$$\varepsilon = \frac{-\mu}{2a} \tag{2.20}$$

Thus, it follows by differentiation that,

$$\frac{da}{dt} = \frac{\mu}{2\varepsilon^2}\frac{d\varepsilon}{dt} = \frac{\mu}{2}\left(\frac{2}{\mu}\right)^2 a^2 \frac{d\varepsilon}{dt} = \frac{2}{\mu} a^2 \frac{d\varepsilon}{dt} = \frac{2}{a n^2}\frac{d\varepsilon}{dt} \tag{2.21}$$

After substitution of $d\epsilon/dt$ from (2.19) and using the definition of V_R from (2.17)

$$\frac{da}{dt} = -2Ba^2\rho\, \frac{(1+e\cos E)^2}{\sqrt{1-e^2\cos^2 E}}\left(1 - d\, \frac{1-e\cos E}{1+e\cos E}\right)^2 \frac{dE}{dt} \tag{2.22}$$

The rate of change of the angular momentum $\bar{h}$ per unit mass is equal to the external moments produced by the perturbing force,

$$\frac{d\bar{h}}{dt} = \bar{r}\times\frac{\bar{D}}{m} = r\bar{R}\times\frac{D}{m} = r\bar{R}\times(\bar{R}R + \bar{S}S + \bar{W}W) = r(-\bar{S}W + \bar{W}S) \tag{2.23}$$

But, the vector $\bar{h}$ can be defined as

$$\bar{h} = h\,\bar{W} \tag{2.24}$$

Thus, differentiating Equation (2.24), one obtains

$$\frac{d\bar{h}}{dt} = \frac{dh}{dt}\,\bar{W} + h\,\dot{\bar{W}}\ {}^{*} \tag{2.25}$$

Comparison of the $\bar{W}$ components in relations (2.23) and (2.25) yields,

$$\frac{dh}{dt} = a(1-e\cos E)\,S = rS \tag{2.26}$$

where S is the component of the perturbing force in the direction of $\bar{S}$ given by Equation (2.16)

$$S = -B a \rho V_R \dot{E}\sqrt{1-e^2}\left[1-d\,\frac{(1-e\cos E)^2}{1-e^2}\right] \tag{2.27}$$

or, after substitution of V_R from Equation (2.17)

$$S = -Ba^2 n\sqrt{1-e^2}\,\rho\sqrt{\frac{1+e\cos E}{1-e\cos E}}\left(1-d\,\frac{1-e\cos E}{1+e\cos E}\right)\left[1-d\,\frac{(1-e\cos E)^2}{1-e^2}\right]\frac{dE}{dt} \tag{2.28}$$

But, since

$$h = \sqrt{\mu a(1-e^2)} = a^2 n\sqrt{1-e^2} \tag{2.29}$$

one has that,

$$e^2 = \frac{1-h^2}{\mu a} \tag{2.30}$$

Thus, differentiation of (2.30) with respect to time, yields

$$2e\,\frac{de}{dt} = -2\left(\frac{h}{\mu a}\right)\frac{dh}{dt} + \left(\frac{h}{\mu a}\right)\frac{h}{a}\,\frac{da}{dt} = \frac{-h}{\mu a}\left(2\,\frac{dh}{dt} - \frac{h}{a}\,\frac{da}{dt}\right) \tag{2.31}$$

---

* $H\dot{w}$ is the component in the direction of $-\bar{S}$

where da/dt is given by relation (2.22), dh/dt by (2.26) and (2.27), while by (2.29),

$$\frac{h}{a} = an\sqrt{1-e^2} \tag{2.32}$$

$$\frac{h}{\mu a} = \sqrt{\frac{1-e^2}{\mu a}} = \sqrt{\frac{1-e^2}{a^2 n}} \tag{2.33}$$

Thus, substitution in Equation (2.31) yields

$$e\frac{de}{dt} = -\left(\frac{\sqrt{1-e^2}}{a^2 n}\right) B a^3 n\sqrt{1-e^2}\,\rho\dot{E}\sqrt{\frac{1+e\cos E}{1-e\cos E}}\left(1-d\,\frac{1-e\cos E}{1+e\cos E}\right)*$$
$$*\left\{(1+e\cos E)\left(1-d\,\frac{1-e\cos E}{1+e\cos E}\right)-(1-e\cos E)\left[1-d\,\frac{(1-e\cos E)^2}{1-e^2}\right]\right\} \tag{2.34}$$

or

$$\frac{de}{dt} = -2Ba(1-e^2)\rho\sqrt{\frac{1+e\cos E}{1-e\cos E}}\left(1-d\,\frac{1-e\cos E}{1+e\cos E}\right)*$$
$$\cdot\left[\cos E - d\,\frac{(1-e\cos E)(2\cos E - e - e\cos^2 E)}{2(1-e^2)}\right]\frac{dE}{dt} \tag{2.35}$$

The motion of the node is the same as the motion of the projection, $\bar{h}_p$, of $\bar{h}$ (angular momentum) on the equatorial plane. Since $\bar{h}_p$ is perpendicular to the node, the motion of $\bar{h}_p$ and the node is produced by the component of $d\bar{h}/dt$ in the direction of the node.

Comparison of the respective $\bar{S}$-components of $d\bar{h}/dt$ in Equations (2.23) and (2.25), after replacing $\bar{W}$ by $d\sigma/dt\ \bar{S}$, yields,*

$$h\frac{d\sigma}{dt} = -rW \tag{2.26}$$

Applying this relation to the component of $d\bar{h}/dt$ in the direction of the node,

$$h\sin i\,\frac{d\Omega}{dt} = rW\sin u \tag{2.37}$$

hence,

$$\frac{d\Omega}{dt} = \frac{rW\sin u}{h\sin i} = \frac{rW\sin u}{a^2 n\sqrt{1-e^2}\sin i} = \frac{(1-e\cos E)W\sin u}{an\sqrt{1-e^2}\sin i} \tag{2.38}$$

---

* $d\sigma$ is the angle through which the angular momentum vector is rotated in time dt

But, from Equation (2.16) and the value for V_R from Equation (2.17), it follows that,

$$W = -B a^2 \Omega_e \sin i \cos u \,(1-e\cos E)^2 \rho \sqrt{\frac{1+e\cos E}{1-e\cos E}} \left(1-d\,\frac{1-e\cos E}{1+e\cos E}\right)\frac{dE}{dt} \tag{2.39}$$

Substitution of this expression for W is relation (2.38) yields,

$$\frac{d\Omega}{dt} = -B a \rho \,\frac{\Omega_e}{n}\,\frac{\sin 2u}{2\sqrt{1-e^2}}\,(1-e\cos E)^2\sqrt{1-e^2\cos^2 E}\left(1-d\,\frac{1-e\cos E}{1+e\cos E}\right)\frac{dE}{dt} \tag{2.40}$$

where

$$(1-e\cos E)^2 \sin 2u = (1-e\cos E)^2 \sin(2\theta+2\omega) = (1-e\cos E)^2(\sin 2\omega\cos 2\theta$$

$$+\cos 2\omega \sin 2\theta) = (1-\cos E)^2 \sin 2\omega \cos 2\theta \qquad **$$

$$= (1-e\cos E)^2 \sin 2\omega\left[\frac{(\cos E - e)^2 - (1-e)^2(1-\cos^2 E)}{(1-e\cos E)^2}\right]$$

Thus, substitution of the expression for $(1 - e\cos E)^2 \sin 2u$ in Equation (2.40) yields,

$$\frac{d\Omega}{dt} = -\frac{B a \rho \Omega_e \sin 2\omega}{2n\sqrt{1-e^2}}\sqrt{1-e^2\cos^2 E}\left(1-d\,\frac{1-e\cos E}{1+e\cos E}\right)\Big[-(1-2e^2)$$
$$- 2e\cos E + (2-e^2)\cos^2 E\Big]\frac{dE}{dt} \tag{2.41}$$

The time rate of change of the orbital inclination is related to $d\Omega/dt$ as will be shown. Indicating by the subscript "o" the conditions in the unperturbed orbital plane,

$$\sin \Delta i = \sin(i - i_o) = \sin i \cos i_o - \cos i \sin i_o \tag{2.42}$$

Now, let ϵ be the inclination of the perturbed orbit relative to the unperturbed orbit, i.e.,

$$\cos \epsilon = \cos i_o \cos i + \sin i_o \sin i \cos\Delta \tag{2.43}$$

But, from orbital relationships,

** The term $(\cos 2\omega \sin 2\theta)$ is an odd function of θ and will contribute nothing to the integral $\int_0^{2\pi}$

$$\sin i = \frac{\sin i_0 \sin u_0}{\sin u} \tag{2.44}$$

$$\tan u_0 = \frac{\sin i_0 \sin i \sin \Delta\Omega}{\cos i_0 \cos \epsilon - \cos i} \tag{2.45}$$

Solving these three equations simultaneously for $\cos i$ yields

$$\cos i = \frac{1}{\sin u} \left(\cos i_0 \sin u_0 \cos\Delta\Omega - \cos u_0 \sin\Delta\Omega \right) \tag{2.46}$$

Substitution of the expressions for $\sin i$ and $\cos i$ into relation (2.42) yields,

$$\sin \Delta i = \frac{\sin i_0}{\sin u} \left[\cos i_0 \sin u_0 (1 - \cos\Delta\Omega) + \cos u_0 \sin\Delta\Omega \right] \tag{2.47}$$

Finally, differentiation with respect to time, and taking the limit as $\Delta\Omega \to 0$, yields,

$$\frac{di}{dt} = \frac{\sin i}{\sin u} \cos u \, \frac{d\Omega}{dt} = 2 \sin i \cos^2 u \left[\frac{1}{\sin 2u} \frac{d\Omega}{dt} \right] \tag{2.48}$$

Substituting the expression for $\frac{d\Omega}{dt}$ from relation (2.40), it follows that

$$\frac{di}{dt} = -Ba \, \frac{\Omega e}{n} \, \frac{\sin i \cos^2 u}{\sqrt{1-e^2}} (1 - e\cos E)^2 \sqrt{1 - e^2\cos^2 E} \left(1 - d\frac{1 - e\cos E}{1 + e\cos E} \right) \frac{dE}{dt} \tag{2.49}$$

where $2\cos^2 u = 1 + \cos 2u = 1 + \cos(2\theta + 2\omega) = 1 + \cos 2\omega \cos 2\theta + \sin 2\omega \sin 2\theta = 1 + \cos 2\omega \cos\theta$ **

$$= 1 + \cos 2\omega \left[\frac{(\cos E - e)^2 - (1-e)^2 (1 - \cos^2 E)}{(1 - e\cos E)^2} \right]$$

$$= 1 + \cos 2\omega \left[\frac{-(1 - 2e^2) - 2e\cos E + (2 - e^2)\cos^2 E}{(1 - e\cos E)^2} \right]$$

so that

$$\frac{di}{dt} = -Ba\rho \frac{\Omega e}{2n} \frac{\sin i}{\sqrt{1-e^2}} (1 - e\cos E)^2 \sqrt{1 - e^2\cos^2 E} \left(1 - d\frac{1 - e\cos E}{1 + e\cos E} \right) \left[1 + \cos 2\omega \frac{(2e^2 - 1) - 2e\cos E + (2 - e^2)\cos^2 E}{(1 - e\cos E)^2} \right] \frac{dE}{dt} \tag{2.50}$$

** The term $\sin 2\omega \sin 2\theta$ is an odd function of θ and will contribute nothing to the integral $\int_0^{2\pi}$

The rate of change of the argument of perigee ω resulting from the motion of the node (assuming that the in-plane perturbing forces are zero) is equal to du/dt.

$$\frac{d\omega}{dt} = \frac{du}{dt}$$

From orbital relationships we have that,

$$\sin i \cos u = \cos i_0 \sin \epsilon + \sin i_0 \cos \epsilon \cos u_0 \qquad (2.51)$$

$$\sin \epsilon = \frac{\sin i \sin \Delta\Omega}{\sin u_0} \qquad (2.52)$$

$$\cos \epsilon = \frac{\sin i}{\sin i_0}\left(\cos \Delta\Omega - \frac{\cos i_0 \cos u_0}{\sin u_0} \sin \Delta\Omega\right) \qquad (2.53)$$

where ϵ is the inclination of the perturbed orbit relative to the unperturbed orbit.

Elimination of ϵ between these three equations yields,

$$\cos u = \cos u_0 \cos \Delta\Omega + \cos i_0 \sin u_0 \sin \Delta\Omega \qquad (2.54)$$

Differentiating with respect to time and taking the limit as $\Delta\Omega \to 0$, one obtains

$$\left(\frac{d\omega}{dt}\right)_w = \frac{du}{dt} = -\cos i \frac{d\Omega}{dt} \qquad (2.55)$$

where $d\Omega/dt$ is given by Equation (2.41).

The subscript "w" in Equation (2.55) indicates that this is the change in ω contributed by the nodal motion which is caused by the component of the perturbing acceleration normal to the orbital plane.

The contribution to the change in ω caused by the R and S components of the perturbing acceleration in the orbital plane is equal to $-(d\theta/dt)^*$, which is the negative change of the true anomaly θ caused by the perturbing acceleration. This derivative must not be confused with $(d\theta/dt)$, which is the rate of change of the true anomaly θ in an unperturbed Kepler orbit. Hence,

$$\left(\frac{d\omega}{dt}\right)_{R,S} = -\left(\frac{d\theta}{dt}\right)^* \qquad (2.56)$$

The perturbing accelerations in the $\bar{R}$ and $\bar{S}$ directions (in the orbital plane) will tend to rotate the perigee in the opposite direction of motion and change the orientation of the velocity vector, which must remain tangent to the instantaneous osculating ellipse at any time. This will result in a change $d\gamma/dt$ of the flight path angle. The rotation of the perigee causes also a change in θ.

From the definition of

$$\gamma = \tan^{-1}\left(\frac{e \sin\theta}{1+e\cos\theta}\right) \tag{2.57}$$

it follows by differentiation that,

$$\frac{d\gamma}{dt} = \frac{e(\cos\theta+e)}{1+e^2+2e\cos\theta}\left(\frac{d\theta}{dt}\right)^* + \frac{\sin\theta}{1+e^2+2e\cos\theta}\frac{de}{dt} \tag{2.58}$$

But since,

$$1+e^2+2e\cos\theta = (1+e\cos\theta)(1+e\cos E) \tag{2.59}$$

Equation (2.58) becomes,

$$\frac{d\gamma}{dt} = \frac{e\cos E}{1+e\cos E}\left(\frac{d\theta}{dt}\right)^* + \frac{\sin E}{\sqrt{1-e^2}\,(1+e\cos E)}\frac{de}{dt} \tag{2.60}$$

Hence,

$$-\left(\frac{d\theta}{dt}\right)^*_{R,S} = \frac{1+e\cos E}{e\cos E}\left[\frac{\sin E}{\sqrt{1-e^2}\,(1+e\cos E)}\frac{de}{dt} - \frac{d\gamma}{dt}\right] \tag{2.61}$$

Next, the value of $d\gamma/dt$ in Equation (2.61) will be determined.

If N is the component of the perturbing acceleration normal to the velocity vector $\bar{V}$ in the orbital plane, then

$$N = R\cos\gamma - S\sin\gamma = V\frac{d\gamma}{dt} \tag{2.62}$$

where,

$$V = an\left(\frac{1+e\cos E}{1-e\cos E}\right)^{\frac{1}{2}} \tag{2.63}$$

$$\sin \gamma = \frac{e \sin E}{\sqrt{1 - e^2 \cos^2 E}} \tag{2.64}$$

$$\cos \gamma = \frac{\sqrt{1-e^2}}{\sqrt{1-e^2 \cos^2 E}} \tag{2.65}$$

Substitution of relations (2.63), (2.64), (2.65), in (2.62) yields

$$\frac{d\gamma}{dt} = \frac{1}{a\, n\,(1+e\cos E)} \left(R \sqrt{1-e^2} - S\, e \sin E\right) \tag{2.66}$$

From Equation (2.16)

$$R = -B a \rho V_R\, e \sin E \frac{dE}{dt} \tag{2.67}$$

$$S = -B a \rho V_R \sqrt{1-e^2} \left(1 - d \frac{(1-e\cos E)^2}{1-e^2}\right) \frac{dE}{dt} \tag{2.68}$$

Thus, substitution of R and S from (2.67), (2.68) and V_R from (2.17) yields,

$$\frac{d\gamma}{dt} = -\frac{B a \rho \sqrt{1-e^2}\, e \sin E}{1+e\cos E} \sqrt{\frac{1+e\cos E}{1-e\cos E}} \left(1 - d\frac{1-e\cos E}{1+e\cos E}\right) d \frac{(1-e\cos E)^2}{1-e^2} \frac{dE}{dt} \tag{2.69}$$

From Equation (2.35)

$$\frac{de}{dt} = -B a \rho (1-e^2) \sqrt{\frac{1+e\cos E}{1-e\cos E}} \left(1 - d\frac{1-e\cos E}{1+e\cos E}\right) \Big[2\cos E - \frac{d}{1-e^2}(1-e\cos E)(2\cos E - e - e\cos^2 E) \Big] \frac{dE}{dt} \tag{2.70}$$

When relations (2.69) and (2.70) are substituted for $d\gamma/dt$ and de/dt in Equation (2.61), it follows that,

$$-\left(\frac{d\theta}{dt}\right)^{*} = -\frac{B a \rho \sqrt{1-e^2} \sin E}{e \cos E} \sqrt{\frac{1+e\cos E}{1-e\cos E}} \left(1 - d\frac{1-e\cos E}{1+e\cos E}\right) \Big[2\cos E - \frac{d}{1-e^2}(1-e\cos E)(2\cos E - e - e\cos^2 E) - e d \frac{(1-e\cos E)^2}{1-e^2} \Big] \frac{dE}{dt} \tag{2.71}$$

$$-\left(\frac{d\theta}{dt}\right)^{*} = Ba\rho\,\frac{\sqrt{1-e^2}}{e}\,\sin E\,\sqrt{\frac{1+e\cos E}{1-e\cos E}}\left(1-d\,\frac{1-e\cos E}{1+e\cos E}\right)\left[2-\frac{d}{1-e^2}(1 - e\cos E)(2-e^2\cos E)\right]\frac{dE}{dt} \tag{2.72}$$

Since $(d\theta/dt)^*$ is an odd function of E (because of the factor sin E), the integration of this term over the interval $(0, 2\pi)$ to derive the secular perturbations will contribute nothing. Therefore, when this reasoning is applied to relation (2.56), it follows that,

$$\int_0^{2\pi}\left(\frac{d\omega}{dt}\right)_{R,S} = \int_0^{2\pi}-\left(\frac{d\theta}{dt}\right)^{*} = 0 \tag{2.73}$$

Hence, the only change in ω over a revolution is caused by the W-component of the perturbing acceleration and is given by Equation (2.55).

2.2.3.3 Determination of Atmospheric Density Allowing for Earth Flattening

Very accurate analytical approximations can be obtained by expressing the variation of the mass density, ρ, in the vicinity of perigee by the osculating exponential atmosphere,

$$\rho = \rho_p\, e^{-K(h-h_p)} \tag{2.74}$$

where ρ is the atmospheric density at altitude h above the earth's spheroidal surface, ρ_p is the atmospheric density at perigee, K = -d/dh $\log_e \rho$) is the inverse of the density scale height, and

$$h = a\,(1 - e\cos E) - R \tag{2.75}$$

$$h_p = a\,(1 - e) - R_p \tag{2.76}$$

The radius, R, of the spheroidal earth at any point, whose geocentric latitude is δ , is given by

$$R = \frac{R_{EQ}}{\sqrt{1 + \Lambda \sin^2\delta}} = R_{EQ}\left(1 - \frac{\Lambda}{2}\sin^2\delta\right) = R_{EQ}\left(1 - \frac{\Lambda}{2}\sin^2 i \sin^2 u\right) \tag{2.77}$$

where,

$$\Lambda = \frac{2f - f^2}{(1-f)^2} = 2f\,\frac{(1 - f/2)}{(1-f)^2} \sim 2f \tag{2.78}$$

and where f is the flattening of the Earth.

Thus, R and R_p are approximately

$$R = R_{EQ}\left(1 - f\sin^2 i \sin^2 u\right) \tag{2.79}$$

$$R_p = R_{EQ}\,(1 - f\,\sin^2 i\,\sin^2\omega) \tag{2.80}$$

In view of relations (2.75), (2.76), (2.79), and (2.80), it follows that,

$$K(h - h_p) = K a e\,(1 - \cos E) + K R_{EQ}\, f \sin^2 i\,(\sin^2 u - \sin^2\omega) \tag{2.81}$$

Defining,

$$C = K a e \tag{2.82}$$

and

$$Q = K R_{EQ}\, f \sin^2 i \tag{2.83}$$

equation (2.74) reduces to,

$$\rho = \rho_p\, e^{-C(1-\cos E)}\, e^{-Q(\sin^2 u - \sin^2\omega)} \tag{2.84}$$

where,

$$\sin^2 u - \sin^2\omega = (\sin u + \sin\omega)(\sin u - \sin\omega)$$

$$= \left(2\sin\frac{u+\omega}{2}\cos\frac{u-\omega}{2}\right)\left(2\cos\frac{u+\omega}{2}\sin\frac{u-\omega}{2}\right) = \sin(u+\omega)\sin(u-\omega)$$

$$= \cos 2\omega\,\sin^2\theta + \sin 2\omega\,\sin\theta\cos\theta = \cos 2\omega\,\sin^2\theta + \sin 2\omega\,\sin\theta\sqrt{1-\sin^2\theta}\,{}^{*}$$

* In the expansion which follows, the odd powers of sine θ are ignored because they contribute nothing to the integral $\int_0^{2\pi}$

Hence,

$$e^{-Q(\sin^2 u \sin^2\omega)} = 1 - Q\cos 2\omega \sin^2\theta + \frac{Q^2}{2}(\cos^2 2\omega \sin^4\theta$$
$$- \sin^2 2\omega \sin^4\theta + \sin^2 2\omega \sin^2\theta) - \frac{Q^3}{6}\ (\cos^3 2\omega \sin^6\theta$$
$$+ 3\cos 2\omega \sin^2 2\omega \sin^4\theta - 3\cos 2\omega \sin^2 2\omega \sin^6\theta)$$
$$+ \frac{Q^4}{24}(\cos^4 2\omega \sin^8\theta + \sin^4 2\omega \sin^8\theta - \sin^2 4\omega \sin^8\theta - 2\sin^2 2\omega \cos^2 2\omega \sin^8\theta \tag{2.85}$$
$$+ \sin^2 4\omega \sin^6\theta + 2\sin^2 2\omega \cos^2 2\omega \sin^6\theta - 2\sin^4 2\omega \sin^6\theta + \sin^4 2\omega \sin^4\theta)$$

Thus, collecting terms in powers of $\sin\Theta$ and setting $\sin\theta = \frac{\sqrt{1-e^2}\sin E}{1-e\cos E}$, one obtains

$$e^{-Q(\sin^2 u - \sin^2\omega)} = 1 + (1-e^2)(-Q\cos 2\omega + \frac{Q^2}{2}\sin^2 2\omega)\left[\frac{\sin E}{1-e\cos E}\right]^2$$
$$+ (1-e^2)^2\left(\frac{Q^2}{2}\cos 4\omega - \frac{Q^3}{2}\cos 2\omega \sin^2 2\omega + \frac{Q^4}{24}\sin^4 2\omega\left[\frac{\sin E}{1-e\cos E}\right]^4 +\right.$$
$$+ (1-e^2)^3\left(-\frac{Q^3}{6}\cos^3 2\omega + \frac{Q^3}{2}\cos 2\omega \sin^2 2\omega + \frac{Q^4}{4}\sin^2 2\omega \cos^2 2\omega\right.$$
$$\left.- \frac{Q^4}{12}\sin^4 2\omega\right)\left[\frac{\sin E}{1-e\cos E}\right]^6 + (1-e^2)^4\left(\frac{Q^4}{24}\cos^4 2\omega - \frac{Q^4}{16}\sin^2 4\omega\right.$$
$$\left.+ \frac{Q^4}{24}\sin^4 2\omega\right)\left[\frac{\sin E}{1-e\cos E}\right]^8 \tag{2.86}$$

Defining for convenience,

$$Q_1 = (1-e^2)\ (-Q\cos 2\omega + \frac{Q^2}{2}\sin^2 2\omega)$$
$$Q_2 = (1-e^2)^2\ \frac{Q^2}{2}\cos 4\omega\ \frac{Q^3}{2}\cos 2\omega \sin^2 2\omega + \frac{Q^4}{24}\sin^4 2\omega)$$
$$Q_3 = (1-e^2)^3\left(-\frac{Q^3}{6}\cos^3 2\omega + \frac{Q^3}{2}\cos 2\omega \sin^2 2\omega + \frac{Q^4}{4}\sin^2 2\omega \cos^2 2\omega - \frac{Q^4}{12}\sin^4 2\omega\right)$$
$$Q_4 = (1-e^2)^4\left(\frac{Q^4}{24}\cos^4 2\omega - \frac{Q^4}{16}\sin^2 4\omega + \frac{Q^4}{24}\sin^4 2\omega\right) \tag{2.87}$$

Equation (2.86) will assume the form,

$$e^{-Q(\sin^2 u - \sin^2 \omega)} = 1 + Q_1\left(\frac{\sin E}{1-e\cos E}\right)^2 + Q_2\left(\frac{\sin E}{1-e\cos E}\right)^4 + Q_3\left(\frac{\sin E}{1-e\cos E}\right)^6 + Q_4\left(\frac{\sin E}{1-e\cos E}\right)^8 \tag{2.88}$$

Substitution of (2.88) in Equation (2.84) yields,

$$\rho = \rho_p\, e^{-c(1-\cos E)}\left[1 + Q_1\left(\frac{\sin E}{1-e\cos E}\right)^2 + Q_2\left(\frac{\sin E}{1-e\cos E}\right)^4 + Q_3\left(\frac{\sin E}{1-e\cos E}\right)^6 + Q_4\left(\frac{\sin E}{1-e\cos E}\right)^8\right] \tag{2.89}$$

Please note that Sterne retains only the coefficients Q_1 and Q_2. His reasoning for doing so is that for close Earth satellites $Q = K\,R_{EQ}\, f \sin^2 i < 0.2$, and hence the powers of $Q > 2$ (which appear in the coefficients Q_3 and Q_4) are small, and the error incurred will be only about 0.16 percent. This observation is quite true for orbital altitudes of about 200 n.mi., and the higher the altitude, the smaller will be the error. However, at 100 n.mi., $Q \sim 0.5$ and not 0.2. Thus, the error resulting from the deletion of the coefficients Q_3 and Q_4 will be about 3.4 percent. Hence, for orbital altitudes below 200 n.mi., terms through Q_4 should be retained.

Sterne introduces a new variable in the derivation of the asymptotic solutions and retains only the coefficients Q_1 and Q_2,

$$y^2 = c\,(1-\cos E)^{**} \tag{2.90}$$

$$\frac{\sin^2 E}{(1-e\cos E)^2} = \frac{(1-\cos E)(1+\cos E)}{[(1-e)+e(1-\cos E)]^2} = \frac{\frac{y^2}{c}\left(2-\frac{y^2}{c}\right)}{(1-e)^2\left(1+\frac{e}{1-e}\frac{y^2}{c}\right)^2}$$

$$= \frac{2}{(1-e)^2}\,\frac{y^2}{c} - \frac{1}{(1-e)^2}\left(\frac{4e}{1-e}+1\right)\frac{y^4}{c^2}$$

$$\frac{\sin^4 E}{(1-e\cos E)^4} = \left[\frac{2}{(1-e)^2}\,\frac{y^2}{c} - \frac{1}{(1-e)^2}\left(\frac{4e}{1-e}+1\right)\frac{y^4}{c^2}\right]^2 = \frac{4}{(1-e)^4}\,\frac{y^4}{c^2} + \cdots \tag{2.91}$$

Substitution of Equation (2.90), (2.91) in relation (2.89) yields,

$$\rho = \rho_p\, e^{-y^2}\left[1 + 2\frac{Q_1}{(1-e)^2}\,\frac{y^2}{c} - \frac{Q_1}{(1-e)^2}\left(\frac{4e}{1-e}+1\right)\frac{y^4}{c^2} + 4\frac{Q_2}{(1-e)^4}\,\frac{y^4}{c^2}\right] \tag{2.92}$$

** $c = Kae$

2.2.3.4 The Average Secular Rates of the Orbital Elements

The time derivatives of the orbital elements a, e, Ω, i, ω, were derived previously and are given by Equations (2.22), (2.35), (2.41), (2.50) and (2.55), respectively. The average secular** rates of the orbital elements are obtained by integrating the respective time derivatives over one orbital period P and dividing by P. These average secular rates per unit time will be denoted by a dot and the subscript "sec."

$$\dot{a}_{sec} = -\frac{2Ba^2}{P}\int_0^{2\pi} \rho \frac{(1+e\cos E)^2}{\sqrt{1-e^2\cos^2 E}}\left(1-d\,\frac{1-e\cos E}{1+e\cos E}\right)^2 dE \tag{2.93}$$

$$\dot{e}_{sec} = -\frac{2Ba(1-e)^2}{P}\int_0^{2\pi}\rho\sqrt{\frac{1+e\cos E}{1-e\cos E}}\left(1-d\,\frac{1-e\cos E}{1+e\cos E}\right)\Big[\cos E - \frac{d}{2(1-e^2)}(1-e\cos E)(2\cos E-e-e\cos^2 E)\Big]\,dE \tag{2.94}$$

$$\dot{\Omega}_{sec} = -\frac{Ba\,\Omega_e\sin 2\omega}{2P\pi\sqrt{1-e^2}}\int_0^{2\pi}\rho\sqrt{1-e^2\cos^2 E}\left(1-d\,\frac{1-e\cos E}{1+e\cos E}\right)\Big[-(1-2e^2) - 2e\cos E+(2-e^2)\cos^2 E\Big]\,dE \tag{2.95}$$

$$\dot{i}_{sec} = -\frac{Ba\,\Omega_e\sin i}{2P\pi\sqrt{1-e^2}}\int_0^{2\pi}\rho(1-e\cos E)^2\sqrt{1-e^2\cos^2 E}\left(1-d\,\frac{1-e\cos E}{1+e\cos E}\right)\Big[1 + \cos 2\omega\,\frac{(2e^2-1)-2e\cos E+(2-e^2)\cos^2 E}{(1-e\cos E)^2}\Big]\,dE \tag{2.96}$$

$$\dot{\omega}_{sec} = -\cos i\,\dot{\Omega}_{sec} \tag{2.97}$$

$$\dot{P} = \frac{3}{2}P\left(\frac{\dot{a}_{sec}}{a}\right) \tag{2.98}$$

** "Secular" means monotonically increasing with time

2.2.3.5 Integration of the Time Rates of the Orbital Elements - Asymptotic Solutions

The asymptotic solutions apply only to eccentric orbits for which $c > 3$. The integration of the time rates of change is performed over a revolution of the satellite. Subsequent division by the orbital period yields the average rate of change in each orbital element throughout the revolution. It should be noted that the orbital elements and the parameters d, c, and Q_1, appearing in the integrands of the respective integrals defining the changes per revolution are considered constant during the interval of variation so as to make possible the integration in closed form.

$$\dot{a}_{SEC} = -\frac{2Ba^2}{P}\int_0^{2\pi} \rho \frac{(1+e\cos E)^2}{\sqrt{1-e^2\cos^2 E}}\left(1-d\,\frac{1-e\cos E}{1+e\cos E}\right)^2 dE \qquad (2.99)$$

$$\dot{a}_{SEC} = -\frac{4Ba^2}{P}\int_0^{\pi} \rho \frac{\left[(1-d)+(1+d)e\cos E\right]^2}{\sqrt{1-e^2\cos^2 E}}\, dE \qquad (2.100)$$

Sterne introduces the new variable,

$$Y^2 = c\,(1-\cos E)$$

where

$$c = Kae$$

so that,

$$\begin{aligned}(1-d)+(1+d)\,e\cos E &= (1-d)+(1+d)e-(1+d)e\frac{Y^2}{c}\\ &= \left[(1-d)+(1-d)e\right]\left[1-\frac{1+d}{(1-d)+(1+d)e}\,e\,\frac{Y^2}{c}\right]\end{aligned} \qquad (2.101)$$

or, if k is defined as

$$k = \frac{1+d}{(1-d)+(1+d)e} \qquad (2.102)$$

$$\left[(1-d)+(1+d)e\cos E\right]^2 = \left[(1-d)+(1+d)e\right]^2\left(1-ke\frac{Y^2}{c}\right)^2 \qquad (2.103)$$

Likewise,

$$1-e^2\cos^2 E = 1-e^2\left(1-\frac{Y^2}{c}\right)^2$$
$$= (1-e)^2\left[1+\frac{2e^2}{1-e^2}\frac{Y^2}{c}-\frac{e^2}{1-e^2}\frac{Y^4}{c^2}\right] \tag{2.104}$$

Finally,

$$dE = \frac{-\frac{2}{c}Y\,dY}{-\sqrt{1-(1-\frac{Y^2}{c})^2}} = \frac{\frac{2}{c}Y\,dY}{\sqrt{2\frac{Y^2}{c}-\frac{Y^4}{c^2}}} = \frac{\frac{2}{c}Y\,dY}{\sqrt{\frac{2}{c}}\,Y\sqrt{1-\frac{Y^2}{2c}}} = \sqrt{\frac{2}{c}}\frac{dY}{\sqrt{1-\frac{Y^2}{2c}}} \tag{2.105}$$

Combining (2.103), (2.104) and (2.105) yields,

$$\frac{[(1-d)+(1+d)e\cos E]^2}{\sqrt{1-e^2\cos^2 E}}dE = \Lambda\sqrt{\frac{2}{c}}\frac{(1-2ke\frac{Y^2}{c}+k^2e^2\frac{Y^4}{c^2})dY}{\sqrt{1+\frac{2e^2}{1-e^2}\frac{Y^2}{c}-\frac{e^2}{1-e^2}\frac{Y^4}{c^2}}\sqrt{1-\frac{Y^2}{2c}}} \tag{2.106}$$

where

$$\Lambda = \frac{[(1-d)+(1+d)e]^2}{\sqrt{1-e^2}} = \frac{(1+e)^2}{\sqrt{1-e^2}}\left(1-d\,\frac{1-e}{1+e}\right)^2 \tag{2.107}$$

Substitution of Equation (2.106) in (2.100) yields, after manipulation in the denominator,

$$\dot{u}_{sec} = -\frac{4Ba^2\Lambda}{p}\sqrt{\frac{2}{c}}\int_0^{\sqrt{2c}}\rho\frac{(1-2\frac{ke}{c}y^2+\frac{k^2e^2}{c^2}y^4)\,dy}{\sqrt{1+\frac{1}{2c}(\frac{4e^2}{1-e^2}-1)y^2-\frac{2}{c^2}(\frac{e^2}{1-e^2})y^4+\frac{1}{2c^3}(\frac{e^2}{1-e^2})y^6}} \tag{2.108}$$

Now, expanding the denominator in powers of y^2, ignoring powers > 4, the integrand becomes, except for the factor ρ,

$$\left[1-\frac{1}{4c}\left(\frac{4e^2}{1-e^2}-1\right)Y^2+\frac{1}{c^2}\left(\frac{e^2}{1-e^2}\right)Y^4+\frac{3}{32c^2}\left(\frac{4e^2}{1-e^2}-1\right)Y^4\right]\left[1\right.$$
$$\left.-2\frac{ke}{c}Y^2+\frac{k^2e^2}{c^2}Y^4\right] = 1+\frac{1}{4c}\left(1-8ke-\frac{4e^2}{1-e^2}\right)Y^2$$
$$+\frac{3}{32c^2}\left[1+\frac{8}{3}e^2\frac{(1+5e^2)}{(1-e^2)^2}-\frac{16}{3}ke\left(\frac{1-5e^2}{1-e^2}\right)+\frac{32}{3}k^2e^2\right]Y^4$$
$$= 1+\frac{1}{4c}f_1Y^2+\frac{3}{32c^2}f_2Y^4 \tag{2.109}$$

where,

$$f_1 = 1 - 8ke - \frac{4e^2}{1-e^2} \qquad (2.110)$$

$$f_2 = 1 + \frac{8}{3} e^2 \frac{(1+5e^2)}{(1-e^2)^2} - \frac{16}{3} k e \left(\frac{1-5e^2}{1-e^2}\right) + \frac{32}{3} k^2 e^2 \qquad (2.111)$$

Thus, substituting the result of multiplication of (2.109) in Equation (2.108) yields,

$$\dot{a}_{SEC} = -\frac{4 B a^2 \Lambda}{p} \sqrt{\frac{2}{C}} \int_0^{\sqrt{2C}} \rho \left(1 + \frac{f_1}{4C} Y^2 + \frac{3 f_2}{32 C^2} Y^4\right) dY \qquad (2.112)$$

But, from Equation (2.92),

$$\rho = \rho_p e^{-Y^2} \left[1 + \frac{2}{c} \frac{Q_1}{(1-e)^2} Y^2 - \frac{1}{c^2} \frac{Q_1}{(1-e)^2} \left(\frac{4e}{1-e} + 1\right) Y^4 + \frac{4}{c^2} \frac{Q_2}{(1-e)^2} Y^4\right] \qquad (2.113)$$

Replacing ρ in the integrand of Equation (2.112) by relation (2.113), and multiplying, will yield the following new integrand, excluding the factor $e^{-y^2} dy$,

$$1 + \frac{1}{4C} \left[f_1 + \frac{8Q_1}{(1-e)^2}\right] Y^2$$
$$+ \frac{3}{32C^2} \left[f_2 - \frac{16}{3} \frac{Q_1}{(1-e)^2} \left(2 + \frac{8e}{1-e} - f_1\right) + \frac{128}{3} \frac{Q_2}{(1-e)^4}\right] Y^4 \qquad (2.114)$$

For the sake of simplicity, the following definitions are introduced,

$$F_1 = f_1 + \frac{8Q_1}{(1-e)^2} \qquad (2.115)$$

$$F_2 = f_2 - \frac{16}{3} \frac{Q_1}{(1-e)^2} \left(2 + \frac{8e}{1-e} - f_1\right) + \frac{128}{3} \frac{Q_2}{(1-e)^4} \qquad (2.116)$$

These definitions reduce the integrand given by Equation (2.114) except for the factor $e^{-y^2} dy$, to

$$1 + \frac{F_1}{4c} Y^2 + \frac{3F_2}{32c^2} Y^4 \tag{2.117}$$

So that Equation (2.112) becomes,

$$\dot{a}_{sec} = -\frac{4Ba^2\Lambda}{P} \sqrt{\frac{2}{c}}\, \rho_p \int_0^{\sqrt{2c}} e^{-Y^2}\left(1 + \frac{F_1}{4c} Y^2 + \frac{3F_2}{32c^2} Y^4\right) dY \tag{2.118}$$

The asymptotic solutions are based on the assumption that when $c > 3$, the upper limit $\sqrt{2c}$ of the integral may be approximated by ∞ without a great loss in accuracy. This assumption is reasonably valid since, for large values of y^2, the order of magnitude of e^{-y^2} is much lower than that of a polynomial in (y^2/c).

Thus, making this approximation,

$$\dot{a}_{sec} \sim -\left(\frac{4Ba^2\Lambda}{P}\right) \sqrt{\frac{2}{c}}\, \rho_p \int_0^{\infty} e^{-Y^2}\left(1 + \frac{F_1}{4c} Y^2 + \frac{3F_2}{32c^2} Y^4\right) dY \tag{2.119}$$

Now, since the integrals introduced can be evaluated as,

$$\int_0^{\infty} e^{-Y^2} dY = \frac{\sqrt{\pi}}{2}$$

$$\int_0^{\infty} e^{-Y^2} Y^2 dY = \frac{1}{2} \frac{\sqrt{\pi}}{2}$$

$$\int_0^{\infty} e^{-Y^2} Y^4 dY = \frac{3}{4} \frac{\sqrt{\pi}}{2} \tag{2.120}$$

The average secular rate of a can be written as

$$\dot{a}_{sec} \sim -\left(\frac{2Ba^2\Lambda}{P}\right) \rho_p \sqrt{\frac{2\pi}{c}} \left[1 + \frac{F_1}{8c} + \frac{9F_2}{128c^2} + \cdots\right] \tag{2.121}$$

The change in the period follows directly from $\dot{a}_{sec}$,

$$\dot{P}_{SEC} = \frac{3}{2} P \left(\frac{\dot{a}_{SEC}}{a}\right) \sim -3Ba\Lambda\rho_p \sqrt{\frac{2\pi}{c}} \left[1 + \frac{F_1}{8C} + \frac{9F_2}{128C^2}\right] \tag{2.122}$$

Similarly,

$$\dot{e}_{SEC} = -\frac{2Ba(1-e^2)}{P} \int_0^{2\pi} \rho \frac{(1+e\cos E)}{\sqrt{1-e^2\cos^2 E}} \left(1 - d\,\frac{1-e\cos E}{1+e\cos E}\right) \left[\cos E - \frac{d}{2(1-e^2)} (1-e\cos E)(2\cos E - e - e\cos^2 E)\right] dE \tag{2.123}$$

$$\dot{e}_{SEC} = -\frac{4Ba(1-e^2)}{P} \int_0^{\pi} \rho \frac{(1-d)+(1+d)e\cos E}{\sqrt{1-e^2\cos^2 E}} \left[\cos E - \frac{d}{2(1-e^2)} (1-e\cos E)(2\cos E - e - e\cos^2 E)\right] dE \tag{2.124}$$

Introducing the new variable y^2, as defined in Equation (2.90) yields,

$$\begin{aligned}
\cos E &- \frac{d}{2(1-e^2)} (1-e\cos E)(2\cos E - e - e\cos^2 E) = 1 - (1-\cos E) \\
&- \left(\frac{1-e}{1+e}\right) d \left[1 + \frac{e}{1-e}(1-\cos E)\right] \left[1 - (1-\cos E) - e\,\frac{(1-\cos E)^2}{2(1-e)}\right] \\
&= 1 - \frac{y^2}{c} - \left(\frac{1-e}{1+e}\right) d \left[1 + \frac{e}{1-e}\,\frac{y^2}{c}\right] \left[1 - \frac{y^2}{c} - \frac{e}{2(1-e)}\,\frac{y^4}{c^2}\right] \\
&= 1 - \frac{y^2}{c} - \left(\frac{1-e}{1+e}\right) d \left[1 - \frac{1-2e}{1-e}\,\frac{y^2}{c} - \frac{3}{2}\,\frac{e}{1-e}\,\frac{y^4}{c^2}\right] = \left(1 - \frac{1-e}{1+e} d\right) \\
&- \left[\frac{(1-d)+(1+d)e}{1+e} + \frac{ed}{1+e}\right] \frac{y^2}{c} + \frac{3}{2}\,\frac{ed}{1+e}\,\frac{y^4}{c^2} \\
&= \left(1 - \frac{1-e}{1+e} d\right) \left[1 - \left(1 + \frac{ed}{(1-d)+(1+d)e}\right) \frac{y^2}{c} + \frac{3}{2}\,\frac{ed}{(1-d)+(1+d)e}\,\frac{y^4}{c^2}\right]
\end{aligned} \tag{2.125}$$

Likewise,

$$\begin{aligned}
(1-d) + (1+d)e\cos E &= \left[(1-d) + (1+d)e\right] \left[1 - \frac{1+d}{(1-d)+(1+d)e}\, e\,\frac{y^2}{c}\right] \\
&= \left[(1-d) + (1+d)e\right] \left(1 - ke\,\frac{y^2}{c}\right)
\end{aligned} \tag{2.126}$$

where,

$$k = \frac{1+d}{(1-d)+(1+d)e} \tag{2.127}$$

Finally,

$$1 - e^2\cos^2 E = 1-e^2\left(1-\frac{Y^2}{C}\right)^2 = (1-e^2)\left[1+\frac{2e^2}{1-e^2}\frac{Y^2}{C}-\frac{e^2}{1-e^2}\frac{Y^4}{C^2}\right] \tag{2.128}$$

$$dE = \sqrt{\frac{2}{C}}\;\frac{dY}{\sqrt{1-\frac{Y^2}{2C}}} \tag{2.129}$$

For the sake of convenience, the polynomial of (2.125) will be designated by P(E), and the coefficients of the transformed P(E) polynomial (resulting from the change of variable) will be denoted as follows:

$$r = 4\left(1+\frac{ed}{(1-d)+(1+d)e}\right)$$

$$s = \frac{16\,ed}{(1-d)+(1+d)e}$$

Using this notation, the polynomial (2.125) will assume the simplified form,

$$P(E) = \left(1-\frac{1-e}{1+e}\,d\right)\left(1-\frac{r}{4C}\,Y^2+\frac{3s}{32C^2}\,Y^4\right) \tag{2.130}$$

Combining relations (2.126), (2.128), (2.129), (2.130) yields the integrand of (2.124), except for ρ,

$$\frac{(1-d)+(1+d)e\cos E}{\sqrt{1-e^2\cos^2 E}}\,P(E)\,dE = \Lambda^*\sqrt{\frac{2}{C}}$$

$$\frac{\left(1-\frac{r}{4C}Y^2+\frac{3s}{32C^2}Y^4\right)\left(1-ke\frac{Y^2}{C}\right)}{\sqrt{1+\frac{2e^2}{1-e^2}\frac{Y^2}{C}-\frac{e^2}{1-e^2}\frac{Y^4}{C^2}}\sqrt{1-\frac{Y^2}{2C}}}\,dY \tag{2.131}$$

where

$$\Lambda^* = \frac{(1-d)+(1+d)e}{\sqrt{1-e^2}}\left(1-\frac{1-e}{1+e}\,d\right) = \sqrt{\frac{1+e}{1-e}}\left(1-\frac{1-e}{1-e}d\right)^2 \tag{2.132}$$

Substitution of Equation (2.131) in (2.124) yields, after the multiplication in the denominator,

$$\dot{e}_{SEC} = -\frac{4Ba(1-e^2)\Lambda^*}{P}\sqrt{\frac{2}{c}}\int_0^{\sqrt{2c}} \rho \frac{\left(1-\frac{r}{4c}Y^2+\frac{3s}{32c^2}Y^4\right)\left(1-\frac{ke}{c}Y^2\right)dY}{\sqrt{1+\frac{1}{2c}\left(\frac{4e^2}{1-e^2}-1\right)Y^2-\frac{2}{c^2}\left(\frac{e^2}{1-e^2}\right)Y^4+\frac{1}{2c^3}\left(\frac{e^2}{1-e^2}\right)Y^6}} \tag{2.133}$$

As before, expanding the denominator in powers of Y^2, and ignoring powers > 4, the integrand becomes, except for the factor ρ ,

$$\left[1-\frac{1}{4c}\left(\frac{4e^2}{1-e^2}-1\right)Y^2+\frac{1}{c^2}\left(\frac{e^2}{1-e^2}\right)Y^4+\frac{3}{32c^2}\left(\frac{4e^2}{1-e^2}-1\right)^2Y^4\right]\left[1-\frac{ke}{c}Y^2\right]\left[1-\frac{r}{4c}Y^2\right.$$
$$\left.+\frac{3s}{32c^2}Y^4\right] = \left[1+\frac{1}{4c}\left(1-\frac{4e^2}{1-e^2}-4ke\right)Y^2+\frac{3}{32c^2}\left\{1+\frac{8}{3}\left(\frac{e^2}{1-e^2}\right)\right.\right.$$
$$\left.\left.-\frac{8}{3}ke\left(1-\frac{4e^2}{1-e^2}\right)\right\}Y^4\right]\left[1-\frac{r}{4c}Y^2+\frac{3s}{32c^2}Y^4\right]$$
$$= 1+\frac{1}{4c}\left(1-\frac{4e^2}{1-e^2}-4ke-r\right)Y^2+\frac{3}{32c^2}\left[1+\frac{8}{3}\left(\frac{e^2}{1-e^2}\right)\right.$$
$$\left.-\frac{8}{3}ke\left(1-\frac{4e^2}{1-e^2}\right)-\frac{2}{3}\left(1-\frac{4e^2}{1-e^2}-4ke\right)r+s\;Y^4\right]$$
$$= 1+\frac{f_1^*}{4c}Y^2+\frac{3f_2^*}{32c^2}Y^4 \tag{2.134}$$

where, after replacing the parameters r and s by their respective values,

$$f_1^* = -\left(3+4ke+\frac{4e^2}{1-e^2}+\frac{4ed}{(1-d)+(1+d)e}\right) \tag{2.135}$$

$$f_2^* = 1+\frac{8}{3}\left(\frac{e^2}{1-e^2}\right)-\frac{8}{3}ke\left(1-\frac{4e^2}{1-e^2}\right)+\frac{16ed}{(1-d)+(1+d)e}$$
$$-\frac{8}{3}\left(1+\frac{ed}{(1-d)+(1+d)e}\right)\left(1-\frac{4e^2}{1-e^2}-4ke\right) \tag{2.136}$$

Substituting the result of the multiplication of (2.134) in Equation (2.133) yields,

$$\dot{e}_{SEC} = -\frac{4Ba(1-e^2)\Lambda^*}{P}\sqrt{\frac{2}{c}}\int_0^{\sqrt{2c}}\rho\left(1+\frac{f_1^*}{4c}Y^2+\frac{3f_2^*}{32c^2}Y^4\right)dY \tag{2.137}$$

where, from Equation (2.92),

$$\rho = \rho_p e^{-Y^2}\left[1 + \frac{2}{c}\frac{Q_1}{(1-e)^2}Y^2 - \frac{1}{c^2}\frac{Q_1}{(1-e)^2}\left(\frac{4e}{1-e}+1\right)Y^4 + \frac{4}{c^2}\frac{Q_2}{(1-e)^4}Y^4\right] \quad (2.138)$$

Replacing ρ in the integrand of Equation (2.137) by relation (2.138), and multiplying, the new integrand, except for $e^{-y^2}dy$, becomes,

$$1 + \frac{1}{4c}\left(f_1^* + \frac{8Q_1}{(1-e)^2}\right)Y^2 + \frac{3}{32c^2}\left[f_2^* - \frac{16}{3}\frac{Q_1}{(1-e)^2}\left(2+\frac{8e}{1-e}-f_1^*\right)\right.$$
$$\left. + \frac{128}{3}\frac{Q_2}{(1-e)^4}\right]Y^4 \quad (2.139)$$

Defining for convenience,

$$F_1^* = f_1^* + \frac{8Q_1}{(1-e)^2} \quad (2.140)$$

$$F_2^* = f_2^* - \frac{16}{3}\frac{Q_1}{(1-e)^2}\left(2+\frac{8e}{1-e}-f_1^*\right) + \frac{128}{3}\frac{Q_2}{(1-e)^4} \quad (2.141)$$

the integrand given by Equation (2.139) becomes, except for the factor $e^{-y^2}dy$,

$$1 + \frac{F_1^*}{4c}Y^2 + \frac{3F_2^*}{32c^2}Y^4 \quad (2.142)$$

Thus,

$$\dot{e}_{SEC} = -\frac{4B_a(1-e^2)\overset{*}{\Lambda}\pi}{p\pi}\sqrt{\frac{2}{c}}\,\rho_p\int_0^{\sqrt{2c}} e^{-Y^2}\left(1+\frac{F_1^*}{4c}Y^2+\frac{3F_2^*}{32c^2}Y^4\right)dY \quad (2.143)$$

Again, as before, replacing the upper limit of the integral by ∞ , yields,

$$\dot{e}_{SEC} \sim -\left(\frac{4B_a(1-e^2)\overset{*}{\Lambda}}{p}\right)\sqrt{\frac{2}{c}}\,\rho_p\int_0^{\infty} e^{-Y^2}\left(1+\frac{F_1}{4c}Y^2+\frac{3F_2^*}{32c^2}Y^4\right)dY \quad (2.144)$$

The asymptotic solution is obtained by using relations (2.120),

$$\dot{e}_{SEC} \sim -\left(\frac{2Ba(1-e^2)\Lambda^*}{P}\right)\rho_p\sqrt{\frac{2\pi}{c}}\left(1+\frac{F_1^*}{8c}+\frac{9F_2^*}{128c^2}\right) \tag{2.145}$$

The secular rate of the nodal longitude, Ω , can be derived in the same manner, from (2.95),

$$\dot{\Omega}_{SEC} = -\frac{Ba\,\Omega_e \sin 2\omega}{2P\pi\sqrt{1-e^2}}\int_0^{2\pi}\rho\sqrt{1-e^2\cos^2E}\left(1-d\,\frac{1-e\cos E}{1+e\cos E}\right)\Big[2e^2-1 - 2e\cos E + (2-e^2)\cos^2E\Big]dE \tag{2.146}$$

$$\dot{\Omega}_{SEC} = -\frac{Ba\,\Omega_e \sin 2\omega}{P\pi\sqrt{1-e^2}}\int_0^{\pi}\rho\sqrt{1-e^2\cos^2E}\;\frac{[(1-d)+(1+d)e\cos E]}{1+e\cos E}\Big[2e^2-1 - 2e\cos E + (2-e^2)\cos^2E\Big]dE \tag{2.147}$$

Introducing the new variable y^2, as defined by Equation (2.90) yields,

$$\begin{aligned}2e^2-1-2e\cos E+(2-e^2)\cos^2E &= (1-e)^2\left[1-2\left(\frac{2+e}{1-e}\right)(1-\cos E) + \frac{2-e^2}{(1-e)^2}(1-\cos E)^2\right] \\ &= (1-e)^2\left[1-2\left(\frac{2+e}{1-e}\right)\frac{y^2}{c}+\frac{2-e^2}{(1-e)^2}\frac{y^4}{c^2}\right]\end{aligned} \tag{2.148}$$

Likewise,

$$\begin{aligned}(1-d)+(1+d)e\cos E &= \Big[(1-d)+(1+d)e\Big]\left[1-\frac{1+d}{(1+d)+(1-d)e}\,e\,\frac{y^2}{c}\right] \\ &= \Big[(1-d)+(1+d)e\Big]\left(1-ke\,\frac{y^2}{c}\right)\end{aligned} \tag{2.149}$$

$$\frac{1}{1+e\cos E} = \frac{1}{(1+e)\left[1-\frac{e}{1+e}\frac{y^2}{c}\right]} = \frac{1}{1+e}\left[1+\left(\frac{e}{1+e}\right)\frac{y^2}{c} + \left(\frac{e}{1+e}\right)^2\frac{y^4}{c^2}\right] \tag{2.150}$$

So that the product of (2.149) and (2.150) becomes,

$$\frac{(1-d)+(1+d)e\cos E}{1+e\cos E} = \left(1-d\,\frac{1-e}{1+e}\right)\left(1-ke\,\frac{Y^2}{c}\right)\left[1+\frac{e}{1+e}\,\frac{Y^2}{c}+\left(\frac{e}{1+e}\right)^2\frac{Y^4}{c^2}\right]$$

$$= \left(1-d\,\frac{1-e}{1+e}\right)\left[1+\left(\frac{e}{1+e}-ke\right)\frac{Y^2}{c}+\left(\frac{e}{1+e}\right)\left(\frac{e}{1+e}-ke\right)\frac{Y^4}{c^2}\right] \quad (2.151)$$

Similarly,

$$1-e^2\cos^2 E = 1-e^2\left(1-\frac{Y^2}{c}\right)^2 = (1-e^2)\left[1+\frac{2e^2}{1-e^2}\,\frac{Y^2}{c}-\frac{e^2}{1-e^2}\,\frac{Y^4}{c^2}\right] \quad (2.152)$$

$$dE = \sqrt{\frac{2}{c}}\,\frac{dY}{\sqrt{1-\frac{Y^2}{2c}}} = \sqrt{\frac{2}{c}}\,\sqrt{1+\frac{Y^2}{2c}+\frac{Y^4}{4c^2}}\,dY \quad (2.153)$$

So that,

$$\sqrt{1-e^2\cos^2 E}\,dE = \sqrt{\frac{2}{c}}\,\sqrt{1-e^2}\,\sqrt{1+\frac{1}{2}\left(1+\frac{4e^2}{1-e^2}\right)\frac{Y^2}{c}+\frac{Y^4}{4c^2}}$$

$$= \sqrt{\frac{2}{c}}\,\sqrt{1-e^2}\left[1+\frac{1}{4}\left(1+\frac{4e^2}{1-e^2}\right)\frac{Y^2}{c}+\frac{1}{32}\left\{\left(3-\frac{8e^2}{1-e^2}\right)-\left(\frac{4e^2}{1-e^2}\right)^2\right\}\frac{Y^4}{c^2}\right]dY \quad (2.154)$$

Multiplication of relations (2.151) and (2.154) yields,

$$\sqrt{1-e^2\cos^2 E}\;\frac{[(1-d)+(1+d)e\cos E]}{1+e\cos E}\,dE$$

$$= \sqrt{\frac{2}{c}}\,\sqrt{1-e^2}\left(1-d\,\frac{1-e}{1+e}\right)\left\{1+\frac{1}{4c}\left[\left(1+\frac{4e^2}{1-e^2}\right)+4\left(\frac{e}{1+e}-ke\right)\right]Y^2\right.$$

$$\left.+\frac{1}{32c^2}\left[8\left(\frac{e}{1+e}-ke\right)\left(1+\frac{4e}{1+e}+\frac{4e^2}{1-e^2}\right)+\left(3-\frac{8e^2}{1-e^2}\right)-\left(\frac{4e^2}{1-e^2}\right)^2\right]Y^4\right\}dY \quad (2.155)$$

Finally, multiplication of (2.155) by (2.148) yields the integrand of (2.147), except for the factor ρ ,

$$\sqrt{1-e^2\cos^2 E}\;\frac{[(1-d)+(1+d)e\cos E]}{1+e\cos E}\left[2e^2-1-2e\cos E+(2-e^2)\cos^2 E\right]dE$$

$$= \sqrt{\frac{2}{c}}\,\sqrt{1-e^2}\left(1-d\,\frac{1-e}{1+e}\right)(1-e)^2\left\{1-\frac{1}{4c}\left[8\left(\frac{2+e}{1-e}\right)-4\left(\frac{e}{1+e}-ke\right)\right.\right.$$

$$- \left(1+ \frac{4e^2}{1-e^2}\right)\Big] Y^2 + \frac{1}{32C^2}\Big[8\left(\frac{e}{1+e} - ke\right)\left(1+\frac{4e}{1+e}+\frac{4e^2}{1-e^2}\right)$$

$$+ \left(3 - \frac{8e^2}{1-e^2}\right) - \left(\frac{4e^2}{1-e^2}\right)^2 - 16\left(\frac{2+e}{1-e}\right)\left(1+\frac{4e^2}{1-e^2}+4\left(\frac{e}{1+e}-ke\right)\right)$$

$$+ \frac{32(2-e^2)}{(1-e)^2}\Big] Y^4 \Big\} dy = \sqrt{\frac{2}{c}}\sqrt{1-e^2}\left(1-d\,\frac{1-e}{1+e}\right)(1-e)^2 \Big\{1 -$$

$$- \frac{1}{4c}\left[15 + \frac{4e(5+6e)}{1-e^2} + 4ke\right]Y^2 + \frac{1}{32c^2}\Big[8\left(\frac{e}{1+e} - ke\right)\left(1+\frac{4e}{1-e^2}\right.$$

$$\left. - 8\,\frac{2+e}{1-e}\right) - 16\left(\frac{2+e}{1-e}\right)\left(1+\frac{4e^2}{1-e^2}\right) + \left(3 - \frac{8e^2}{1-e^2}\right) - \left(\frac{4e^2}{1-e^2}\right)^2$$

$$+ \frac{32(2-e^2)}{(1-e)^2}\Big] Y^4 \Big\} dY \qquad (2.156)$$

Defining for convenience,

$$f_1^{**} = -\left[15 + \frac{4e(5+6e)}{1-e^2} + 4ke\right] \qquad (2.157)$$

$$f_2^{**} = 8\left(\frac{e}{1+e} - ke\right)\left(1+\frac{4e}{1-e^2} - 8\,\frac{2+e}{1-e}\right) - 16\left(\frac{2+e}{1-e}\right)\left(1+\frac{4e^2}{1-e^2}\right)$$

$$+ \left(3 - \frac{8e^2}{1-e^2}\right) - \left(\frac{4e^2}{1-e^2}\right)^2 + 32\,\frac{(2-e^2)}{(1-e)^2} \qquad (2.158)$$

where,

$$k = \frac{1+d}{(1-d)+(1+d)e} \qquad (2.159)$$

the integrand of Equation (2.147) becomes, except for the factor ρ,

$$\sqrt{\frac{2}{c}}\sqrt{1-e^2}\left(1-d\,\frac{1-e}{1+e}\right)(1-e)^2\left[1+\frac{f_1^{**}}{4C}Y^2 + \frac{f_2^{**}}{32C^2}Y^4\right]dY \qquad (2.160)$$

Substitution of relation (2.160) for the integrand in Equation (2.147) yields,

$$\dot{\Omega}_{sec} = -\frac{B_a\,\Omega_e\,\mathrm{SIN}\,2\omega}{P\,n}\left(1-d\,\frac{1-e}{1+e}\right)(1-e)^2\sqrt{\frac{2}{c}}\int_0^{\sqrt{2c}}\rho\left[1+\frac{f_1^{**}}{4c}Y^2 + \frac{f_2^{**}}{32c^2}Y^4\right]dY \qquad (2.161)$$

* Sterne has an error in his expression for f_1^{**}

Now, from Equation (2.92),

$$\rho = \rho_P\, e^{-y^2}\left[1 + \frac{2}{C}\,\frac{Q_1}{(1-e)^2}\,y^2 - \frac{1}{C^2}\,\frac{Q_1}{(1-e)^2}\left(\frac{4e}{1-e}+1\right)y^4 + \frac{4}{C^2}\,\frac{Q_2}{(1-e)^4}\,y^4\right] \quad (2.162)$$

Replacing ρ in the integrand of Equation (2.161) by relation (2.162), and multiplying, the new integrand, except for $e^{-y^2}dy$, becomes

$$1 + \frac{1}{4C}\left(f_1^{**} + \frac{8Q_1}{(1-e^2)}\right)y^2 + \frac{1}{32C^2}\left[f_2^{**} - \frac{16Q_1}{(1-e)^2}\left(2+\frac{8e}{1-e}-f_1^{**}\right) + \frac{128Q_2}{(1-e)^4}\right]y^4 \quad (2.163)$$

Finally, defining

$$F_1^{**} = f_1^{**} + \frac{8Q_1}{(1-e)^2} \quad (2.164)$$

$$F_2^{**} = f_2^{**} - \frac{16Q_1}{(1-e)^2}\left(2+\frac{8e}{1-e}-f_1^{**}\right) + \frac{128Q_2}{(1-e)^4} \quad (2.165)$$

the integrand given by Equation (2.163) becomes, except for the factor $e^{-y^2}dy$,

$$1 + \frac{F_1^{**}}{4C}\,y^2 + \frac{F_2^{**}}{32C^2}\,y^4 \quad (2.166)$$

and hence,

$$\dot{\Omega}_{SEC} = -\left(\frac{Ba\,\Omega_e \sin 2\omega}{Pn}\right)\left(1-d\,\frac{1-e}{1+e}\right)(1-e)^2\sqrt{\frac{2}{C}}\,\rho_P\int_0^{\sqrt{2C}} e^{-y^2}\left(1+\frac{F_1^{**}}{4C}\,y^2+\frac{F_2^{**}}{32C^2}\,y^4\right)dy \quad (2.167)$$

As before, replacing the upper limit of the integral by ∞ yields,

$$\dot{\Omega}_{SEC} \sim -\left(\frac{Ba\,\Omega_e \sin 2\omega}{Pn}\right)\left(1-d\,\frac{1-e}{1+e}\right)(1-e)^2\sqrt{\frac{2}{C}}\,\rho_P\int_0^{\infty} e^{-y^2}\left(1+\frac{F_1^{**}}{4C}\,y^2+\frac{F_2^{**}}{32C^2}\,y^4\right)dy \quad (2.168)$$

and the asymptotic solution is obtained by using relations (2.120),

$$\dot{\Omega}_{SEC} \sim -\left(\frac{Ba\,\Omega_e \sin 2\omega}{2Pn}\right)\left(1-d\,\frac{1-e}{1+e}\right)(1-e)^2\rho_P\sqrt{\frac{2\pi}{C}}\left(1+\frac{F_1^{**}}{8C}+\frac{3F_2^{**}}{128C^2}\right) \quad (2.169)$$

The secular change in the inclination is obtained from Equation (2.96),

$$i_{SEC} = -\frac{Ba\Omega_e \sin i}{\rho_\pi \sqrt{1-e^2}} \int_0^\pi \rho (1-e\cos E)^2 \left[\sqrt{1-e^2\cos^2 E}\left(1-d\,\frac{1-e\cos E}{1+e\cos E}\right) dE\right]$$

$$-\left(\frac{Ba\Omega_e \sin i}{\rho_\pi\sqrt{1-e^2}}\right)\cos 2\omega \int_0^\pi \rho \sqrt{1-e^2\cos^2 E}\left(1-d\,\frac{1-e\cos E}{1+e\cos E}\right)\Big[2e^2-1-2e\cos E \qquad (2.170)$$

$$+(2-e^2)\cos^2 E\Big] dE$$

The integrand of the first integral is shown for convenience as a product of 2 factors: the first factor is:

$$(1-e\cos E)^2 = \left[(1-e)+e(1-e\cos E)\right]^2 = (1-e)^2\left[1+\frac{e}{1-e}(1-\cos E)\right]^2$$

$$= (1-e)^2\left[1+2\frac{e}{1-e}\,\frac{y^2}{C}+\left(\frac{e}{1-e}\right)^2\frac{y^4}{C^2}\right] \qquad (2.171)$$

The second factor of the integrand (the one in the brackets) is given by Equation (2.155),

$$\sqrt{1-e^2\cos^2 E}\left(1-d\,\frac{1-e\cos E}{1+e\cos E}\right)dE = \sqrt{\frac{2}{C}}\sqrt{1-e^2}\left(1-d\,\frac{1-e}{1+e}\right)\Bigg\{1+\frac{1}{4C}\Bigg[\left(1+\frac{4e^2}{1-e^2}\right)$$

$$+4\left(\frac{e}{1+e}-ke\right)\Bigg]y^2+\frac{1}{32C^2}\Bigg[8\left(\frac{e}{1+e}-ke\right)\left(1+\frac{4e}{1+e}+\frac{4e^2}{1-e^2}\right)$$

$$+\left(3-\frac{8e^2}{1-e^2}\right)-\left(\frac{4e^2}{1-e^2}\right)^2\Bigg]y^4\Bigg\}dy \qquad (2.172)$$

Multiplication of Equation (2.171) and (2.172) yields,

$$\sqrt{\frac{2}{C}}\sqrt{1-e^2}\left(1-d\,\frac{1-e}{1+e}\right)(1-e)^2\Bigg\{1+\frac{1}{4C}\Bigg[\left(1+\frac{4e^2}{1-e^2}\right)+4\left(\frac{e}{1+e}-ke\right)+\frac{8e}{1-e}\Bigg]y^2+$$

$$+\frac{1}{32C^2}\Bigg[8\left(\frac{e}{1+e}-ke\right)\left(1+\frac{4e}{1+e}+\frac{4e^2}{1-e^2}+\frac{8e}{1-e}\right)+16\left(\frac{e}{1-e}\right)\left(1+\frac{4e^2}{1-e^2}\right)$$

$$+\left(3-\frac{8e^2}{1-e^2}\right)-\left(\frac{4e^2}{1-e^2}\right)^2+32\left(\frac{e}{1-e}\right)^2\Bigg]y^4\Bigg\}dy \qquad (2.173)$$

Thus defining,

$$f_1^{***} = \left(1+\frac{4e^2}{1-e^2}\right)+4\left(\frac{e}{1+e}-ke\right)+\frac{8e}{1-e} = 1+\frac{4e(3+2e)}{1-e^2}-4ke \qquad (2.174)$$

$$f_2^{***} = 8\left(\frac{e}{1+e} - ke\right)\left(1 + \frac{4e(3+2e)}{1-e^2}\right) + 16\left(\frac{e}{1-e}\right)\left(1 + \frac{4e^2}{1-e^2}\right) + \left(3 - \frac{8e^2}{1-e^2}\right) - \left(\frac{4e^2}{1-e^2}\right)^2 + 32\left(\frac{e}{1-e}\right)^2 \quad (2.175)$$

Equation (2.173), which is the transformed integrand of the first integral in Equation (2.170), except for the factor ρ , becomes,

$$\sqrt{\frac{2}{C}}\sqrt{1-e^2}\left(1-d\,\frac{1-e}{1+e}\right)(1-e)^2\left[1 + \frac{f_1^{***}}{4C}\,y^2 + \frac{f_2^{***}}{32C^2}\,y^4\right]dy \qquad (2.176)$$

Substitution of (2.176) for the integrand of the first integral in Equation (2.170) yields,

$$(\dot{i}_{SEC})_{PART\,1} = -\left(\frac{Ba\Omega_e \sin i}{P_n}\right)\left(1-d\,\frac{1-e}{1+e}\right)(1-e)^2\sqrt{\frac{2}{C}}\int_0^{\sqrt{2C}} \rho\left[1 + \frac{f_1^{***}}{4C}\,y^2 + \frac{f_2^{***}}{32C^2}\,y^4\right]dy \quad (2.177)$$

where, from Equation (2.92),

$$\rho = \rho_p\, e^{-y^2}\left[1 + \frac{2}{C}\,\frac{Q_1}{(1-e)^2}\,y^2 - \frac{1}{C^2}\,\frac{Q_1}{(1-e)^2}\left(\frac{4e}{1-e}+1\right)y^4 + \frac{4}{C^2}\,\frac{Q_2}{(1-e)^4}\,y^4\right] \quad (2.178)$$

Replacing ρ in Equation (2.177) by relation (2.178) yields the new integrand, except for the factor $e^{-y^2}dy$,

$$1 + \frac{1}{4C}\left(f_1^{***} + \frac{8Q_1}{(1-e)^2}\right)y^2 + \frac{1}{32C^2}\left[f_2^{***} - \frac{16Q_1}{(1-e)^2}\left(2 + \frac{8e}{1-e} - f_1^{***}\right) + \frac{128Q_2}{(1-e)^4}\right]y^4 \qquad (2.179)$$

Defining for convenience,

$$F_1^{***} = f_1^{***} + \frac{8Q_1}{(1-e)^2} \qquad (2.180)$$

$$F_2^{***} = f_2^{***} - \frac{16Q_1}{(1-e)^2}\left(2 + \frac{8e}{1-e} - f_1^{***}\right) + \frac{128Q_2}{(1-e)^4} \qquad (2.181)$$

the integrand given by Equation (2.179) becomes, except for the factor $e^{-y^2}dy$,

$$1 + \frac{F_1^{***}}{4c} Y^2 + \frac{F_2^{***}}{32c^2} Y^4 \tag{2.182}$$

So that,

$$\left(\dot{i}_{sec}\right)_{PART\,1} = -\left(\frac{B_a\,\Omega_e \sin i}{P n}\right)\left(1-d\,\frac{1-e}{1+e}\right)(1-e)^2\sqrt{\frac{2}{c}}\,P_P\int_0^{\sqrt{2c}} e^{-Y^2}\Big[1 + \frac{F_1^{***}}{4c}Y^2 + \frac{F_2^{***}}{32c^2}Y^4\Big]\,dY \tag{2.183}$$

Finally, replacing the upper limit of the integral by ∞ yields,

$$\left(\dot{i}_{sec}\right)_{PART\,1} \sim -\left(\frac{B_a\,\Omega_e \sin i}{P n}\right)\left(1-d\,\frac{1-e}{1+e}\right)(1-e)^2\sqrt{\frac{2}{c}}\,P_P\int_0^{\infty} e^{-Y^2}\Big[1 + \frac{F_1^{***}}{4c}Y^2 + \frac{F_2}{32c^2}Y^4\Big]\,dY \tag{2.184}$$

and the asymptotic solution is obtained by using relations (2.120),

$$\left(\dot{i}_{sec}\right)_{PART\,1} \sim -\left(\frac{B_a\,\Omega_e \sin i}{2P n}\right)\left(1-d\,\frac{1-e}{1+e}\right)(1-e)^2 P_P\sqrt{\frac{2\pi}{c}}\left(1 + \frac{F_1^{***}}{8c} + \frac{3F_2^{***}}{128c^2}\right) \tag{2.185}$$

Evaluation of the second integral in Equation (2.170) proceeds as follows:

$$\left(\dot{i}_{sec}\right)_{PART\,2} = -\left(\frac{B_a\,\Omega_e \sin i}{P n\,\sqrt{1-e^2}}\right)\cos 2\omega\int_0^{\pi} \rho\sqrt{1-e^2\cos^2 E}\;* \\ *\,\frac{[(1-d)+(1+d)e\cos E]}{1+e\cos E}\Big[2c^2-1-2e\cos E+(2-e^2)\cos^2 E\Big]\,dE \tag{2.186}$$

The integrand of this integral is exactly the same as the integrand of $\dot{\Omega}_{sec}$ given by (2.147). The only difference between $\dot{\Omega}_{sec}$, as defined by Equation (2.147), and $(\dot{i}_{sec})_{part}$ is in the coefficients preceding the respective integrals. The former integral has the factor $\sin 2\omega$ and the later has the factor $\sin i \cos 2\omega$. Hence, when this difference is accounted for, the asymptotic solution of integral (2.186) is obtained from that for $\dot{\Omega}_{sec}$, which is given by Equation (2.169),

$$\left(\dot{i}_{sec}\right)_{PART\,2} \sim -\left(\frac{B_a\cdot\Omega_e \sin i}{2P n}\right)\cos 2\omega\left(1-d\,\frac{1-e}{1+e}\right)(1-e)^2 P_P\sqrt{\frac{2\pi}{c}}\left(1+\frac{F_1^{***}}{8c}+\frac{3F_2^{***}}{128c^2}\right) \tag{2.187}$$

Hence, the total perturbation in orbital inclination amounts to,

$$\dot{i}_{SEC} = \left(\dot{i}_{SEC}\right)_{PART\,1} + \left(\dot{i}_{SEC}\right)_{PART\,2} \qquad RAD/SEC \tag{2.188}$$

Combining relations (2.185) and (2.187), one has that.

$$\dot{i}_{SEC} \sim -\left(\frac{B_a\,\Omega_e \sin i}{2P\,n}\right)\left(1 - d\,\frac{1-e}{1+e}\right)(1-e)^2 \rho_P \sqrt{\frac{2\pi}{c}}\left[\left(1 + \frac{F_1^{***}}{8c} + \frac{3F_2^{***}}{128c^2}\right) + \cos 2\omega \left(1 + \frac{F_1^{**}}{8c} + \frac{3F_2^{**}}{128c^2}\right)\right] \tag{2.189}$$

The final orbit element considered is defined by Equation (2.97),

$$\dot{\omega}_{SEC} = -\cos i\ \dot{\Omega}_{SEC} \qquad RAD/SEC \tag{2.190}$$

where $\dot{\Omega}_{sec}$ is given by Equation (2.169).

2.2.3.6 An Alternate Technique Leading to Standard-Form Solutions in Terms of Bessel Functions

In deriving the asymptotic solutions for eccentric orbits for which $c > 3$, Sterne introduced a new variable $y^2 = c(1 - \cos E)$ to reduce the integrals to a suitable algebraic form for integration.

A simpler and more elegant technique can be used by expanding the integrands directly in powers of the eccentric anomaly, E, and expressing the solutions in terms of modified Bessel functions of the first kind, $I_0(c)$ and $I_1(c)$ of the zero and first orders. Using this technique, the solutions are expressed in a standard form which is applicable to both eccentric orbits ("Asymptotic Solutions") and nearly circular orbits ("General Solutions"). Indeed, to obtain the "Asymptotic Solutions" for eccentric orbits from the standard form of solutions, the modified Bessel functions, $I_0(c)$ and $I_1(c)$, are replaced by the corresponding asymptotic series expansions. Likewise, in order to obtain the "General Solutions" for nearly circular orbits, the modified Bessel functions, $I_0(c)$ and $I_1(c)$, are replaced by the corresponding regular series expansions. In view of this fact, it is irrelevant whether the analysis is originally performed with the "General Solutions" or the "Asymptotic Solutions" in mind. However, the former concept is more convenient for our purposes and will be applied here. The "Asymptotic Solutions" will then be obtained as indicated.

The average secular rates of the orbital elements are obtained by integrating the respective time derivatives over the orbital period P and dividing by P.

If the average secular rate of any of the six orbital elements is denoted by $\dot{\nu}_{sec}$, it will have the following general form according to Equations (2.93) through (2.98),

$$\dot{\nu}_{SEC} = - \left(\frac{CONST}{P}\right) * 2 \int_0^{\pi} \rho f(E)\, dE \tag{2.191}$$

where

$$f(E) = \sum_{n=0}^{\infty} \alpha_n \cos^n E = \alpha_0 + \alpha_1 \cos E + \alpha_2 \cos^2 E + \alpha_3 \cos^3 E + \alpha_4 \cos^4 E + \cdots \tag{2.192}$$

and where powers of $\cos E > 4$ are neglected because the respective coefficients α_n contain powers of e generally of order $(n-1)$ or n. (In Sterne's analysis, powers of $e > 3$ are not retained.)

It remains to express the density ρ in terms of the eccentric anomaly E. From Equation (2.89),

$$\rho = \rho_p\, e^{-c(1-\cos E)} \left[1 + Q_1 \left(\frac{\sin E}{1 - e\cos E}\right)^2 + Q_2 \left(\frac{\sin E}{1-e\cos E}\right)^4 \right] \tag{2.193}$$

In the ensuing expansion of the functions of E (which multiply Q_1 and Q_2) in powers of $\cos E$, only terms containing powers of $e < 3$, for the first function, and powers of $e < 2$, for the second function, will be retained. This step is taken since Q_1 and Q_2 are of the order of Q and Q^2, respectively.

$$\begin{aligned}\left(\frac{\sin E}{1-e\cos E}\right)^2 &= (1-\cos^2 E)(1 + 2e\cos E + 3e^2\cos^2 E) \\ &= 1 + 2e\cos E - (1-3e^2)\cos^2 E - 2e\cos^3 E - 3e^2\cos^4 E\end{aligned} \tag{2.194}$$

$$\begin{aligned}\left(\frac{\sin E}{1-e\cos E}\right)^4 &= (1 - 2\cos^2 E + \cos^4 E)(1 + 4e\cos E) \\ &= 1 + 4e\cos E - 2\cos^2 E - 8e\cos^3 E + \cos^4 E + 4e\cos^5 E\end{aligned} \tag{2.195}$$

With these expansions, Equation (2.193) becomes,

$$P = e^{-c} P_p \Big\{ e^{c\cos E} + Q_1 e^{c\cos E}\left[1 + 2e\cos E - (1-3e^2)\cos^2 E - 2e\cos^3 E - 3e^2\cos^4 E\right] + Q_2 e^{c\cos E}\left[1 + 4e\cos E - 2\cos^2 E - 8e\cos^3 E + \cos^4 E + 4e\cos^5 E\right]\Big\} \tag{2.196}$$

Thus, substitution of relation (2.196) in Equation (2.191) yields,

$$\dot{\nu}_{SEC} = -\left(\frac{CONST}{P}\right) \cdot 2e^{-c} P_p \Big\{ \int_0^{\pi} e^{c\cos E}\left[\alpha_0 + \alpha_1\cos E + \alpha_2\cos^2 E + \alpha_3\cos^3 E + \alpha_4\cos^4 E\right] dE + Q_1 \int_0^{\pi} e^{c\cos E}\left[1 + 2e\cos E - (1-3e^2)\cos^2 E - 2e\cos^3 E - 3e^2\cos^4 E\right]\left[\alpha_0 + \alpha_1\cos E + \alpha_2\cos^2 E\right] dE + Q_2 \int_0^{\pi} e^{c\cos E}\left[1 + 4e\cos E - 2\cos^2 E - 8e\cos^3 E + \cos^4 E + 4e\cos^5 E\right]\left[\alpha_0 + \alpha_1\cos E\right] dE \tag{2.197}$$

Now, the second and third integrals become,

$$Q_1 \int_0^{\pi} e^{c\cos E}\left[\alpha_0 + (2\alpha_0 e + \alpha_1)\cos E + (-\alpha_0 + 3\alpha_0 e^2 + 2\alpha_1 e + \alpha_2)\cos^2 E - (2\alpha_0 e + \alpha_1)\cos^3 E - (3\alpha_0 e^2 + 2\alpha_1 e + \alpha_2)\cos^4 E\right] dE \tag{2.198}$$

$$Q_2 \int_0^{\pi} e^{c\cos E}\left[\alpha_0 + (4\alpha_0 e + \alpha_1)\cos E - 2\alpha_0\cos^2 E - 2(4\alpha_0 e + \alpha_1)\cos^3 E + \alpha_0\cos^4 E + (4\alpha_0 e + \alpha_1)\cos^5 E\right] dE \tag{2.199}$$

The following integrals, modified (regular) Bessel functions $I_0(c)$ and $I_1(c)$ of order 0 and 1 (first kind), can be utilized to evaluate $\dot{\nu}_{sec}$,

$$\int_0^{\pi} e^{c\cos E}\, dE = \pi I_0(c) \tag{2.200}$$

$$\int_0^{\pi} e^{c\cos E}\cos E\, dE = \pi I_1(c) \tag{2.201}$$

$$\int_0^{\pi} e^{c\cos E}\cos^2 E\, dE = \pi\left[I_0(c) - \frac{I_1(c)}{c}\right] \tag{2.202}$$

$$\int_0^{\pi} e^{c\cos E}\cos^3 E\, dE = \pi\left[-\frac{I_0(c)}{c} + \left(1+\frac{2}{c^2}\right) I_1(c)\right] \tag{2.203}$$

$$\int_0^{\pi} e^{c\cos E}\cos^4 E\, dE = \pi\left[\left(1+\frac{3}{c^2}\right) I_0(c) - \left(\frac{2}{c}+\frac{6}{c^3}\right) I_1(c)\right] \tag{2.204}$$

$$\int_0^{\pi} e^{c\cos E}\cos^5 E\, dE = \pi\left[\left(-\frac{2}{c}-\frac{12}{c^3}\right) I_0(c) + \left(1+\frac{7}{c^2}+\frac{24}{c^4}\right) I_1(c)\right] \tag{2.205}$$

Multiplication of relations (2.200) through (2.204) by the coefficients of the first integrand yields the solution of the first integral in Equation (2.197),

$$\begin{aligned}
&\int_0^{\pi} e^{c\cos E}\left[\alpha_0 + \alpha_1\cos E + \alpha_2\cos^2 E + \alpha_3\cos^3 E + \alpha_4\cos^4 E\right] dE \\
&= \pi\left[\left(\alpha_0+\alpha_2+\alpha_4 - \frac{\alpha_3}{c} + \frac{3\alpha_4}{c^2}\right) I_0(c)\right. \\
&\qquad \left. + \left(\alpha_1+\alpha_3 - \frac{\alpha_2}{c} - \frac{2\alpha_4}{c} + \frac{2\alpha_3}{c^2} - \frac{6\alpha_4}{c^3}\right) I_1(c)\right]
\end{aligned} \tag{2.206}$$

Multiplication of relations (2.202) through (2.204) by the coefficients of the second integrand, and retaining only terms with powers of c in the denominators, yields the solution of the second integral in Equation (2.197); more precisely, of integral (2.198),

$$\begin{aligned}
&Q_1\int_0^{\pi} e^{c\cos E}\left[\alpha_0 + (2\alpha_0 e+\alpha_1)\cos E + (-\alpha_0+3\alpha_0 e^2+2\alpha_1 e+\alpha_2)\cos^2 E\right. \\
&\qquad\qquad \left. - (2\alpha_0 e+\alpha_1)\cos^3 E - (3\alpha_0 e^2+2\alpha_1 e+\alpha_2)\cos^4 E\right] dE \\
&= \pi\frac{Q_1}{c}\left\{\left[(2\alpha_0 e+\alpha_1) - \frac{3}{c}(3\alpha_0 e^2+2\alpha_1 e+\alpha_2)\right] I_0(c)\right. \\
&\qquad \left. + \left[\alpha_0 + (3\alpha_0 e^2+2\alpha_1 e+\alpha_2) - \frac{2}{c}(2\alpha_0 e+\alpha_1) + \frac{6}{c^2}(3\alpha_0 e^2+2\alpha_1 e+\alpha_2)\right] I_1(c)\right\}
\end{aligned} \tag{2.207}$$

Multiplication of relations (2.203) through (2.205) by the coefficient of the third integrand and retaining only terms with powers of $c > 1$ in the denominators, yields the solution of the third integral in Equation (2.197); more precisely, of integral (2.199),

$$
\begin{aligned}
&Q_2 \int_0^{\pi} e^{c\cos E}\Big[\alpha_0 + (4\alpha_0 e + \alpha_1)\cos E - 2\alpha_0 \cos^2 E \\
&\qquad - 2(4\alpha_0 e + \alpha_1)\cos^3 E + \alpha_0 \cos^4 E + (4\alpha_0 e + \alpha_1)\cos^5 E\Big]\,dE = \\
&3\pi \frac{Q_2}{c^2}\Big\{\Big[\alpha_0 - \frac{4}{c}(4\alpha_0 e + \alpha_1)\Big] I_0(c) + \Big[(4\alpha_0 e + \alpha_1) - \frac{2\alpha_0}{c} \\
&\qquad + \frac{8}{c^2}(4\alpha_0 e + \alpha_1)\Big] I_1(c)\Big\}
\end{aligned}
\tag{2.208}
$$

Substituting relations (2.206), (2.207), and (2.208) into Equation (2.197), and combining all terms with $I_0(c)$ as a factor on one side, and those with $I_1(c)$ as a factor on another side, the following "General Solutions" are obtained for the case of $c \leq 3$; that is, for nearly-circular orbits,

$$
\begin{aligned}
\dot{\nu}_{sec} = -\left(\frac{Const}{p}\right) 2\pi e^{-c} \rho_p \Big[&\Big(a_0 + \frac{Q_1}{c} a_1 + \frac{3Q_2}{c^2} a_2\Big) I_0(c) \\
&+ \Big(b_0 + \frac{Q_1}{c} b_1 + \frac{3Q_2}{c^2} b_2\Big) I_1(c)\Big]
\end{aligned}
\tag{2.209}
$$

where

$$a_0 = \alpha_0 + \alpha_2 + \alpha_4 - \frac{\alpha_3}{c} + \frac{3\alpha_4}{c^2} \tag{2.210}$$

$$a_1 = (2\alpha_0 e + \alpha_1) - \frac{3}{c}(3\alpha_0 e^2 + 2\alpha_1 e + \alpha_2) \tag{2.211}$$

$$a_2 = \alpha_0 - \frac{4}{c}(4\alpha_0 e + \alpha_1) \tag{2.212}$$

$$b_0 = \alpha_1 + \alpha_3 - \frac{\alpha_2}{c} - \frac{2\alpha_4}{c} + \frac{2\alpha_3}{c^2} - \frac{6\alpha_4}{c^3} \tag{2.213}$$

$$b_1 = \alpha_0 + (3\alpha_0 e^2 + 2\alpha_1 e + \alpha_2) - \frac{2}{c}(2\alpha_0 e + \alpha_1) + \frac{6}{c^2}(3\alpha_0 e^2 + 2\alpha_1 e + \alpha_2) \tag{2.214}$$

$$b_2 = (4\alpha_0 e+\alpha_1) - \frac{2\alpha_0}{c} + \frac{8}{c^2}(4\alpha_0 e+\alpha_1) \quad (2.215)$$

In order to obtain the asymptotic solutions for eccentric orbits ($c > 3$), the modified (regular) Bessel functions, $I_0(c)$ and $I_1(c)$, are replaced by the following asymptotic definitions for $n = 0$ and $n = 1$,

$$I_n(c) \sim \frac{e^c}{\sqrt{2\pi c}}\left[1+\sum_{n=1}^{\infty}(-1)^n \frac{(4n^2-1^2)(4n^2-3^2)(4n^2-5^2)\cdots}{n!\,(8c)^n}\right] \sim \frac{e^c}{\sqrt{2\pi c}}G_n(c) \quad (2.216)$$

Hence, the asymptotic solutions are:

$$\dot{\nu}_{sec} = -\left(\frac{CONST}{P}\right)\rho_p\sqrt{\frac{2\pi}{c}}\left[\left(a_0+\frac{Q_1}{c}a_1+\frac{3Q_2}{c^2}a_2\right)G_0(c) + \left(b_0+\frac{Q_1}{c}b_1+\frac{3Q_2}{c^2}b_2\right)G_1(c)\right] \quad (2.217)$$

where the coefficients (a_0, a_1, a_2) and (b_0, b_1, b_2) are the same for both types of solutions.

In order to obtain the solutions for the individual secular changes of the orbital elements, it is necessary only to determine the respective coefficients for each case and introduce them in relations (2.210) through (2.215). The respective solutions will then be given by Equation (2.209) for near-circular orbits ($c \leq 3$) and by Equation (2.217) for eccentric orbits ($c > 3$).

2.2.3.6.1 The α_n Coefficient for $\dot{a}_{sec}$

Equation (2.93) may be rewritten in the form,

$$\dot{a}_{sec} = -\left(\frac{2B_a}{P}\right)\times 2\int_0^{\pi}\rho\,\frac{[(1-d)+(1+d)\,e\cos E]^2}{\sqrt{1-e^2\cos^2 E}} \quad (2.218)$$

Defining,

$$j = \frac{1+d}{1-d} \quad (2.219)$$

$$\left[(1-d)+(1+d)\,e\cos E\right]^2 = (1-d)^2(1+2\,j\,e\cos E+j^2e^2\cos^2 E) \quad (2.220)$$

$$(1 - e^2 \cos^2 E)^{-1/2} = 1 + \frac{e^2}{2} \cos^2 E + \ldots \tag{2.221}$$

and multiplying relations (2.220) and (2.221), neglecting powers of $e > 3$ and substituting the resulting product in the integral (2.218) yields,

$$\dot{a}_{sec} = - \left[\frac{2Ba^2(1-d)^2}{p}\right] \times 2\int_0^{\pi} \rho \left[1 + 2je\cos E + e^2(j^2 + \tfrac{1}{2})\cos^2 E + je^3\cos^3 E\right] dE \tag{2.222}$$

Thus,

$$\begin{cases} \alpha_0 = 1 \\ \alpha_1 = 2je \\ \alpha_2 = e^2(j^2 + \tfrac{1}{2}) \\ \alpha_3 = je^3 \\ \alpha_4 = 0 \\ \dfrac{CONST}{p} = \dfrac{2Ba^2(1-d)^2}{p} \end{cases} \tag{2.223}$$

When the values of these coefficients are substituted in relations (2.210) through (2.215), it follows that,

$$
\begin{cases}
a_0 = 1 + e^2\left(j^2 + \frac{1}{2}\right) - \dfrac{je^3}{c} \\
a_1 = 2e(j+1) - \frac{3}{c}\, e^2\left(j^2 + 4j + \frac{7}{2}\right) \\
a_2 = 1 - \frac{8}{c} e\,(j+2) \\
b_0 = 2je + je^3 - \frac{e^2}{c}\left(j^2 + \frac{1}{2}\right) + \frac{2}{c^2}\, je^3 \\
b_1 = 1 + e^2\left(j^2 + 4j + \frac{7}{2}\right) - \frac{4}{c}\, e\,(j+1) + \frac{6}{c^2}\, e^2\left(j^2 + 4j + \frac{7}{2}\right) \\
b_2 = 2e(j+2) - \frac{2}{c} + \frac{16}{c^2}\, e\,(j+2)
\end{cases}
$$

(2.224)

Knowing $\dot{a}_{sec}$, the change in the orbital period is obtained from the relation,

$$\dot{P} = \frac{3}{2}\, P \left(\frac{\dot{a}_{sec}}{a}\right) \quad \text{(non-dimensional)}$$

$$\dot{P} = -3B\,(1-d)^2 \times 2\pi e^{-c} \rho_p \left[\left(a_0 + \frac{Q_1}{c} a_1 + \frac{3Q_2}{c^2} a_2\right) I_0(c) + \left(b_0 + \frac{Q_1}{c} b_1 + \frac{3Q_2}{c^2} b_2\right) I_1(c)\right] \qquad (2.225)$$

2.2.3.6.2 The α_n Coefficients for $\dot{e}_{sec}$

Equation (2.94) may be rewritten in the form,

$$\dot{e}_{sec} = -\left[\frac{2B_a\,(1-e^2)}{P}\right] \times 2\int_0^{\pi} \rho \left[\frac{(1-d)+(1+d)\,e\cos E}{\sqrt{1-e^2\cos^2 E}}\right]\left[\cos E - \frac{d}{2(1-e^2)}\,(1-e\cos E)(2\cos E - e - e\cos^2 E)\right] dE \qquad (2.226)$$

Series expansion in terms of cosE yields,

$$(1-e^2\cos^2E)^{-1/2} = 1+\frac{e^2}{2}\cos^2E+\ldots \tag{2.227}$$

$$\frac{(1-d)+(1+d)e\cos E}{\sqrt{1-e^2\cos^2E}} = (1-d)+e(1+d)\cos E+\frac{e^2}{2}(1-d)\cos^2E+\frac{e^3}{2}\cos^3E \tag{2.228}$$

$$\left(\frac{(1-d)+(1+d)e\cos E}{\sqrt{1-e^2\cos^2E}}\right)\cos E = (1-d)\cos E+e(1+d)\cos^2E+\frac{e^2}{2}(1-d)\cos^3E + \frac{e^3}{2}\cos^4E \tag{2.229}$$

It will suffice to assume that,

$$\frac{d}{2(1-e^2)} = \frac{d}{2}$$

Also, since $d \ll 1$ (approximately 0.06), terms containing e^3d are neglected.

Hence, the second term in the second brackets of the integrand in (2.226) becomes,

$$-\frac{d}{2}(1-e\cos E)(2\cos E-e-e\cos^2E) = \frac{d}{2}\left[e-(2+e^2)\cos E + 3e\cos^2E-e^2\cos^3E\right] \tag{2.230}$$

Multiplication of (2.228) and (2.230) yields,

$$\left[\frac{(1-d)+(1+d)e\cos E}{\sqrt{1-e^2\cos^2E}}\right]\frac{d}{2}\left[e-(2+e^2)\cos E+3e\cos^2E-e^2\cos^3E\right]$$
$$= \frac{d}{2}\left[(e(1-d)-2(1-d-e^2d)\cos E+e(1-5d)\cos^2E + e^2(1+5d)\cos^3E\right] \tag{2.2.31}$$

Addition of Equation (2.229) and (2.231) yields the integrand in (2.226), except for the factor ρ,

$$\dot{e}_{SEC} = -\left[\frac{2B_a(1-e^2)}{p}\right] \times 2\int_0^{\pi} \rho \left[\frac{ed}{2}(1-d) + \left[(1-d)^2 + e^2 d^2\right]\cos E + \frac{e}{2}(1-d)(2+5d)\cos^2 E + \frac{e^2}{2}(1+5d^2)\cos^3 E + \frac{e^3}{2}\cos^4 E\right] dE \tag{2.232}$$

Thus,

$$\begin{aligned}
\alpha_0 &= \frac{ed}{2}(1-d) \\
\alpha_1 &= (1-d)^2 + e^2 d^2 \\
\alpha_2 &= \frac{e}{2}(1-d)(2+5d) \\
\alpha_3 &= \frac{e^2}{2}(1+5d^2) \\
\alpha_4 &= \frac{e^3}{2} \\
\frac{\text{CONST}}{p} &= \frac{2B_a(1-e)^2}{p}
\end{aligned} \tag{2.233}$$

These coefficients, when substituted in relations (2.210) through (2.215) yield,

$$\left\{\begin{aligned}
a_0 &= e(1-d)(1+3d) + \frac{e^3}{2} - \frac{e^2}{2c}(1+5d^2) + \frac{3e^3}{2c^2} \\
a_1 &= (1-d)^2 + e^2 d - \frac{3}{2c}e(1-d)(6+d) \\
a_2 &= \frac{ed}{2}(1-d) - \frac{4}{c}\left[(1-d)^2 + e^2 d(2-d)\right] \\
b_0 &= (1-d)^2 + \frac{e^2}{2}(1+7d^2) - \frac{e}{2c}(1-d)(2+5d) - \frac{e^3}{c} + \frac{e^2}{c^2}(1+5d^2) - \frac{3e^3}{c^3} \\
b_1 &= \frac{ed}{2}(1-d) + \frac{e}{2}(1-d)(6+d) - \frac{2}{c}\left[(1-d)^2 + e^2 d\right] + \frac{3}{c^2}e(1-d)(6+d) \\
b_2 &= (1-d)^2 + e^2 d(2-d) - \frac{ed}{c}(1-d) + \frac{8}{c^2}\left[(1-d)^2 + e^2 d(2-d)\right]
\end{aligned}\right. \tag{2.234}$$

2.2.3.6.3 The α_n Coefficients for $\dot{\Omega}_{sec}$

Equation (2.95) may be rewritten in the form,

$$\dot{\Omega}_{sec} = -\left[\frac{B_a \Omega_e \sin 2\omega}{2P_n \sqrt{1-e^2}}\right] \times 2\int_0^{\pi} \rho \sqrt{1-e^2\cos^2 E}\left[\frac{(1-d)+(1+d)e\cos E}{1+e\cos E}\right]\Big[-(1-2e^2) - 2e\cos E + (2-e^2)\cos^2 E\Big] dE \tag{2.235}$$

Series expansion in terms of cosE yields,

$$(1+e\cos E)^{-1} = 1 - e\cos E + e^2\cos^2 E - e^3\cos^3 E + \dots \tag{2.236}$$

$$\frac{(1-d)+(1+d)e\cos E}{1+e\cos E} = (1-d) + 2ed\cos E - 2e^2 d\cos^2 E + \dots \tag{2.237}$$

$$\sqrt{1-e^2\cos^2 E} = 1 - \frac{e^2}{2}\cos^2 E + \dots \tag{2.238}$$

Multiplying (2.237) and (2.238), it follows that,

$$\frac{(1-d)+(1+d)e\cos E}{1+e\cos E}\sqrt{1-e^2\cos^2 E} = (1-d) + 2ed\cos E - \frac{e^2}{2}(1+3d)\cos^2 E + \dots \tag{2.239}$$

Finally, multiplication of (2.239) by the second brackets of the integrand in (2.235) yields,

$$\begin{aligned}\frac{(1-d)+(1+d)e\cos E}{1+e\cos E}\sqrt{1-e^2\cos^2 E}\left[-(1-2e^2) - 2e\cos E + (2-e^2)\cos^2 E\right] \\ = -(1-2e^2)(1-d) - 2e\cos E + \left[2(1-d) - \frac{e^2}{2}(1+3d)\right]\cos^2 E \\ + (4ed + e^3)\cos^3 E - e^2(1+3d)\cos^4 E + \dots\end{aligned} \tag{2.240}$$

where, again, the terms containing e^3d were neglected.

Substituting relation (2.240) for the integrand in Equation (2.235) yields,

$$\dot{\Omega}_{SEC} = -\left[\frac{Ba\,\Omega_e\,SIN2\omega}{2P_n\sqrt{1-e^2}}\right] \times 2\int_0^{\pi} \rho\,\Big[-(1-2e^2)(1-d) - 2e\,COSE$$
$$+\left(2(1-d)-\frac{e^2}{2}(1+3d)\right)COS^2E + (4ed+e^3)COS^3E - e^2(1+3d)COS^4E\Big]\,dE \qquad (2.241)$$

Thus,

$$\begin{cases} \alpha_0 = -(1-2e^2)(1-d) \\ \alpha_1 = -2e \\ \alpha_2 = 2(1-d) - \frac{e^2}{2}(1+3d) \\ \alpha_3 = 4ed + e^3 \\ \alpha_4 = -e^2(1+3d) \\ \frac{CONST}{P} = \frac{Ba\,\Omega_e\,SIN2\omega}{2P_n\sqrt{1-e^2}} \end{cases} \qquad (2.242)$$

Substituting these coefficients in relations (2.210) through (2.215) will yield,

$$\begin{cases} a_0 = (1-d) + \frac{e^2}{2}(1-13d) - \frac{e}{c}(4d+e^2) - \frac{3}{c^2}e^2(1+3d) \\ a_1 = -2e\left[1+(1-d)(1-2e^2)\right] - \frac{3}{c}\left[2(1-d) - \frac{3}{2}e^2(5-d)\right] \\ a_2 = -(1-2e^2)(1-d) + \frac{8}{c}e(3-2d-4e^2) \\ b_0 = -2e\left(1-2d-\frac{e^2}{2}\right) - \frac{2}{c}(1-d) + \frac{5e^2}{2c}(1+3d) \\ \qquad + \frac{2e}{c^2}(4d+e^2) + \frac{6}{c^3}e^2(1+3d) \\ b_1 = (1-d) - \frac{e^2}{2}(11+d) + \frac{4e}{c}\left[1+(1-d)(1-2e^2)\right] + \frac{6}{c^2}\left[2(1-d) - \frac{3}{2}e^2(5-d)\right] \\ b_2 = -2e\left[1+2(1-d)(1-2e^2)\right] + \frac{2}{c}(1-2e^2)(1-d) \\ \qquad - \frac{16e}{c^2}\left[1+2(1-d)(1-2e^2)\right] \end{cases} \qquad (2.243)$$

2.2.3.6.4 The α_n Coefficients for i_{sec}

Equation (2.96) may be rewritten in the form,

$$\dot{i}_{sec} = -\left[\frac{Ba\Omega_e \sin i}{2P_n\sqrt{1-e^2}}\right]\times 2\int_0^{\pi} \rho \left\{(1-e\cos E)^2\sqrt{1-e^2\cos^2 E}\left[\frac{(1-d)+(1+d)e\cos E}{1+e\cos E}\right] + \right.$$
$$\left. + \cos 2\omega \sqrt{1-e^2\cos^2 E}\left[\frac{(1-d)+(1+d)e\cos E}{1+e\cos E}\right]\left[2e^2-1-2e\cos E+(2-e^2)\cos^2 E\right]\right\} dE$$

(2.244)

However, the second part of the integrand is the same as the integrand of $\dot{\Omega}_{sec}$ in Equation (2.235), except for the factor cos 2ω. Therefore, the respective integrated solution will be the one found for $\dot{\Omega}_{sec}$, except for the constant term. Hence, the α_n coefficients for the second part of the integrand of Equation (2.244) are exactly those in (2.242) and the corresponding a_n and b_n are those in (2.243).

The first part of the integrand in (2.244) is the product of,

$$(1-e\cos E)^2 = 1-2e\cos E+e^2\cos^2 E \qquad (2.245)$$

and a second factor, the expansion of which is given by relation (2.239),

$$(1-e\cos E)^2\sqrt{1-e^2\cos^2 E}\left[\frac{(1-d)+(1+d)e\cos E}{1+e\cos E}\right]$$
$$= (1-d)-2e(1-2d)\cos E+\frac{e^2}{2}(1-13d)\cos^2 E+e^3\cos^3 E \qquad (2.246)$$

Thus, substituting relation (2.246) for the first part of the integrand in (2.244), and remembering that the second part of the integrand is the same as the integrand of $\dot{\Omega}_{sec}$ in (2.235) except for the factor cos 2ω, yields,

$$\dot{i}_{sec} = -\left[\frac{Ba\ \Omega_e \sin i}{2P_n\ \sqrt{1-e}}\right]\times 2\int_0^{\pi} \rho\left[(1-d)-2e(1-2d)\cos E+\frac{e^2}{2}(1-13d)\cos^2 E\right.$$
$$\left. + e^3\cos^3 E+\cos 2\omega(\text{INTEGRAND OF } \dot{\Omega})\right] dE \qquad (2.247)$$

Hence, the α_n coefficients for Part 1 are,

$$
\text{Part 1}\quad
\begin{cases}
\alpha_0 = 1-d \\
\alpha_1 = -2e(1-2d) \\
\alpha_2 = \dfrac{e^2}{2}(1-13d) \\
\alpha_3 = e^3 \\
\alpha_4 = 0 \\
\dfrac{CONST}{P} = \dfrac{B_a\, r_e \sin i}{2pn\sqrt{1-e^2}}
\end{cases}
\tag{2.248}
$$

Substitution of these coefficients in relations (2.210) through (2.215) yields

$$
\text{Part 1}\quad
\begin{cases}
a_0 = (1-d) + \dfrac{e^2}{2}(1-13d) - \dfrac{e^3}{c} \\
a_1 = 2ed + \dfrac{3e^2}{2c}(1+3d) \\
a_2 = (1-d) - \dfrac{8e}{c} \\
b_0 = -2e\left(1-2d-\dfrac{e^2}{2}\right) - \dfrac{e^2}{2c}(1-13d) + \dfrac{2e^3}{c^2} \\
b_1 = (1-d) - \dfrac{e^2}{2}(1+3d) - \dfrac{4ed}{c} - \dfrac{3e^2}{c^2}(1+3d) \\
b_2 = 2e - \dfrac{2}{c}(1-d) + \dfrac{16e}{c^2}
\end{cases}
\tag{2.249}
$$

Attaching to the a_n, b_n coefficients for $\dot{\Omega}_{sec}$ in relations (2.243), the subscript " Ω " for proper identification, and multiplying each of them by the factor $\cos 2\omega$, yields the coefficients a_n, b_n for i_{sec} by adding $(\cos 2\omega)a_{n_{\dot{\Omega}}}$ and $(\cos 2\omega)b_{n_{\dot{\Omega}}}$ to the corresponding coefficients in (2.249), that is,

$$
\text{Part 1} + \text{Part 2}\quad
\begin{cases}
a_{0i} = a_0 + \cos 2\omega \, a_{0\dot{\Omega}} \\
a_{1i} = a_1 + \cos 2\omega \, a_{1\dot{\Omega}} \\
a_{2i} = a_2 + \cos 2\omega \, a_{2\dot{\Omega}} \\
b_{0i} = b_0 + \cos 2\omega \, b_{0\dot{\Omega}} \\
b_{1i} = b_1 + \cos 2\omega \, b_{1\dot{\Omega}} \\
b_{2i} = b_2 + \cos 2\omega \, b_{2\dot{\Omega}}
\end{cases}
\tag{2.250}
$$

2.2.3.6.5 The α_n Coefficients for $\dot{\omega}_{sec}$

These coefficients are the same as those given for $\dot{\Omega}_{sec}$. The two solutions are related as follows:

$$\dot{\omega}_{sec} = -\cos i \, \dot{\Omega}_{sec} \tag{2.251}$$

2.2.4 <u>Analytical Development of Kalil's Technique (General Solutions)</u>

2.2.4.1 Reduction of the Time Rates of the Orbital Elements to Integrable Form

The "General Solutions" apply only to near-circular orbits for which $c \leq 3$. Kalil uses basically the approach employed by Sterne for the derivation of the drag acceleration vector, $\frac{\bar{D}}{m}$; that is

$$\frac{\bar{D}}{m} = -B_a \rho V_R \left[\bar{R} e \sin E + \bar{S} \sqrt{1-e^2} \left(1 - d \frac{(1-e\cos E)^2}{1-e^2} \right) + \bar{W} \frac{\Omega_e}{n} \sin i \cos u \, (1-e\cos E)^2 \right] \dot{E} \tag{2.252}$$

where V_R is the relative velocity of the satellite with respect to the atmosphere,

$$V_R = a n \sqrt{\frac{1+e\cos E}{1-e\cos E}} \left(1 - d \frac{1-e\cos E}{1+e\cos E} \right) \tag{2.253}$$

Kalil also uses the same definitions for the secular rates of the orbital elements a, e, P, as derived by Sterne and summarized in Equations (2.93), (2.94), and (2.98),

$$\dot{a}_{SEC} = -\frac{2Ba^2}{P}\int_0^{2\pi}\rho\,\frac{(1-e\cos E)^2}{\sqrt{1-e^2\cos^2 E}}\left(1-\quad d\,\frac{1-e\cos E}{1+e\cos E}\right)^2 dE \qquad (2.254)$$

$$\dot{e}_{SEC} = -\frac{2B_a(1-e^2)}{P}\int_0^{2\pi}\rho\,\sqrt{\frac{1+e\cos E}{1-e\cos E}}\left(1-d\,\frac{1-e\cos E}{1+e\cos E}\right)\Big[\cos E - \frac{d}{2(1-e^2)}(1-e\cos E)(2\cos E-e-e\cos^2 E)\Big]dE \qquad (2.255)$$

$$\dot{P} = -3Ba\int_0^{2\pi}\rho\,\frac{(1-e\cos E)^2}{\sqrt{1-e^2\cos^2 E}}\left(1-\quad d\,\frac{1-e\cos E}{1+e\cos E}\right)^2 dE \qquad (2.256)$$

Kalil, however, does not present in his paper solutions for $\dot{\Omega}_{sec}$, $\dot{i}_{sec}$, and $\dot{\omega}_{sec}$. For the sake of completeness, however, the solutions for these three orbital elements will be included in this analysis; and, for this reason, the respective definitions will be transcribed in the form presented by Sterne by Equations (2.95), (2.96), and (2.97),

$$\dot{\Omega}_{SEC} = -\frac{B_a\,\Omega_e\sin 2\omega}{2P\cdot n\,\sqrt{1-e^2}}\int_0^{2\pi}\rho\sqrt{1-e^2\cos^2 E}\left(1-\quad d\,\frac{1-e\cos E}{1+e\cos E}\right)\Big[(2e^2-1) - 2e\cos E+(2-e^2)\cos^2 E\Big]dE \qquad (2.257)$$

$$\dot{i}_{SEC} = -\frac{B a\,\Omega_e\sin i}{2Pn\sqrt{1-e^2}}\int_0^{2\pi}\rho\sqrt{1-e^2\cos^2 E}\left(1-\quad d\,\frac{1-e\cos E}{1+e\cos E}\right)\Big\{(1-e\cos E)^2 + \cos 2\omega\Big[(2e^2-1)-2e\cos E+(2-e^2)\cos^2 E\Big]\Big\}dE \qquad (2.258)$$

$$\dot{\omega}_{SEC} = -\cos i\,\dot{\Omega}_{SEC} \qquad (2.259)$$

Let $\dot{\nu}_{sec}$ represent the secular rate in any of the six orbital elements,

$$\dot{\nu} = -\left(\frac{CONST}{P}\right)\int_0^{2\pi} \rho f(E)\, dE \tag{2.260}$$

where f(E) represents the polynomial expansion in powers of cosE of the integrands (except for ρ) appearing under the integrals in relations (2.254) through (2.259).

$$f(E) = \alpha_0 + \alpha_1 \cos E + \alpha_2 \cos^2 E + \alpha_3 \cos^3 E + \alpha_4 \cos^4 E + \cdots \tag{2.261}$$

Powers of $\cos E > 4$ are not retained in the expansion of f(E), because the α_n coefficients generally contain powers of e of the order of n or (n-1), and powers of $e > 3$ are neglected in this analysis. In fact, Kalil even neglects the power e^3.

Once again, the density ρ will be given by Equation (2.89); that is

$$\rho = \rho_p e^{-c(1-\cos E)}\left[1 + Q_1\left(\frac{\sin E}{1-e\cos E}\right)^2 + Q_2\left(\frac{\sin E}{1-e\cos E}\right)^4 + Q_3\left(\frac{\sin E}{1-e\cos E}\right)^6 + Q_4\left(\frac{\sin E}{1-e\cos E}\right)^8\right] \tag{2.262}$$

where the coefficients Q_1, Q_2, Q_3, Q_4 are given by Equation (2.87),

$$\begin{aligned}
Q_1 &= (1-e^2)\left(-Q\cos 2\omega + \frac{Q^2}{2}\sin 2\omega\right) \\
Q_2 &= (1-e^2)^2\left(\frac{Q^2}{2}\cos 4\omega - \frac{Q^3}{2}\cos 2\omega \sin^2 2\omega + \frac{Q^4}{24}\sin^4 2\omega\right) \\
Q_3 &= (1-e^2)^3\left(-\frac{Q^3}{6}\cos^3 2\omega + \frac{Q^3}{2}\cos 2\omega \sin^2 2\omega + \frac{Q^4}{4}\sin^2 2\omega \cos^2 2\omega - \frac{Q^4}{12}\sin^4 2\omega\right) \\
Q_4 &= (1-e^2)^4\left(\frac{Q^4}{24}\cos^4 2\omega - \frac{Q^4}{16}\sin^2 4\omega + \frac{Q^4}{24}\sin^4 2\omega\right)
\end{aligned} \tag{2.263}$$

Kalil appears to have three errors in these coefficients, which have been corrected: (1) in Q_1 he has $(1 - e^2)^2$ instead of $(1 - e^2)$; (2) in Q_2 he has $(1 - e^2)$ instead of $(1 - e^2)^2$; (3) in the third term of the expression for Q_3, he has $Q^4/24$, instead of $Q_4/4$.

The term Q was already defined. However, the definition is transcribed here for completeness.

$$Q = K R_{EQ} f \sin^2 i \tag{2.264}$$

where f is the flattening of the Earth and K is the inverse of the density scale height.

At this point, the functions containing the eccentric anomaly in Equation (2.262) are each expanded in a series. However, in the expansion of $[\sin E/(1-e\cos E)]^2$ powers of e^2 are retained; whereas in the remaining three expansions, only the first-order powers in e are retained. This procedure was adopted due to the order of the respective coefficients Q_i, i = 1, 2, 3, 4.

$$\left(\frac{\sin E}{1-e\cos E}\right)^2 = (1-\cos^2 E)(1+2e\cos E+3e^2\cos^2 E = 1+2e\cos E - (1-3e^2)\cos^2 E - 2e\cos^3 E - 3e^2\cos^4 E$$

$$\left(\frac{\sin E}{1-e\cos E}\right)^4 = (1-2\cos^2 E+\cos^4 E)(1+4e\cos E) = 1+4e\cos E - 2\cos^2 E - 8e\cos^3 E + \cos^4 E + 4e\cos^5 E$$

$$\left(\frac{\sin E}{1-e\cos E}\right)^6 = (1-\cos^2 E)^3(1+6e\cos E) = (1-3\cos^2 E+3\cos^4 E-\cos^6 E)(1+6e\cos E)$$
$$= 1+6e\cos E - 3\cos^2 E - 18e\cos^3 E + 3\cos^4 E + 18e\cos^5 E - \cos^6 E - 6e\cos^7 E$$

$$\left(\frac{\sin E}{1-e\cos E}\right)^8 = (1-\cos^2 E)^4(1+8e\cos E) = (1-4\cos^2 E+6\cos^4 E$$
$$- 4\cos^6 E + \cos^8 E)(1+8e\cos E) = 1+8e\cos E - 4\cos^2 E$$
$$- 32e\cos^3 E + 6\cos^4 E + 48e\cos^5 E - 4\cos^6 E$$
$$- 32e\cos^7 E + \cos^8 E + 8e\cos^9 E$$

(2.265)

Substitution of (2.265) into (2.262) yields,

$$
\begin{aligned}
= \; e^{-c}\Big\{ & e^{c\cos E} + Q_1 e^{c\cos E}\left[1+2e\cos E-(1-3e^2)\cos^2 E-2e\cos^3 E-3e^2\cos^4 E\right] \\
& + Q_2 e^{c\cos E}\left[1+4e\cos E-2\cos^2 E-8e\cos^3 E+\cos^4 E+4e\cos^5 E\right] \\
& + Q_3 e^{c\cos E}\left[1+6e\cos E-3\cos^2 E-18e\cos^3 E+3\cos^4 E+18e\cos^5 E \right. \\
& \qquad \left. -\cos^6 E-6e\cos^7 E\right] \\
& + Q_4 e^{c\cos E}\left[1+8e\cos E-4\cos^2 E-32e\cos^3 E+6\cos^4 E \right. \\
& \qquad \left. +48e\cos^5 E-4\cos^6 E-32e\cos^7 E+\cos^8 E+8e\cos^9 E\right]\Big\}
\end{aligned}
\tag{2.266}
$$

and the substitution of relations (2.266) and (2.261) into (2.260) yields,

$$
\begin{aligned}
\dot{\nu}_{SEC} = -\left(\frac{CONST}{P}\right)\rho_P\, e^{-c}\Big\{ & \int_0^{2\pi} e^{c\cos E}\left[\alpha_0+\alpha_1\cos E+\alpha_2\cos^2 E+\alpha_3\cos^3 E \right. \\
& \qquad \left. +\alpha_4\cos^4 E\right] dE \\
& + Q_1\int_0^{2\pi} e^{c\cos E}\left[\alpha_0+(2\alpha_0 e+\alpha_1)\cos E+(-\alpha_0+3\alpha_0 e^2+2\alpha_1 e+\alpha_2)\cos^2 E \right. \\
& \qquad \left. -(2\alpha_0 e+\alpha_1)\cos^3 E-(3\alpha_0 e^2+2\alpha_1 e+\alpha_2)\cos^4 E\right] dE \\
& + Q_2\int_0^{2\pi} e^{c\cos E}\left[\alpha_0+(4\alpha_0 e+\alpha_1)\cos E-2\alpha_0\cos^2 E-2(4\alpha_0 e+\alpha_1)\cos^3 E \right. \\
& \qquad \left. +\alpha_0\cos^4 E+(4\alpha_0 e+\alpha_1)\cos^5 E\right] dE \\
& + Q_3\int_0^{2\pi} e^{c\cos E}\left[\alpha_0+(6\alpha_0 e+\alpha_1)\cos E-3\alpha_0\cos^2 E-3(6\alpha_0 e+\alpha_1)\cos^3 E \right. \\
& \qquad +3\alpha_0\cos^4 E+3(6\alpha_0 e+\alpha_1)\cos^5 E-\alpha_0\cos^6 E \\
& \qquad \left. -(6\alpha_0 e+\alpha_1)\cos^7 E\right] dE \\
& + Q_4\int_0^{2\pi} e^{c\cos E}\left[\alpha_0+(8\alpha_0 e+\alpha_1)\cos E-4\alpha_0\cos^2 E-4(8\alpha_0 e+\alpha_1)\cos^3 E \right. \\
& \qquad +6\alpha_0\cos^4 E+6(8\alpha_0 e+\alpha_1)\cos^5 E-4\alpha_0\cos^6 E \\
& \qquad \left. -4(8\alpha_0 e+\alpha_1)\cos^7 E+\alpha_0\cos^8 E+(8\alpha_0 e+\alpha_1)\cos^9 E\right] dE\Big\}
\end{aligned}
\tag{2.267}
$$

Note that in the multiplication of the function f(E) which is defined by Equation (2.261) by the polynomial adjoint to Q_1 in Equation (2.266), the terms containing $\alpha_2 e$ or higher were not retained; while in the multiplication of f(E) by the polynomials adjoint to Q_2, Q_3, Q_4, the terms containing $\alpha_1 e$ were not retained.

2.2.4.2 Kalil's Integration Procedure

Kalil integrates Equation (2.267) in the following manner:

A. The five integrals in Equation (2.267) are combined in one single integral by collecting terms of the same powers in cosE.

B. The powers $\cos^n E$ are then converted to multiple angles cos n E by the application of the following transformation table:

$$2\cos^2 E = 1 + \cos 2E$$

$$4\cos^3 E = 3\cos E + \cos 3E$$

$$8\cos^4 E = 3 + 4\cos 2E + \cos 4E$$

$$16\cos^5 E = 10\cos E + 5\cos 3E + \cos 5E$$

$$32\cos^6 E = 10 + 15\cos 2E + 6\cos 4E + \cos 6E$$

$$64\cos^7 E = 35\cos E + 21\cos 3E + 7\cos 5E + \cos 7E$$

C. The terms in cos nE of the same multiple angles are collected.

D. The integration is performed term by term through the use of modified Bessel functions of the first kind. The individual integrals are defined as follows

$$\int_0^{2\pi} e^{c\cos E} \cos nE \, dE = 2\pi I_n(C)$$

$$n = 0, 1, 2, 3, \cdots n$$

E. The final solutions are of the form,

$$\dot{\nu}_{sec} = -\frac{(\text{CONST})}{p} \, 2\pi \rho_p e^{-C} \sum_{n=0}^{n} A_n I_n(C)$$

where the coefficients A_n are functions of α_n, c, Q_1, Q_2, Q_3, Q_4.

It becomes obvious, from the foregoing presentation of the required algebraic manipulations for deriving the final solutions, that Kalil's approach is cumbersome, tedious, and also inconvenient. In addition, the form in which the final solutions are presented has the following disadvantages:

A. It involves the modified Bessel functions of all orders. To reduce the Bessel functions of higher orders to those of the zero and first orders, the recurrent reduction formula

$$I_{n+1} = I_{n-1} - \frac{2n}{c} I_n$$

must be used, which means additional algebraic manipulations, collecting terms, determining the new coefficients of $I_0(c)$ and $I_1(c)$, etc.

B. The Q_1, Q_2, Q_3, Q_4 parameters do not appear explicitly in a suitable form for switching off some of them when it is desired, as - for instance - for comparison with Sterne, who retains only Q_1 and Q_2**. Instead, these parameters enter in an intricate form in the A_n coefficients.

C. The fundamental rule that the power of the parameter c, appearing in connection with Q_1, Q_2, Q_3, Q_4, should never be lower than the order of the subscript of the respective Q_n, is not obvious in Kalil's form of the final solutions.

2.2.4.3 Alternate Integration Procedure

For the reasons listed and in order to avoid the inherent, lengthy algebraic manipulations, the five integrals in Equation (2.267) will not be combined; rather, the integration of the individual terms in each of the five integrals will be performed directly (without converting the powers of cosE to multiple angles) by the use of the table of integrals presented on the next page in terms of modified Bessel functions of the zero and first orders.

The multiplication of the individual member integrals by their respective coefficients is indicated at the margin of the table of integrals. There are five column of coefficients corresponding to the five polynomials in Equations (2.267). It will be noted that not all of the coefficients of the last four polynomials appear at the margin of the table of integrals. The reason for this being that the products of the coefficients of the four polynomials (having the parameters Q_1, Q_2, Q_3, Q_4 as factors) by the corresponding integrals should not contain powers of c lower than the order of the subscript of the respective Q_n.

The integration is extended up to terms containing $\cos^9 E$. Kalil neglects in his derivations powers of $e > 2$. In this analysis, only powers of $e > 3$

** or if it is desired to neglect the oblateness of the atmosphere in which case $Q_1 = Q_2 = Q_3 = Q_4 = 0$

are neglected, except for the terms having Q_1 as factors (where powers of e^3 are not retained) and for terms having Q_2, Q_3, Q_4 as factors (where only the first power of e is retained).

Using this alternate integration procedure and the table of integrals, the integrated solutions of Equation (2.267) will assume the following form, once all terms having I_0 and I_1 as a factor are collected.

$$\frac{1}{2\pi}\int_0^{2\pi} e^{c\cos E}\, dE = I_0$$

$$\frac{1}{2\pi}\int_0^{2\pi} e^{c\cos E}\cos E\, dE = I_1$$

$$\frac{1}{2\pi}\int_0^{2\pi} e^{c\cos E}\cos^2 E\, dE = I_0 - \frac{I_1}{c}$$

$$\frac{1}{2\pi}\int_0^{2\pi} e^{c\cos E}\cos^3 E\, dE = -\frac{I_0}{c} + \left(1+\frac{2}{c^2}\right)I_1$$

$$\frac{1}{2\pi}\int_0^{2\pi} e^{c\cos E}\cos^4 E\, dE = \left(1+\frac{3}{c^2}\right)I_0 - \frac{2}{c}\left(1+\frac{3}{c^2}\right)I_1$$

	Q_1	Q_2	Q_3
α_0			
α_1			
α_2	$(-\alpha_0+3\alpha_0 e^2 + 2\alpha_1 e+\alpha_2)$		
α_3	$-(2\alpha_0 e+\alpha_1)$	$-2(4\alpha_0 e+\alpha_1)$	
α_4	$-(3\alpha_0 e^2+2\alpha_1 e + \alpha_2)$	α_0	$3\alpha_0$

	Q_2	Q_3	Q_4
$\frac{1}{2\pi}\int_0^{2\pi} e^{c\cos E}\cos^5 E\,dE = -\frac{2}{c}\left(1+\frac{6}{c^2}\right)I_0+\left(1+\frac{7}{c^2}+\frac{24}{c^4}\right)I_1$	$(4\alpha_0 e+\alpha_1)$	$3(6\alpha_0 e+\alpha_1)$	$6(8\alpha_0 e+\alpha_1)$
$\frac{1}{2\pi}\int_0^{2\pi} e^{c\cos E}\cos^6 E\,dE = \left(1+\frac{9}{c^2}+\frac{60}{c^4}\right)I_0-\frac{3}{c}\left(1+\frac{11}{c^2}+\frac{40}{c^4}\right)I_1$		$-\alpha_0$	$-4\alpha_0$
$\frac{1}{2\pi}\int_0^{2\pi} e^{c\cos E}\cos^7 E\,dE = -\frac{3}{c}\left(1+\frac{17}{c^2}+\frac{120}{c^4}\right)I_0 + \left(1+\frac{15}{c^2}+\frac{192}{c^4}+\frac{720}{c^6}\right)I_1$		$-(6\alpha e+\alpha_1)$	$-4(8\alpha_0 e+\alpha_1)$
$\frac{1}{2\pi}\int_0^{2\pi} e^{c\cos E}\cos^8 E\,dE = \left(1+\frac{18}{c^2}+\frac{345}{c^4}+\frac{2520}{c^6}\right)I_0 - \frac{4}{c}\left(1+\frac{24}{c^2}+\frac{330}{c^4}+\frac{1260}{c^6}\right)I_1$			α_0
$\frac{1}{2\pi}\int_0^{2\pi} e^{c\cos E}\cos^9 E\,dE = -\frac{4}{c}\left(1+\frac{33}{c^2}+\frac{675}{c^4}+\frac{5040}{c^6}\right)I_0 + 1+\frac{26}{c^2}+\frac{729}{c^4}+\frac{10440}{c^6}+\frac{40320}{c^8})$			$(8\alpha_0 e+\alpha_1)$

Thus, the general expression for the change in any single element assumes the form,

$$\dot{\nu}_{sec} = -\left(\frac{CONST}{P}\right) * 2\pi \rho_p e^{-c} \left\{ \left[a_0 + \frac{a_1}{c} a_1 + 3\frac{a_2}{c^2} a_2 + 15\frac{a_3}{c^3} a_3 + 105\frac{a_4}{c^4} a_4 \right] I_0(c) + \left[b_0 + \frac{a_1}{c} b_1 + 3\frac{a_2}{c^2} b_2 + 15\frac{a_3}{c^3} b_3 + 105\frac{a_4}{c^4} b_4 \right] I_1(c) \right\} \quad (2.268)$$

where,

$$a_0 = \alpha_0 + \alpha_2 + \alpha_4 - \frac{\alpha_3}{c} + \frac{3\alpha_4}{c^2}$$

$$a_1 = (2\alpha_0 e + \alpha_1) - \frac{3}{c}(3\alpha_0 e^2 + 2\alpha_1 e + \alpha_2)$$

$$a_2 = \alpha_0 - \frac{4}{c}(4\alpha_0 e + \alpha_1)$$

$$a_3 = (6\alpha_0 e + \alpha_1) - \frac{4}{c}\alpha_0 + \frac{24}{c^2}(6\alpha_0 e + \alpha_1)$$

$$a_4 = \alpha_0 - \frac{12}{c}(8\alpha_0 e + \alpha_1) + \frac{24}{c^2}\alpha_0 - \frac{192}{c^3}(8\alpha_0 e + \alpha_1)$$

$$b_0 = \alpha_1 + \alpha_3 - \frac{\alpha_2}{c} - \frac{2\alpha_4}{c} + \frac{2\alpha_3}{c^2} - \frac{6\alpha_4}{c^3}$$

$$b_1 = \alpha_0 + (3\alpha_0 e^2 + 2\alpha_1 e + \alpha_2) - \frac{2}{c}(2\alpha_0 e + \alpha_1) + \frac{6}{c^2}(3\alpha_0 e^2 + 2\alpha_1 e + \alpha_2)$$

$$b_2 = (4\alpha_0 e + \alpha_1) - \frac{2}{c}\alpha_0 + \frac{8}{c^2}(4\alpha_0 e + \alpha_1)$$

$$b_3 = \alpha_0 - \frac{8}{c}(6\alpha_0 e + \alpha_1) + \frac{8}{c^2}\alpha_0 - \frac{48}{c^3}(6\alpha_0 e + \alpha_1)$$

$$b_4 = (8\alpha_0 e + \alpha_1) - \frac{8}{c}\alpha_0 + \frac{72}{c^2}(8\alpha_0 e + \alpha_1) - \frac{48}{c^3}\alpha_0 + \frac{384}{c^4}(8\alpha_0 e + \alpha_1) \quad (2.269)$$

2.2.4.3.1 The α_n, a_n, b_n Coefficients for $\dot{a}_{sec}$

$$
\begin{aligned}
\alpha_0 &= 1 \\
\alpha_1 &= 2je \\
\alpha_2 &= e^2\left(j^2 + \tfrac{1}{2}\right) \\
\alpha_3 &= je^3 \\
\alpha_4 &= 0 \qquad\qquad (2.270) \\
CONST &= 2Ba^2(1-d)^2 \qquad\qquad j = \frac{1+d}{1-d}
\end{aligned}
$$

$$
\begin{aligned}
a_0 &= 1 + e^2\left(j^2 + \tfrac{1}{2}\right) - \frac{je^3}{c} \\
a_1 &= 2e(j+1) - \frac{3}{c}e^2\left(j^2 + 4j + \tfrac{7}{2}\right) \\
a_2 &= 1 - \frac{8}{c}e(j+2) \\
a_3 &= 2e(j+3) - \frac{4}{c} + \frac{48}{c^2}e(j+3) \\
a_4 &= 1 - \frac{24}{c}e(j+4) + \frac{24}{c^2} - \frac{384}{c^3}e(j+4) \\
b_0 &= 2je + je^3 - \frac{e^2}{c}\left(j^2 + \tfrac{1}{2}\right) + 2\frac{je^3}{c^2} \\
b_1 &= 1 + e^2\left(j^2 + 4j + \tfrac{7}{2}\right) - \frac{4}{c}e(j+1) + \frac{6}{c^2}e^2\left(j^2 + 4j + \tfrac{7}{2}\right) \\
b_2 &= 2e(j+2) - \frac{2}{c} + \frac{16}{c^2}e(j+2) \\
b_3 &= 1 - \frac{16}{c}e(j+3) + \frac{8}{c^2} - \frac{96}{c^3}e(j+3) \\
b_4 &= 2e(j+4) - \frac{8}{c} + \frac{144}{c^2}e(j+4) - \frac{48}{c^3} + \frac{768}{c^4}e(j+4) \qquad (2.271)
\end{aligned}
$$

2.2.4.3.2 The α_n, a_n b_n Coefficients for $\dot{P}_{sec}$

They are the same as for $\dot{a}_{sec}$, because

$$\dot{P} = \frac{3}{2} P \left(\frac{\dot{a}_{SEC}}{a}\right) \tag{2.272}$$

The constant factor for P is obtained as follows,

$$\left(\frac{CONST}{P}\right)_{\dot{P}} = \frac{3}{2} \frac{P}{a}\left(\frac{CONST}{P}\right)_{\dot{a}} = 3Ba(1-d)^2 \tag{2.273}$$

2.2.4.3.3 The α_n, a_n, b_n Coefficients for $\dot{e}_{sec}$

$$\begin{aligned}
\alpha_0 &= \frac{ed}{2}(1-d) \\
\alpha_1 &= (1-d)^2 + e^2d^2 \\
\alpha_2 &= \frac{ed}{2}(1-d)(2+5d) \\
\alpha_3 &= \frac{e^2}{2}(1+5d^2) \\
\alpha_4 &= \frac{e^3}{2} \\
CONST &= 2Ba(1-e^2)
\end{aligned} \qquad (2.274$$

$$a_0 = e(1-d)(1+3d) + \frac{e^3}{2} - \frac{e^2}{2c}(1+5d^2) + \frac{3}{2c^3}\, e^3$$

$$a_1 = (1-d)^2 + e^2d - \frac{3}{2C}\, e(1-d)(6+d)$$

$$a_2 = \frac{ed}{2}(1-d) - \frac{4}{C}\left[(1-d)^2 + e^2d(2-d)\right]$$

$$a_3 = (1-d)^2 + e^2d(3-2d) - \frac{2}{C}ed(1-d) + \frac{24}{C^2}\left[(1-d)^2 + e^2d(3-2d)\right]$$

$$a_4 = \frac{ed}{2}(1-d) - \frac{12}{C}\left[(1-d)^2 + e^2d(4-3d)\right] + \frac{12}{C^2}ed(1-d) - \frac{192}{C^3}\left[(1-d)^2 + e^2d(4-3d)\right]$$

$$b_0 = (1-d)^2 + \frac{e^2}{2}(1-7d^2) - \frac{e}{2c}(1-d)(2+5d) - \frac{e^3}{c} + \frac{e^2}{c^2}(1+5d^2) - \frac{3}{c^3}e^3$$

$$b_1 = e(1-d)(3+d) - \frac{2}{c}\left[(1-d)^2 + e^2 d\right] + \frac{3}{c^2}e(1-d)(6+d)$$

$$b_2 = (1-d)^2 + e^2 d(2-d) - \frac{ed}{c}(1-d) + \frac{8}{c^2}\left[(1-d)^2 + e^2 d(2-d)\right]$$

$$b_3 = \frac{ed}{2}(1-d) - \frac{8}{c}\left[(1-d)^2 + e^2 d(3-2d)\right] + \frac{4}{c^2}ed(1-d) - \frac{48}{c^3}\left[(1-d)^2 + e^2 d(3-2d)\right]$$

$$b_4 = \left[(1+d)^2 + e^2 d(4-3d)\right] - \frac{4}{c}ed(1-d) + \frac{72}{c^2}\left[(1-d)^2 + e^2 d(4-3d)\right] - \frac{24}{c^3}ed(1-d) + \frac{384}{c^4}\left[(1-d)^2 + e^2 d(4-3d)\right]$$

(2.275)

2.2.4.3.4 The α_n, a_n, b_n Coefficients for $\dot{\Omega}_{sec}$

$$\alpha_0 = -(1-d)(1-2e^2)$$
$$\alpha_1 = -2e$$
$$\alpha_2 = 2(1-d) - \frac{e^2}{2}(1+3d)$$
$$\alpha_3 = 4ed + e^3$$
$$\alpha_4 = -e^2(1+3d)$$
$$CONST = \frac{B a \Omega_e \sin 2\omega}{2n\sqrt{1-e^2}}$$

(2.276)

$$
\begin{aligned}
a_0 &= (1-d) + \frac{e^2}{2}(1-13d) - \frac{e}{c}(4d+e^2) - \frac{3}{c^2}e^2(1+3d) \\
a_1 &= -2e\left[1+(1-d)(1-2e^2)\right] - \frac{3}{c}\left[2(1-d) - \frac{3}{2}e^2(5-d)\right] \\
a_2 &= -(1-d)(1-2e^2) + \frac{8}{c}e\left[1+2(1-d)(1-2e^2)\right] \\
a_3 &= -2e\left[1+3(1-d)(1-2e^2)\right] + \frac{4}{c}(1-d)(1-2e^2) - \frac{48}{c^2}e\left[1+3(1-d)(1-2e^2)\right] \\
a_4 &= -(1-d)(1-2e^2) + \frac{24}{c}e\left[1+4(1-d)(1-2e^2)\right] - \frac{24}{c^2}(1-d)(1-2e^2) \\
&\quad + \frac{384}{c^3}e\left[1+4(1-d)(1-2e^2)\right]
\end{aligned}
$$

$$
\begin{aligned}
b_0 &= -2e(1-2d) + e^3 - \frac{2}{c}(1-d) + \frac{5}{2c}e^2(1+3d) + \frac{2}{c^2}e(4d+e^2) + \frac{6}{c^3}e^2(1+3d) \\
b_1 &= (1-d) - \frac{e^2}{2}(11+d) + \frac{4}{c}e\left[1+(1-d)(1-2e^2)\right] + \frac{6}{c^2}\left[2(1-d) - \frac{3}{2}e^2(5-d)\right] \\
b_2 &= -2e\left[1+2(1-d)(1-2e^2)\right] + \frac{2}{c}(1-d)(1-2e^2) - \frac{16}{c^2}e\left[1+2(1-d)(1-2e^2)\right] \\
b_3 &= -(1-d)(1-2e^2) + \frac{16}{c}e\left[1+3(1-d)(1-2e^2)\right] - \frac{8}{c^2}(1-d)(1-2e^2) + \frac{96}{c^3}e\left[1+3(1-d)(1-2e^2)\right] \\
b_4 &= -2e\left[1+4(1-d)(1-2e^2)\right] + \frac{8}{c}(1-d)(1-2e^2) - \frac{144}{c^2}e\left[1+4(1-d)(1-2e^2)\right] \\
&\quad + \frac{48}{c^3}(1-d)(1-2e^2) - \frac{768}{c^4}e\left[1+4(1-d)(1-2e^2)\right]
\end{aligned}
\tag{2.277}
$$

2.2.4.3.5 The α_n, a_n, b_n Coefficients for $\dot{i}_{sec}$

$$\alpha_0 = (1-d) \qquad + \cos 2\omega * (\alpha_0 \text{ OF } \dot{\Omega}_{SEC})$$

$$\alpha_1 = -2e(1-2d) + \cos 2\omega * (\alpha_1 \text{ OF } \dot{\Omega}_{SEC})$$

$$\alpha_2 = \frac{e^2}{2}(1-13d) + \cos 2\omega * (\alpha_2 \text{ OF } \dot{\Omega}_{SEC})$$

$$\alpha_3 = e^3 \qquad + \cos 2\omega * (\alpha_3 \text{ OF } \dot{\Omega}_{SEC})$$

$$\alpha_4 = 0 \qquad + \cos 2\omega * (\alpha_4 \text{ OF } \dot{\Omega}_{SEC})$$

$$\text{CONST} = \frac{Ba\,\Omega_e \sin i}{2n\sqrt{1-e^2}} \tag{2.278}$$

$$a_0 = (1-d) + \frac{e^2}{2}(1-13d) - \frac{e^3}{C} + (a_0 \text{ OF } \dot{\Omega}_{SEC}) * \cos 2\omega$$

$$a_1 = 2ed + \frac{3}{2C}e^2(1+3d) + (a_1 \text{ OF } \dot{\Omega}_{SEC}) * \cos 2\omega$$

$$a_2 = (1-d) - \frac{8}{C}e + (a_2 \text{ OF } \dot{\Omega}_{SEC}) \cos 2\omega$$

$$a_3 = 2e(2-d) - \frac{4}{C}(1-d) + \frac{48}{C^2}e(2-d) + (a_3 \text{ OF } \dot{\Omega}_{SEC}) * \cos 2\omega$$

$$a_4 = (1-d) - \frac{24}{C}e(3-2d) + \frac{24}{C^2}(1-d) - \frac{384}{C^3}e(3-2d) + (a_4 \text{ OF } \dot{\Omega}_{SEC}) * \cos 2\omega$$

$$b_0 = -2e(1-2d) + e^3 - \frac{e^2}{2C}(1-13d) + \frac{2}{C^2}e^3 + (b_0 \text{ OF } \dot{\Omega}_{SEC}) * \cos 2\omega$$

$$b_1 = (1-d) - \frac{e^2}{2}(1+3d) - \frac{4}{C}ed - \frac{3}{C^2}e^2(1+3d) + (b_1 \text{ OF } \dot{\Omega}_{SEC}) * \cos 2\omega$$

$$b_2 = 2e - \frac{2}{C}(1-d) + \frac{16}{C^2}e + (b_2 \text{ OF } \dot{\Omega}_{SEC}) * \cos 2\omega$$

$$b_3 = (1-d) - \frac{16}{C}e(2-d) + \frac{8}{C^2}(1-d) - \frac{96}{C^3}e(2-d) + (b_3 \text{ OF } \dot{\Omega}_{SEC}) * \cos 2\omega$$

$$b_4 = 2e(3-2d) - \frac{8}{C}(1-d) + \frac{144}{C^2}(3-2d) - \frac{48}{C^3}(1-d)$$

$$+ \frac{768}{C^4}e(3-2d) + (b_4 \text{ OF } \dot{\Omega}_{SEC}) * \cos 2\omega \tag{2.279}$$

2.2.4.3.6 The α_n, a_n, b_n Coefficients for $\dot{\omega}_{sec}$

They are the same as for $\dot{\Omega}_{sec}$ because

$$\dot{\omega}_{SEC} = -\cos i \, \dot{\Omega}_{SEC} \tag{2.280}$$

2.2.4.4 Reduction of Kalil's Solution of $\dot{a}_{sec}$ for Comparison

Kalil's original solution for $\dot{a}_{sec}$ is given in the following form,

$$\dot{a}_{SEC} = -2Ba^2(1-d)^2 * 2\pi\rho_p \, e^{-c}\left[B_0 I_0 + B_1 I_1 + B_2 I_2 + B_3 I_3 + B_4 I_4 + B_5 I_5\right] \tag{2.281}$$

However, reduction of the higher orders of the modified Bessel functions to the zero and first orders yields,

$$\begin{aligned}\dot{a}_{SEC} = -2Ba^2(1-d)^2 * 2\pi\rho_p \, e^{-c}\Big\{&\left[B_0 + B_2 - \frac{4}{C}B_3 + \left(1+\frac{24}{C^2}\right)B_4 - \frac{12}{C}\left(1+\frac{16}{C^2}\right)B_5\right]I_0(C) \\ &+ \left[B_1 - \frac{2}{C}B_2 + \left(1+\frac{8}{C^2}\right)B_3 - \frac{8}{c}\left(1+\frac{6}{C^2}\right)B_4 + \left(1+\frac{72}{c^2}+\frac{384}{C^4}\right)B_5\right]I_1(c)\Big\}\end{aligned} \tag{2.282}$$

$\dot{a}_{SEC}$	I_0	I_1
$B_0 = 1 + e^2(j^2 + \frac{1}{2})$	1	0
$B_1 = 2je - \frac{e^2}{C}(j^2 + \frac{1}{2}) + \frac{Q_1}{C} + \frac{Q_1}{C}e^2(j^2 + 4 \quad + \frac{7}{2})$	0	1
$B_2 = 2\frac{Q_1}{C}e(j+1) - 3\frac{Q_1}{C^2}e^2(j^2 + 4j + \frac{7}{2}) + 3\frac{Q_2}{C^2}$	1	$-\frac{2}{C}$
$B_3 = 6\frac{Q_2}{C^2}e(j+2) + 15\frac{Q_3}{C^3}$	$-\frac{4}{C}$	$(1 + \frac{8}{C^2})$
$B_4 = 30\frac{Q_3}{C^3}e(j+3) + 15\frac{Q_3}{C^4}e^2(j^2 + 12j + \frac{43}{2}) + 105\frac{Q_4}{C^4}$	$(1 + \frac{24}{C^2})$	$-\frac{8}{C}(1 + \frac{6}{C^2})$
$B_5 = 210\frac{Q_4}{C^4}e(j+4)$	$-\frac{12}{C}(1 + \frac{16}{C^2})$	$(1 + \frac{72}{C^2} + \frac{384}{C^4})$

Thus,

$$
\begin{aligned}
\dot{a}_{SEC} = -2Ba^2(1-d)^2 * 2\pi\rho_p e^{-c}\Bigg\{\Bigg[&1 + e^2(j^2 + \tfrac{1}{2}) + \frac{Q_1}{C}\Big\{2e(j+1) \\
&- \frac{3}{C}e^2(j^2 + 4j + \tfrac{7}{2})\Big\} + \frac{3Q_2}{C^2}\Big\{1 - \frac{8}{C}e(j+2)\Big\} + \\
&+ \frac{15Q_3}{C^3}\Big\{2e(j+3) - \frac{4}{C} + \frac{48}{C^2}e(j+3) + (1 + \frac{24}{C^2})\frac{e^2}{C}(j^2 + 12j + \tfrac{43}{2})\Big\} \\
&+ \frac{105Q_4}{C^4}\Big\{1 - \frac{24}{C}e(j+4) + \frac{24}{C^2} - \frac{384}{C^3}e(j+4)\Big\}\Bigg] I_0 \\
&+ \Bigg[2je - \frac{e^2}{C}(j^2 + \tfrac{1}{2}) + \frac{Q_1}{C}\Big\{1 + e^2(j^2 + 4j + \tfrac{7}{2}) - \frac{4}{C}e(j+1) \\
&+ \frac{6}{C^2}e^2(j^2 + 4j + \tfrac{7}{2})\Big\} + \frac{3Q_2}{C^2}\Big\{2e(j+2) - \frac{2}{C} + \frac{16}{C^2}e(j+2)\Big\} \\
&+ \frac{15Q_3}{C^3}\Big\{1 - \frac{16}{C}e(j+3) + \frac{8}{C^2} - \frac{96}{C^3}e(j+3) - \frac{8}{C^2}(1 + \frac{6}{C^2})e^2(j^2 + 12j + \tfrac{43}{2}) \\
&+ \frac{105Q_4}{C^4}\Big\{2e(j+4) - \frac{8}{C} + \frac{144}{C^2}e(j+4) - \frac{48}{C^3} + \frac{768}{C^4}e(j+4)\Big\}\Bigg] I_1\Bigg\}
\end{aligned}
$$

Kalil has a typographical error in the first term of the B_3 coefficient. He has $(6Q_2/c)$, while the correct form should be $(6Q_2/c^2)$. This fact is substantial by virtue by the argument that the power of c should never be lower than the order of the subscript of the respective Q_n coefficients. He also has an extraneous e^2 term as multiplier of Q_3 in the B_4 coefficient, which is odd since e^2 terms do not have to appear even in connection with Q_2.

2.3 THE EFFECT OF LUNI-SOLAR PERTURBATIONS ON THE ORBIT OF AN EARTH SATELLITE

2.3.1 Basic Review of the Problem

2.3.1.1 Definition of the Disturbing Force

By the attraction of the disturbing force, both the earth and the satellite obtain an acceleration in the direction of the disturbing body. Hence, the specific disturbing force, acting on the artificial earth satellite, is equal to the geometric difference (vector difference) of the direct disturbing attraction acting on the satellite and the indirect disturbing attraction by which the satellite would be acted upon if it were placed at the earth's center.

The disturbing acceleration $\bar{Q}$ of the moon or sun is normally defined by**

$$\bar{Q} = -G m_D \left[\frac{\bar{r}}{\rho^3} - \bar{r}_D \left(\frac{1}{\rho^3} - \frac{1}{r_D^3} \right) \right]$$

where $\bar{r}$, $\bar{r}_D$ are the geocentric position vectors of the satellite and the disturbing body, respectively, ρ is the distance from the disturbing body to the satellite, G the universal gravitational constant, and m_D the mass of the disturbing body.

Defining by $\bar{R}$ and $\bar{D}$ to be the unit directions along the position vectors $\bar{r}$ and $\bar{r}_D$, the vector $\bar{Q}$ can be written in a more convenient form for subsequent resolution into components.

$$\bar{Q} = -G m_D r \left[\frac{\bar{R}}{\rho^3} - \left(\frac{r_D}{r} \right) \left(\frac{1}{\rho^3} - \frac{1}{r_D^3} \right) \bar{D} \right]$$

The parameter $1/p^3$ will now be eliminated through the use of the law of cosines,

$$\frac{1}{\rho^3} = \frac{1}{r_D^3} \left[1 + \left(\frac{r}{r_D} \right)^2 - 2 \left(\frac{r}{r_D} \right) \cos \Phi \right]^{-3/2}$$

** The Symbol Q was also used to denote the perturbing potential of the Earth where Φ is the angle subtended by the unit vectors $\bar{R}$ and $\bar{D}$; that is, $\bar{R} \cdot \bar{D} = \cos \Phi$. Binomial expansion of the function in the brackets, now yields the relation in its most useable form

$$\frac{1}{\rho^3} = \frac{1}{r_D^3}\left[1 + 3\left(\frac{r}{r_D}\right)\cos\Phi - \frac{3}{2}\left(\frac{r}{r_D}\right)^2\left(1 - 5\cos^2\Phi\right)\right]$$

Substituting $1/\rho^3$ from this expansion and setting $K = Gm_D/r_D^3$,

$$\bar{Q} = Kr\left\{\left[1 + 3\left(\frac{r}{r_D}\right)\cos\Phi\right]\bar{R} - \left[3\cos\Phi - \frac{3}{2}\left(\frac{r}{r_D}\right)\left(1 - 5\cos^2\Phi\right)\right]\bar{D}\right\}$$

where powers of (r/r_D) higher than 1 are neglected. It is assumed that the geocentric distance r of the satellite is never greater than approximately 1/10 of the earth-moon distance r_D. (If greater distances are assumed, additional terms must be considered).

The disturbing force $\bar{Q}$ will now be resolved in the directions $\bar{R}$, $\bar{S}$, $\bar{W}$; where $\bar{R}$ is a unit vector along $\bar{r}$, $\bar{S}$ is a unit vector in the osculating plane 90° ahead from $\bar{R}$, and $\bar{W}$ is the unit normal to the osculating plane of the satellite. If (A, B, C) are the direction cosines of the unit vector $\bar{D}$ (the pointing of the disturbing body), relative to the $\bar{N}$, $\bar{M}$, $\bar{W}$ orbital frame of the satellite, and u* the argument of latitude of the satellite, then

$$\bar{D} = \bar{N}A + \bar{M}B + \bar{W}C$$

$$\bar{R} = \bar{N}\cos u^* + \bar{M}\sin u^*$$

$$\bar{S} = -\bar{N}\sin u^* + \bar{M}\cos u^*$$

Thus, the components of the disturbing force $\bar{Q}$ in the $\bar{R}$, $\bar{S}$, $\bar{W}$ directions are,

$$R = -Kr\left[(1 - 3\cos^2\Phi) + \frac{3}{2}\left(\frac{r}{r_D}\right)\cos\Phi\,(3 - 5\cos^2\Phi)\right]$$

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$$S = 3Kr\left[\cos\Phi - \frac{1}{2}\left(\frac{r}{r_D}\right)\left(1-5\cos^2\Phi\right)\right]\left(-A\sin u^* + B\cos u^*\right)$$

$$W = 3Kr\left[\cos\Phi - \frac{1}{2}\left(\frac{r}{r_D}\right)\left(1-5\cos^2\Phi\right)\right](C)$$

where

$$\cos\Phi = A\cos u^* + B\sin u^*$$

2.3.1.2 The Effect of the Disturbing Force on Orbital Decay

The magnitude of the effect of the disturbing body on the orbit of an artificial earth satellite depends on the position of the disturbing body in its orbit. The disturbing effect of the sun (or the moon) also depends on the orientation of the satellite orbit and its nodal position with respect to the orbital plane of the disturbing body. These effects on orbital precession of the satellite may support of oppose the effects of each other or the earth's perturbative tendency. Thus, the apsidal rotation, caused by luni-solar disturbing forces, is by far more complex than in the case of perturbations caused by earth oblateness. First of all, the perigee does not move uniformly; secondly, the apogee moves differently and in a less pronounced manner.

Fortunately from the standpoint of most analyses for close Earth satellites, the uncertainties in the coefficients of the earth's potential function overshadow these perturbations, thus they can generally be neglected. However, these effects become more and more significant with increasing distance of the satellite's orbit from the earth's center. For highly eccentric satellite orbits with apogee radius of about 1/10 of the Earth-moon distance, the affect of luni-solar perturbations on nodal precession and apsidal rotation approaches rapidly the order of magnitude of the effect of Earth oblateness.

The significance of luni-solar perturbations on the orbit of an artificial Earth satellite was pointed out by Kozai after a detailed examination of the orbit of Vanguard I. He found that the perigee height displayed a significant periodic variation which could be attributed neither to atmospheric drag effect nor to any of the harmonics of the Earth's gravitational field. Such perturbations may become particularly significant when resonance occurs; that is, when perigee moves in step with the sun and the moon. In such cases, a progressive change in perigee height may amount to the order of 1 NM per day over a period of several years. When resonance occurs, the eccentricity is the most important orbital element, since any change in it affects the perigee radius, which influences the satellite's lifetime. By expanding the expression of the averaged rate of change of the ecdentricity over a revolution and including all the perturbing influences in it, G. E. Cook has determined the 15 possibilities for resonance to occur.

2.3.2 Review of the Available Literature

2.3.2.1 General Comments on the Papers Reviewed

The effects of the disturbance of a third body on the orbit of an artificial Earth satellite have been investigated recently in several papers. Most of these papers, however, are subject to certain limitations or are applicable to circular orbits only. Some authors (e.g., Spitzer, Reference 3.1) use the simplified lunar theory and, thereby, introduced the assumptions inherent to such theory of small eccentricity and small inclination of the satellite's orbit to the orbit of the disturbing body. Most authors, however, use general perturbation techniques applicable to artificial satellites, without limitations as to eccentricity and orbital inclination of the satellite. However, some of these papers give explicit expressions only for the secular terms [Kozai (Reference 3.2), Blitzer (Reference 3.3), Lorell (Reference 3.4)] while others fail to give general results [Musen (Reference 3.5), Upton (Reference 3.6), Bailie and Musen (Reference 3.7)] and concern themselves with the effects on particular satellites.

The greatest inconvenience of almost all the papers, from the point of view of applicability and of combining the perturbation effects due to various disturbing forces, lies in the choice of the reference plane which, in most cases, is taken as the plane of the disturbing body [Moe (Referencd 3.8), Geyling (Reference 3.9), Penzo (Reference 3.10), etc.]. The inconvenience of such a reference plane increases with the number of disturbing bodies. Since, in each case, a different reference plane and consequently transformation of variables must be used. Only a few of the papers use the inertial earth-equatorial system as the reference frame and, among them, the paper of G. E. Cook (Reference 3.11) deserves special attention.

As a rule, it appears that all the theories on the subject are of first order, the assumption being that the ratio of the satellite's radial distance (r) from the center of the earth to the earth-moon distance (r_D) ≤ 0.1, so that all terms greater than the first power in (r/r_D) may be neglected in the expansion of the disturbing function. A further simplification is introduced by assuming that the disturbing body is fixed during one revolution of the satellite. Such simplification makes possible the integration of the rates of change of the osculating elements and is justified by the fact that the mean motion of the satellite is by far greater than the mean motion of the disturbing body. However, care must be taken to assure that the results are accurate for satellite motions near resonance. For these cases (if not for all), a time average position for the disturbing body should be employed.

2.3.2.2 Methods and Techniques

The method most commonly used is an extension of the general perturbations theory. The rates of change of the osculating elements are defined by Lagrange's planetary equations in terms of the R. S, W components of the disturbing force, and the respective changes per revolution caused by the disturbing force are obtained in closed form by direct integration with respect to the true anomaly. The great majority of the pepers assume that the disturbing body is fixed during one revolution of the satellite and, then, restrict the applicability of their methods to the case when the geocentric orbital radius of the satellite $\leq 1/10$ of the earth-moon distance.

A paper by A. V. Egorova (Reference 3.12), also based on the principles of the general perturbations theory, uses a degree of sophistication which, by all standards, appears to be questionable and inefficient. This paper expands the disturbing function in powers of eccentricity of the satellite (this step is quite unreasonable for many applications due to the large values of the eccentricity involved). The author then tires to avoid the difficulty by performing the integration "by parts" with respect to the eccentric anomaly and using the true anomaly of the disturbing body as the variable of differentiation. For the sun, the integration by parts is done only once; for the moon, the integration is done twice. In both cases, the residual integrals are neglected.

A few authors, like Geyling, use the Hamiltonian approach and present the effects of the luni-solar disturbing forces in terms of variations in the satellite's position. The disturbing body in these analyses is not assumed fixed during one revolution of the satellite. Rather, circular orbits are considered for these bodies with respect to the Earth.

In all of the papers reviewed, with the exception of those by Cook and Kozai, the luni-solar perturbations on the satellite orbit are evaluated with respect to the orbital plane of the disturbing body as the plane of reference; that is, the lunar orbit plane and the ecliptic. This approach constitutes a great inconvenience since, for most of the cases of interest, the main source of perturbation is due to the oblateness of the Earth. For these cases, it would be necessary to determine the respective perturbations caused by each disturbing body (sun, moon) over a revolution of the satellite, resolve these perturbations individually into a common reference frame (the inertial earth-equatorial frame), add the resultant perturbations, and adjust the orbital elements before continuing the process for the next revolution of the satellite.

2.3.2.3 Integration Procedures

To make possible the analytical integration of the rates of change of the osculating elements, the disturbing body is assumed to be fixed during one revolution of the satellite and the time argument in Lagrange's definitions of the rates of change is eliminated in favor of the true anomaly through the relation,

$$dt = \frac{r^2}{h} d\eta$$

where h is the angular momentum per unit mass and η is the true anomaly.

The integration is performed with respect to η over one revolution of the satellite. Normally, both secular and periodic terms are combined, but the two types can be separated at the expense of a reasonable amount of algebraic manipulations, if desired. However, some authors give explicit expressions for the secular terms but fail to give expressions for the periodic terms.

The solutions are generally of the first order. However, G. E. Cook does include second-order terms for the argument of perigee. It is of interest that the solutions are functions of the direction cosiness (A, B, C) of the disturbing body pointing with respect to the $\overline{N}$, $\overline{M}$, $\overline{W}$ orbital frame of the satellite, where $\overline{N}$ is the node of the satellite's orbital plane at the reference plane.

2.3.2.4 Critical Evaluation of the Papers Reviewed

2.3.2.4.1 The Method Based on General Perturbations

This theory is based on general perturbations principles and the integration of Lagrange's planetary equations with respect to the true anomaly over a revolution of the satellite. The perturbations in the osculating elements are evaluated either in the inertial earth-equatorial frame of reference or with respect to the orbital plane of the disturbing body.

2.3.2.4.1.1 The Work of G. E. Cook (Reference 3.11). Assumptions: The disturbing body is fixed during one revolution of the satellite; the ratio (r/r_D) of the geocentric radial distance of the satellite to that of the disturbing body $\leq 1/10$.

Completeness: Complete first order theory with respect to (r/r_D) of the disturbing function; higher order terms included only for the argument of perigee, no explicit expressions are given for the secular perturbations; rather, they are combined together with the periodic perturbations in the solutions. The changes in the orbital elements are evaluated with respect to the inertial earth-equatorial frame of reference.

Evaluation: The analysis is simple, easy to follow, and provides clear geometrical interpretation of the problem, the solutions are concise and meaningful. The greatest advantage of Cook's work lies in the choice of the earth-equatorial inertial system as the reference frame. The fact that he does not separate the secular and periodic terms is not a serious disadvantage, since this can be easily accomplished. However, no real need for such separation seems to exist. It would not be a difficult problem even to extend the analysis to include higher order terms for all of the osculating elements, rather than for the argument of perigee only.

2.3.2.4.1.2 The Work of Y. Kozai (Reference 3.2). Assumptions: The geocentric radial distance of the satellite is very small as compared with that of the moon (Kozai does not say how small; however, ratios less than 1/10 should satisfy this restriction). The first term of the disturbing function may be neglected. The inclination of the orbital plane of the disturbing body to the earth's equator is constant over a year.

Completeness: Incomplete first-order theory, since solutions for the secular perturbations only are given. They are evaluated with respect to the earth-equatorial frame of reference.

Evaluation: Analysis of this paper is hampered by the lack of definitions and by the lengthy form of the disturbing function. However, evaluation is further complicated by the fact that no indication is given as to whether the disturbing body is or is not assumed fixed during one revolution of the satellite. Finally, no reference is made as to how the argument of latitude and the nodal longitude of the disturbing body (the moon) are to be determined. (The inclination of the lunar orbit to the earth's equator is defined in terms of the inclination and the node relative to the ecliptic and the obliquity and is assumed constant over a year. This assumption is not encountered in any other theory.) These factors notwithstanding, the greatest disadvantage of Kozai's paper is its incompleteness, since explicit solutions are presented for the secular rates of change only.

2.3.2.4.1.3 The Work of J. Lorell (Reference 3.4). Assumptions: No assumptions are specified in the paper. However, from a comparison of Lorell's results with those for the secular changes reported in other papers, it was deduced that the usual first-order theory assumptions were made; that is, the disturbing body is fixed during one revolution of the satellite, and the geocentric radius of the satellite r is much smaller than that of the disturbing body.

Completeness: Incomplete first order theory. Only expressions for the secular changes in the osculating elements are presented with respect to the orbital plane of the disturbing body.

Evaluation: The paper lists only the secular rates of change in the osculating elements. No derivations are presented. Neither the assumptions nor the form of the disturbing function are spelled out. The bulk of the paper is devoted to the graphical description of orbit behavior. Thus, from the analytical point of view, Lorell's paper is of little appeal.

2.3.2.4.1.4 The Work of M. Moe (Reference 3.8). Assumptions: The disturbing body is fixed during one revolution of the satellite; the geocentric radius of the satellite is $\leq 1/10$ of the earth-moon distance.

Completeness: Complete first-order theory. Solutions incorporating both secular and periodic changes are given for all orbital elements, except the mean anomaly. The reference plane is the orbital plane of the disturbing body.

Evaluation: The analysis is simple and based on the geometrical interpretation of the problem. An estimate of the error due to the neglect of higher order terms in the expansion of the disturbing function is also presented. Both the periodic and secular terms are combined together in the solutions. If it were not for the inconvenient choice of the reference plane which is the orbital plane of the disturbing body, Moe's paper would be most appropriate for engineering purposes.

2.3.2.4.1.5 The Work of P. Penzo (Reference 3.10). Assumptions: r_D (geocentric radius of the disturbing body) is much greater than r (the geocentric radius of the satellite) and the disturbing body is fixed during the interval of variation.

Completeness: A complete first-order theory presenting the combined secular and periodic perturbations in all osculating elements, with the exception of the mean anomaly. These perturbations are first evaluated with respect to the plane defined by the pointing of the satellite's perigee and that of the disturbing body and, then they are transformed into the frame defined by the orbital plane of the disturbing body, the X-axis being in the direction of the satellite's ascending node.

Evaluation: The greatest disadvantage of Penzo's paper lies in the inconvenient and peculiar frame of reference with respect to which the changes in the osculating elements are evaluated. The transformation from this reference frame to that of the orbital plane of the disturbing body and/or to the earth's equatorial frame is very cumbersome. Further, the transformed solutions relative to the former frame do not provide a clear geometrical interpretation.

2.3.2.4.2 The Method Based on Hamiltonian Canonical Equations

This theory is based on Hamilton's canonical equations and a time dependent moving coordinate frame always centered at the satellite's position in unperturbed motion. The effect of the disturbing force is presented in terms of variations in the satellite's position in Cartesian coordinates of the moving frame.

2.3.2.4.2.1 The Work of F. T. Geyling (Reference 3.9). Assumptions: Circular motion of the satellite; the ratio (r/r_D) of the geocentric radial distance of the satellite to that of the disturbing body is assumed small, but it is not specified how small (again, a ratio of approximately 1/10 should suffice as an upper limit).

Completeness: A first-order theory in Cartesian position coordinates ξ, η, ζ relative to the moving, time dependent, frame whose origin always coincides with the position of the satellite in unperturbed elliptic motion. ξ points radially outward and η is in the plane of the nominal orbit in the direction of anomalistic motion. Any perturbation in the path will result in satellite displacements about the origin of this moving frame. The effect of the variation in ζ is to change the orientation of the orbital plane. The effects of the variations in ξ and η are restricted to changes in orbit shape and timing. The disturbing body is not assumed fixed during one revolution of the satellite; and, therefore, the disturbing function is time dependent. No explicit expressions are given for the changes in the osculating elements.

Evaluation: Geyling's paper may be considered outstanding inasmuch as the disturbing body is not considered fixed during one revolution of the satellite. However, the greatest disadvantage lies in the failure to present explicit expressions for the changes in the osculating elements. Even more serious is the limitation which restricts application to circular orbits.

2.3.2.5 Selection of Paper for Detailed Development

Based on a review of these papers, the following paper was selected for detailed analytical development, G. E. Cook: "Luni-Solar Perturbations of the Orbit of an Earth Satellite," The Geophysical Journal of the Royal Astronomical Society, April 1962, Vol. 6, No. 3. This solution is the result of the observation that the analysis is complete, rigorous and straightforward.

Further, the changes in the osculating elements are evaluated with respect to the inertial earth-equatorial frame of reference making it possible to combine these changes with those produced by other perturbation forces.

2.3.3 Analytical Development of G. E. Cook's Approach

2.3.3.1 The Disturbing Force Due to a Third Body

Let $\bar{r}$ and $\bar{r}_D$ be the geocentric position vectors of the satellite and the disturbing body, respectively, and $\bar{\rho}$ the position vector of the satellite with respect to the disturbing body, so that,

$$\bar{\rho} = \bar{r} - \bar{r}_D \tag{3.1}$$

Also, let $\bar{R}_S$ and $\bar{R}_E$ define the positions of the satellite and the earth with respect to the center of masses O (see Figure 2), so that,

$$\bar{r} = \bar{R}_S - \bar{R}_E \tag{3.2}$$

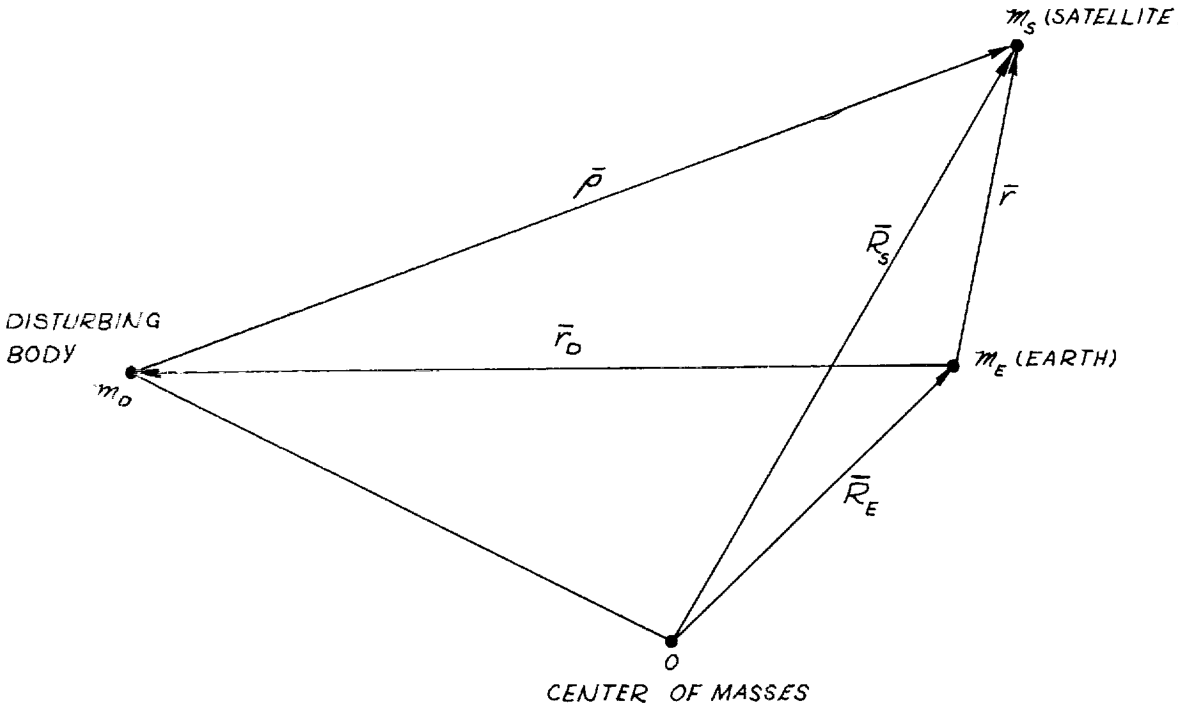


Figure 2. Geometry of the Three-Body Problem

The satellite is attracted by the disturbing body of mass m_D and by the earth of mass m_E. The differential equation of the satellite, including these two forces can be written as,

$$m_S \ddot{\bar{R}}_S = -G m_D m_S \frac{\bar{\rho}}{\rho^3} - G m_E m_S \frac{\bar{r}}{r^3} \tag{3.3}$$

where m_S is the mass of the satellite and G is the universal gravitational constant. Similarly, the earth is attracted by the disturbing body and the satellite, and the differential equation of the earth, including these two forces, is of the form,

$$m_E \ddot{\bar{R}}_E = G m_D m_E \frac{\bar{r}_D}{r_D^3} + G m_S m_E \frac{\bar{r}}{r^3} \tag{3.4}$$

Thus, subtracting (3.4) from (3.3), and differentiating (3.2), yields,

$$\ddot{\bar{r}} + G(m_E + m_S)\frac{\bar{r}}{r^3} = -G m_D \left(\frac{\bar{\rho}}{\rho^3} + \frac{\bar{r}}{r_D^3}\right) \qquad (3.5)$$

Denoting the right hand side of this equation as

$$\bar{Q} = -G m_D \left(\frac{\bar{\rho}}{\rho^3} + \frac{\bar{r}_D}{r_D^3}\right) \qquad (3.6)$$

and replacing $\bar{\rho}$ by relation (3.1), yields,

$$\bar{Q} = -G m_D\left[\frac{\bar{r}}{\rho^3} - \bar{r}_D\left(\frac{1}{\rho^3} - \frac{1}{r_D^3}\right)\right] = -G m_D\, r\left[\frac{\bar{R}}{\rho^3} - \left(\frac{r_D}{r}\right)\left(\frac{1}{\rho^3} - \frac{1}{r_D^3}\right)\bar{D}\right] \qquad (3.7)$$

where $\bar{R}$ and $\bar{D}$ are unit vectors in the directions of $\bar{r}$ and $\bar{r}_D$, respectively. Now, from the law of cosines,

$$\rho^2 = r_D^2 + r^2 - 2 r r_D \cos\Phi = r_D^2\left[1 + \left(\frac{r}{r_D}\right)^2 - 2\left(\frac{r}{r_D}\right)\cos\Phi\right]$$

whence,

$$\frac{1}{\rho^3} = \frac{1}{r_D^3}\left[1 + \left(\frac{r}{r_D}\right)^2 - 2\left(\frac{r}{r_D}\right)\cos\Phi\right]^{-3/2} = \frac{1}{r_D^3}\left[1 + 3\left(\frac{r}{r_D}\right)\cos\Phi - \frac{3}{2}\left(\frac{r}{r_D}\right)^2\left(1 - 5\cos^2\Phi\right)\right] \qquad (3.8)$$

Thus, substituting (3.8) in (3.7) and neglecting powers of $(r/r_D) > 1$, yields

$$\bar{Q} = -K r\left\{\left[1 + 3\left(\frac{r}{r_D}\right)\cos\Phi\right]\bar{R} - \left[3\cos\Phi - \frac{3}{2}\left(\frac{r}{r_D}\right)\left(1 - 5\cos^2\Phi\right)\right]\bar{D}\right\} \qquad (3.9)$$

where,

$$K = \frac{G m_D}{r_D^3} \qquad (3.10)$$

and Φ is the angle subtended by the vectors $\bar{r}$ and $\bar{r}_D$ or $\bar{R}$ and $\bar{D}$.

In order to determine cos Φ and the components of the disturbing force $\bar{Q}$ in the $\bar{R}$, $\bar{S}$, $\bar{W}$ directions (which will be defined later), the following preliminary derivations are required: Let $\bar{N}_D$, $\bar{M}_D$, $\bar{W}_D$ be the unit pointings of the orbital frame of the disturbing body; $\bar{N}_D$ being the node of the orbit of the disturbing body at the earth's equator,

$$\begin{aligned} \bar{N}_D &= \bar{i}\cos\Omega_D + \bar{j}\sin\Omega_D + \bar{k}(0) \\ \bar{M}_D &= -\bar{i}\cos i_D \sin\Omega_D + \bar{j}\cos i_D \cos\Omega_D + \bar{k}\sin i_D \\ \bar{W}_D &= \bar{i}\sin i_D \sin\Omega_D - \bar{j}\sin i_D \cos\Omega_D + \bar{k}\cos i_D \end{aligned} \tag{3.11}$$

where $\bar{i}$, $\bar{j}$, $\bar{k}$ represent the unit vectors of the earth-equatorial inertial frame; $\bar{i}$ pointing to the vernal equinox and $\bar{k}$ aligned with the earth's spin axis. (i_D, Ω_D) are the orientation elements of the orbital plane of the disturbing body relative to the earth's equator.

From the ephemeris, the geocentric position and velocity vectors $\bar{r}_D$, $\dot{\bar{r}}_D$ of the disturbing body are obtained, from which the pointing of the unit normal $\bar{W}_D$ is determined,

$$\bar{W}_D = \frac{\bar{r}_D \times \dot{\bar{r}}_D}{|\bar{r}_D \times \dot{\bar{r}}_D|} = i X_w + j Y_w + k Z_w \tag{3.12}$$

Comparison of the respective components of $\bar{W}_D$ in (3.11) and (3.12) yields,

$$\begin{aligned} \cos i_D &= Z_w \\ \sin\Omega_D &= \frac{X_w}{\sin i_D} \\ \cos\Omega_D &= -\frac{Y_w}{\sin i_D} \end{aligned} \tag{3.13}$$

Since the geocentric vector $\bar{r}_D$ determines the right ascension α_D and declination δ_D of the disturbing body, the argument of latitude u_o can be calculated as follows:

$$\begin{aligned} \cos u_o &= \cos\delta_D \cos(\alpha_D - \Omega_D) \\ \sin u_o &= \frac{\sin\delta_D}{\sin i_D} \end{aligned} \tag{3.14}$$

Now the unit direction $\bar{D}$ of the disturbing body can be defined in the $\bar{N}_D \bar{M}_D \bar{W}_D$ frame,

$$\bar{D} = \bar{N}_D \cos u_D + \bar{M}_D \sin u_D + \bar{W}(0) \tag{3.15}$$

Substitution of the definitions of $\bar{N}_D$ and $\bar{M}_D$ from (3.11) into (3.15) yields,

$$\begin{aligned} D = {} & i\,(\cos u_D \cos \Omega_D - \cos i_D \sin u_D \sin \Omega_D) \\ & + j\,(\cos u_D \sin \Omega_D + \cos i_D \sin u_D \cos \Omega_D) + k \sin i_D \sin u_D \end{aligned} \tag{3.16}$$

Let $\bar{N}$, $\bar{M}$, $\bar{W}$ be the orbital frame of the satellite; $\bar{N}$ is the respective node at the earth's equator,

$$\begin{aligned} \bar{N} &= \bar{i} \cos \Omega + \bar{j} \sin \Omega + \bar{k}(0) \\ \bar{M} &= -\bar{i} \cos i \sin \Omega + \bar{j} \cos i \cos \Omega + \bar{k} \sin i \\ \bar{W} &= \bar{i} \sin i \sin \Omega - \bar{j} \sin i \cos \Omega + \bar{k} \cos i \end{aligned} \tag{3.17}$$

The unit pointing $\bar{D}$ of the disturbing body (as defined by Equation 3.16) is now transformed to the orbital frame $\bar{N}$, $\bar{M}$, $\bar{W}$ of the satellite, by forming the dot products of $\bar{D}$, by $\bar{N}$, $\bar{M}$, $\bar{W}$, respectively.

$$\begin{aligned} \bar{D} = {} & \bar{N}\Big[\cos u_D \cos(\Omega - \Omega_D) + \cos i_D \sin u_D \sin(\Omega - \Omega_D)\Big] \\ & + \bar{M}\Big[\cos i \Big\{-\cos u_D \sin(\Omega - \Omega_D) + \cos i_D \sin u_D \cos(\Omega - \Omega_D)\Big\} \\ & \qquad + \sin i\,(\sin i_D \sin u_D)\Big] \\ & + \bar{W}\Big[\sin i \Big\{\cos u_D \sin(\Omega - \Omega_D) - \cos i_D \sin u_D \cos(\Omega - \Omega_D)\Big\} \\ & \qquad + \cos i\,(\sin i_D \sin u_D)\Big] \\ = {} & \bar{N}A + \bar{M}B + \bar{W}C \end{aligned} \tag{3.18}$$

Now the $\bar{R}$, $\bar{S}$, $\bar{W}$ directions can be defined in the $\bar{N}$, $\bar{M}$, $\bar{W}$ orbital frame of the satellite,

$$\bar{R} = \bar{N}\cos u^* + \bar{M}\sin u^*$$
$$\bar{S} = -\bar{N}\sin u^* + \bar{M}\cos u^* \qquad (3.19)$$
$$\bar{W} = \bar{W}$$

where u* is the argument of latitude of the satellite. Finally, the direction cosines of the unit vector $\bar{D}$, as defined by (3.18) with respect to the $\bar{R}$, $\bar{S}$, $\bar{W}$ pointings given by (3.19), are

$$\cos\Phi = \bar{D}\cdot\bar{R} = A\cos u^* + B\sin u^*$$
$$\bar{D}\cdot\bar{S} = -A\sin u^* + B\cos u^* \qquad (3.20)$$
$$\bar{D}\cdot\bar{W} = C$$

The components of the disturbing force $\bar{Q}$, which is defined by Equation (3.9), in the $\bar{R}$, $\bar{S}$, $\bar{W}$ directions can now be obtained by forming the dot products of $\bar{Q}$ and $\bar{R}$, $\bar{S}$, $\bar{W}$ and then making use of relations (3.20). Let R, S, W be the components of $\bar{Q}$ in the directions of $\bar{R}$, $\bar{S}$, $\bar{W}$, then,

$$R = \bar{Q}\cdot\bar{R} = -Kr\left[(1-3\cos^2\Phi) + \frac{3}{2}\left(\frac{r}{r_0}\right)\cos\Phi\,(3-5\cos^2\Phi)\right]$$
$$= -Kr\left\{1-\frac{3}{2}(A^2+B^2) - 3AB\sin 2u^* - \frac{3}{2}(A^2-B^2)\cos 2u^* \right. \qquad (3.21)$$
$$\left. + \frac{3}{2}\left(\frac{r}{r_0}\right)(A\cos u^* + B\sin u^*)\left[3-5(A\cos u^* + B\sin u^*)^2\right]\right\}$$

$$S = \bar{Q}\cdot\bar{S} = 3Kr\left[\cos\Phi - \frac{1}{2}\left(\frac{r}{r_0}\right)(1-5\cos^2\Phi)\right](\bar{D}\cdot\bar{S})$$
$$(3.22)$$
$$= 3Kr\left\{AB\cos 2u^* - \frac{1}{2}(A^2-B^2)\sin 2u^* \right.$$
$$\left. + \frac{1}{2}\left(\frac{r}{r_0}\right)(A\sin u^* - B\cos u^*)\left[1-5(A\cos u^* + B\sin u^*)^2\right]\right\}$$

$$W = \bar{Q}\cdot\bar{W} = 3Kr\left[\cos\Phi - \frac{1}{2}\left(\frac{r}{r_D}\right)(1-5\cos^2\Phi)\right](\bar{D}\cdot\bar{W})$$

$$= 3KCr\left\{(A\cos u^* + B\sin u^*) - \frac{1}{2}\left(\frac{r}{r_D}\right)\left[1-5(A\cos u^* + B\sin u^*)^2\right]\right\} \tag{3.23}$$

Hence, the vector $\bar{Q}$ assumes the following form in the $\bar{R}$, $\bar{S}$, $\bar{W}$, frame of reference.

$$\begin{aligned}\bar{Q} = 3Kr\Big\{&\left[AB\sin 2u^* + \frac{1}{2}(A^2-B^2)\cos 2u^* + \frac{1}{2}(A^2+B^2) - \frac{1}{3}\right]\bar{R}\\ &+\left[AB\cos 2u^* - \frac{1}{2}(A^2-B^2)\sin 2u^*\right]\bar{S} + C(A\cos u^* + B\sin u^*)\bar{W}\\ &+\frac{1}{2}\left(\frac{r}{r_D}\right)\Big[-(A\cos u^* + B\sin u^*)\left[3-5(A\cos u^* + B\sin u^*)^2\right]\bar{R}\\ &+(A\sin u^* - B\cos u^*)\left[1-5(A\cos u^* + B\sin u^*)^2\right]\bar{S}\\ &-C\left[1-5(A\cos u^* + B\sin u^*)^2\right]\bar{W}\Big]\Big\}\end{aligned} \tag{3.24}$$

Note that the first part of $\bar{Q}$ represents first order effects; the second part represents second order effects. Also note that, by Eq. (3.18),

$$\begin{aligned} A &= \cos u_D \cos(\Omega-\Omega_D) + \cos i_D \sin u_D \sin(\Omega-\Omega_D)\\ B &= \cos i\left[-\cos u_D \sin(\Omega-\Omega_D) + \cos i_D \sin u_D \cos(\Omega-\Omega_D)\right]\\ &\quad + \sin i\,(\sin i_D \sin u_D)\\ C &= \sin i\left[\cos u_D \sin(\Omega-\Omega_D) - \cos i_D \sin u_D \cos(\Omega-\Omega_D)\right]\\ &\quad + \cos i\,(\sin i_D \sin u_D)\end{aligned} \tag{3.25}$$

A, B, C, are the direction cosines of the pointing of the disturbing body (unit vector $\bar{D}$) with respect to the geocentric $\bar{N}$, $\bar{M}$, $\bar{W}$, orbital frame of the satellite, where $\bar{N}$ is the node of the satellite's orbital plane at the earth's equator. In the definitions of A, B, C, the non-subscripted elements i and Ω pertain to the satellite and the ones with subscript "D" pertain to the orbit of the disturbing body. All the orbital elements of both the satellite and the disturbing body are defined with respect to the $\bar{i}$, $\bar{j}$, $\bar{k}$, geocentric (inertial) earth-equatorial frame.

2.3.3.2 Lagrange's Planetary Equations

Lagrange's planetary equations express the rate of change of the osculating elements in terms of the components of the perturbing force R, S, W. R is the component in the direction of the geocentric radius vector of the satellite, S is at right angle to R in the osculating plane, and W is normal to the osculating plane.

The Lagrange equations are:

$$\frac{da}{dt} = \frac{2a^2}{h}\left[e \sin\eta\, R + (1+e\cos\eta)\, S\right]$$

$$\frac{de}{dt} = \frac{p}{h}\left[(\sin\eta) R + \left(\cos\eta + \frac{\cos\eta + e}{1+e\cos\eta}\right) S\right]$$

$$\frac{di}{dt} = \frac{r\cos u^*}{h} W$$

$$\frac{d\Omega}{dt} = \frac{r\sin u^*}{h\sin i} W \qquad (3.26)$$

$$\frac{d\omega}{dt} = -\frac{p}{he}\left[(\cos\eta) R - \left(1 + \frac{1}{1+e\cos\eta}\right)\sin\eta\, S\right] - \cos i\, \frac{d\Omega}{dt}$$

$$\frac{dM}{dt} = n - \sqrt{1-e^2}\left[\frac{2}{h} r R + \left(\frac{d\omega}{dt} + \cos i\, \frac{d\Omega}{dt}\right)\right]$$

where h is the angular momentum per unit mass, n is the mean motion, u* the argument of latitude, and η is the true anomaly of the satellite.

2.3.3.3 Integration of the Time Rates of the Osculating Elements

Cook's theory is a simplified first order theory which neglects (except in the calculations of the change in the argument of perigee) the second part of the disturbing function Q (as defined by Equation 3.24), which has (r/r_D) as a factor. Further, it is assumed that the angular velocities (mean motions) of the sun and moon are small enough, as compared to the angular velocity of the satellite, to consider the disturbing body (sun and moon) to be fixed during one revolution of the satellite. This implies $r \ll r_D$. In fact, the theory is limited to satellites for which $r/r_D \leq 1/10$; that is, the radial distance of the satellite from the earth's center should not exceed one-tenth of the moon's distance from the earth. The simplifying assumption, that the disturbing body is fixed during one revolution of the satellite, makes possible the integration of the instantaneous rates of change of the orbital elements over one revolution of the satellite to obtain the respective changes per revolution.

The error incurred by neglecting the higher order part of the disturbing function $\overline{Q}$ (Equation 3.24) is of the order $(r/r_D)^{3/2}$. Since $(r/r_D) < 0.1$, the error is less than 3 percent. The error incurred by assuming that the body is fixed for one satellite period can be largely eliminated by placing the body at the time average position for the interval in question.

As a first step in the solution, the time argument in Lagrange's equations is replaced by the true anomaly through the relation,

$$h = r^2\left(\frac{d\eta}{dt} + \frac{d\omega}{dt} + \cos i \,\frac{d\Omega}{dt}\right) \tag{3.27}$$

whence,

$$\frac{dt}{d\eta} = \frac{\frac{r^2}{h}}{1 - \frac{r^2}{h}\left(\frac{d\omega}{dt} + \cos i \,\frac{d\Omega}{dt}\right)} \tag{3.28}$$

However, since the main changes in ω and Ω are those due to the earth's gravitational field, and these changes are proportional to J (second harmonic), the retaining of $(d\omega/dt + \cos i \; d\Omega/dt)$ in Equation (3.28) would result in

coupling with the effects of the gravitational field of the disturbing body, and such coupling effects are not considered in this first order theory. Hence, within the range of accuracy of Cook's theory, it will be valid to set,

$$dt = \frac{r^2}{h} d\eta$$

In the ensuing integration process, the integrals of the following functions are needed,

$$(1+e\cos\eta)^{-3} = (1+3e^2+\frac{45}{8}e^4) - 3e(1+\frac{5}{2}e^2)\cos\eta + 3e^2(1+\frac{5}{2}e^2)\cos 2\eta$$
$$-\frac{5}{2}e^3\cos 3\eta + \frac{15}{8}e^4\cos 4\eta \qquad (3.29)$$

$$(1+e\cos\eta)^{-4} = (1+5e^2+\frac{105}{8}e^4) - 4e(1+\frac{15}{4}e^2)\cos\eta + 5e^2(1+\frac{7}{2}e^2)\cos 2\eta$$
$$-5e^3\cos 3\eta + \frac{35}{8}e^4\cos 4\eta$$

$$\begin{aligned}
\int_0^{2\pi}(1+e\cos\eta)^{-3}\cos\eta\, d\eta &= -(2\pi)\,\frac{3}{2}e(1+\frac{5}{2}e^2) \\
\int_0^{2\pi}(1+e\cos\eta)^{-3}\cos 2\eta\, d\eta &= (2\pi)\,\frac{3}{2}e^2(1+\frac{5}{2}e^2) \\
\int_0^{2\pi}(1+e\cos\eta)^{-4}\cos\eta\, d\eta &= -(2\pi)\,2e(1+\frac{15}{4}e^2) \\
\int_0^{2\pi}(1+e\cos\eta)^{-4}\cos 2\eta\, d\eta &= (2\pi)\,\frac{5}{2}e^2(1+\frac{7}{2}e^2) \\
\int_0^{2\pi}(1+e\cos\eta)^{-4}\cos 3\eta\, d\eta &= -(2\pi)\,\frac{5}{2}e^3
\end{aligned} \qquad (3.30)$$

2.3.3.3.1 The Change in the Semi-Major Axis

$$\begin{aligned}
da &= 2a^2\,\frac{r^2}{h^2}\Big[Re\sin\eta + S(1+e\cos\eta)\Big]d\eta \\
&= \frac{6Ka^2}{h^2}r^3\Big\{\Big[AB\cos 2\omega - \frac{1}{2}(A^2-B^2)\sin 2\omega\Big]e\sin\eta\sin 2\eta \\
&\quad -\frac{e}{3}\sin\eta\Big[1-\frac{1}{2}(A^2+B^2)\Big] + \Big[AB\cos 2\omega - \frac{1}{2}(A^2-B^2)\sin 2\omega\Big](1+e\cos\eta)\cos 2\eta\Big\}d\eta
\end{aligned} \qquad (3.31)$$

$$\Delta a = \frac{6Ka(1-e^2)^2}{n^2}\Big[AB\cos 2\omega - \frac{1}{2}(A^2-B^2)\cos 2\omega\Big]\int_0^{2\pi}(1+e\cos\eta)^{-3}(e\cos\eta+\cos 2\eta)\,d\eta$$

By relations (3.30),

$$\int_0^{2\pi} e\cos\eta(1+e\cos\eta)^{-3}d\eta + \int_0^{2\pi}\cos 2\eta(1+e\cos\eta)^{-3}d\eta = 0$$

Therefore,

$$\Delta a = 0 \tag{3.32}$$

2.3.3.3.2 The Change in Eccentricity

$$\begin{aligned}
de &= p\,\frac{r^2}{h^2}\left[R\sin\eta + S\left(\cos\eta + \frac{\cos\eta + e}{1+e\cos\eta}\right)\right]d\eta \\
&= \frac{3Kp}{h^2}r^3\left\{\left[AB\cos 2\omega - \frac{1}{2}(A^2-B^2)\sin 2\omega\right]\sin\eta\sin 2\eta - \frac{1}{3}\sin\eta\left[1-\frac{1}{2}(A^2-B^2)\right]\right. \\
&\quad + \left[AB\cos 2\omega - \frac{1}{2}(A^2-B^2)\sin 2\omega\right]\cos\eta\cos 2\eta \\
&\quad \left. + \left[AB\cos 2\omega - \frac{1}{2}(A^2-B^2)\sin 2\omega\right]\left(\frac{\cos\eta\cos 2\eta + e\cos 2\eta}{1+e\cos\eta}\right)\right\} d\eta
\end{aligned} \tag{3.33}$$

$$\begin{aligned}
\Delta e &= \frac{3K(1-e^2)^3}{n^2}\left[AB\cos 2\omega - \frac{1}{2}(A^2-B^2)\sin 2\omega\right]\int_0^{2\pi}\left[(1+e\cos\eta)^{-3}\cos\eta\right. \\
&\quad \left. + (1+e\cos\eta)^{-4}\left(\frac{1}{2}\cos\eta + \frac{1}{2}\cos 3\eta + e\cos 2\eta\right)\right]d\eta
\end{aligned}$$

By relations (3.30), the value of this integral is:

$$2\pi\left[-\frac{5}{2}e\left(1+\frac{5}{2}e^2\right)\right] \sim 2\pi\left[-\frac{5}{2}e(1-e^2)^{-5/2}\right]$$

$$\Delta e = -\frac{15\pi K e\sqrt{1-e^2}}{n^2}\left[AB\cos 2\omega - \frac{1}{2}(A^2-B^2)\sin 2\omega\right] \tag{3.34}$$

2.3.3.3.3 The Change in Inclination

$$\begin{aligned}
di &= \frac{r^3}{h^2}W\cos u^* d\eta = \frac{3KC}{h^2}r^4\left[A\cos u^* + B\sin u^*\right]\cos u^* d\eta \\
&= \frac{3KC}{2h^2}r^4\left[A + (A\cos 2u^* + B\sin 2u^*)\right]d\eta \\
&= \frac{3KC}{2h^2}r^4\left[A + (A\cos 2\omega + B\sin 2\omega)\cos 2\eta\right]d\eta
\end{aligned} \tag{3.35}$$

$$\Delta i = \frac{3KC(1-e^2)^3}{2n^2}\int_0^{2\pi}\left[A(1+e\cos\eta)^{-4} + (A\cos 2\omega + B\sin 2\omega)\cos 2\eta(1+e\cos\eta)^{-4}\right]d\eta$$

By relations (3.30),

$$\int_0^{2\pi} (1+e\cos\eta)^{-4} d\eta = 2\pi\left(1+5e^2+\tfrac{105}{8}e^4\right) \sim 2\pi\left(1+\tfrac{3}{2}e^2\right)\left(1-e^2\right)^{-7/2}$$

$$\int_0^{2\pi} (1+e\cos\eta)^{-4}\cos 2\eta\, d\eta = 2\pi\tfrac{5}{2}e^2\left(1+\tfrac{7}{2}e^2\right) \sim 2\pi\tfrac{5}{2}e^2(1-e^2)^{-7/2}$$

Therefore,

$$\Delta i = \frac{3\pi KC}{2n^2\sqrt{1-e^2}}\left[A(2+3e^2)+5e^2(A\cos 2\omega + B\sin 2\omega)\right] \quad (3.36)$$

2.3.3.3.4 The Change in Nodal Longitude

$$\begin{aligned} d\Omega &= \frac{r^3}{h^2} W \frac{\sin u}{\sin i} d\eta = \frac{3KC}{h^2}\frac{r^4}{\sin i}\left[A\cos u^* + B\sin u^*\right]\sin u^* d\eta \\ &= \frac{3KC}{2h^2}\frac{r^4}{\sin i}\left[(A\sin 2u^* - B\cos 2u^*)+B\right]d\eta \\ &= \frac{3KC}{2h^2}\frac{r^4}{\sin i}\left[B+(A\sin 2\omega - B\cos 2\omega)\cos 2\eta\right]d\eta \end{aligned} \quad (3.37)$$

$$\Delta\Omega = \frac{3KC(1-e^2)^3}{2n^2\sin i}\int_0^{2\pi}\left[B(1+e\cos\eta)^{-4} + (A\sin 2\omega - B\cos 2\omega)\cos 2\eta(1+e\cos\eta)^{-4}\right]d\eta$$

Using relations (3.30) one obtains,

$$\Delta\Omega = \frac{3\pi KC}{2n^2\sqrt{1-e^2}\sin i}\left[B(2+3e^2)+5e^2(A\sin 2\omega - B\cos 2\omega)\right] \quad (3.38)$$

2.3.3.3.5 The Change in Argument of Perigee

$$\begin{aligned} d\omega + \cos i\, d\Omega &= -\frac{p}{e}\frac{r^2}{h^2}\left[R\cos\eta - S\left(1+\frac{1}{1+e\cos\eta}\right)\sin\eta\right]d\eta \\ &= -\frac{3Kp}{h^2 e} r^3\Big\{\left[AB\sin 2\omega + \tfrac{1}{2}(A^2-B^2)\cos 2\omega\right]\cos\eta\cos 2\eta \\ &\quad -\tfrac{1}{3}\left[1-\tfrac{3}{2}(A^2+B^2)\right]\cos\eta \quad +\left[AB\sin 2\omega + \tfrac{1}{2}(A^2-B^2)\cos 2\omega\right]\Big[\sin\eta\sin 2\eta \\ &\quad + \frac{\sin\eta\sin 2\eta}{1+e\cos\eta}\Big]\Big\}d\eta \end{aligned} \quad (3.39)$$

$$\Delta\omega + \cos i\,\Delta\Omega = -\frac{3K(1-e^2)^3}{n^2 e}\Big\{\Big[AB\sin 2\omega + \tfrac{1}{2}(A^2-B^2)\cos 2\omega\Big] X$$

$$\int_0^{2\pi}\Big[(1+e\cos\eta)^{-3}\cos\eta + (1+e\cos\eta)^{-4}\big(\tfrac{1}{2}\cos\eta - \tfrac{1}{2}\cos 3\eta\big)\Big]d\eta$$

$$-\tfrac{1}{3}\Big[1-\tfrac{3}{2}(A^2+B^2)\Big]\int_0^{2\pi}(1+e\cos\eta)^{-3}\cos\eta\, d\eta\Big\}$$

By relations (3.30),

The first integral $= -(2\pi)\frac{5}{2}e\left(1+\frac{5}{2}e^2\right) \sim -(2\pi)\,\frac{5}{2}e\left(1-e^2\right)^{-5/2}$

The second integral $= -(2\pi)\,\frac{3}{2}e\left(1+\frac{5}{2}e^2\right) \sim -(2\pi)\,\frac{3}{2}e\left(1-e^2\right)^{-5/2}$

Therefore

$$\Delta\omega + \cos i\,\Delta\Omega = \frac{3\pi K\sqrt{1-e^2}}{n^2}\Big\{5\Big[AB\sin 2\omega + \tfrac{1}{2}(A^2-B^2)\cos 2\omega\Big] - \Big[1-\tfrac{3}{2}(A^2+B^2)\Big]\Big\} + O\Big[\Big(\frac{r}{r_0}\Big)^2\Big] \quad (3.40)$$

The second-order terms of the disturbing function $\overline{Q}$ become significant for the argument of perigee, especially for moderately small eccentricities.

$$O\Big[\Big(\frac{r}{r_0}\Big)^2\Big]_{TERM} = \frac{15\pi K}{2n^2 e}\Big(\frac{a}{r_0}\Big)(A\cos\omega + B\sin\omega)\Big[1-\tfrac{5}{4}(A^2+B^2)\Big] \quad (3.41)$$

2.3.3.3.6. The Change in Mean Anomaly

$$dM = n\,dt = -\sqrt{1-e^2}\left[\frac{2r^3}{k^2} R\,d\eta + (\Delta\omega + \cos i\;\Delta\Omega)\right]$$

$$2\,\frac{r^3}{k^2} R\,d\eta = \frac{6K}{k^2}\, r^4 \left\{\left[AB\sin 2u^* + \frac{1}{2}(A^2-B^2)\cos 2u^*\right] - \frac{1}{3}\left[1 - \frac{3}{2}(A^2+B^2)\right]\right\} d\eta \quad (3.42)$$

$$= \frac{6K}{k^2}\, r^4 \left\{\left[AB\sin 2\omega + \frac{1}{2}(A^2-B^2)\cos 2\omega\right]\cos 2\eta - \frac{1}{3}\left[1 - \frac{3}{2}(A^2+B^2)\right]\right\} d\eta$$

$$\int_0^{2\pi} 2\,\frac{r^3}{k^2} R\,d\eta = \frac{6K(1-e^2)^3}{n^2}\left\{\left[AB\sin 2\omega + \frac{1}{2}(A^2-B^2)\cos 2\omega\right]\int_0^{2\pi}(1+e\cos\eta)^{-4}\cos 2\eta\,d\eta - \frac{1}{3}\left[1-\frac{3}{2}(A^2+B^2)\right]\int_0^{2\pi}(1+e\cos\eta)^{-4}d\eta\right\}$$

By relations (3.30),

The first integral $= (2\pi)\frac{5}{2}e^2\left(1+\frac{7}{2}e^2\right) \sim (2\pi)\frac{5}{2}e^2(1-e^2)^{-7/2} \sim (2\pi)\frac{5}{2}\frac{e^2}{1-e^2}(1-e^2)^{-5/2}$

The second integral $= (2\pi)\left(1+5e^2+\frac{105}{8}e^4\right) \sim (2\pi)(1-e^2)^{-5} \sim (2\pi)\left(1+\frac{5}{2}e^2\right)(1-e^2)^{-5/2}$

$$\int_0^{2\pi} 2\,\frac{r^3}{k^2} R\,d\eta = \frac{3\pi K\sqrt{1-e^2}}{n^2}\left\{\left[AB\sin 2\omega + \frac{1}{2}(A^2-B^2)\cos 2\omega\right](10e^2) - \frac{4}{3}\left[1-\frac{3}{2}(A^2+B^2)\right]\left(1+\frac{5}{2}e^2\right)\right\} \quad (3.43)$$

Hence, upon introducing the value of ($\Delta\omega$ + cos i $\Delta\Omega$) from (3.40) and (3.41) one has,

$$\Delta M = n\ell + \frac{3\pi K(1-e^2)}{n^2}\left\{\left[1-\frac{3}{2}(A^2+B^2)\right]\frac{7+10e^2}{3} - 10e^2\left[AB\sin 2\omega + \frac{1}{2}(A^2-B^2)\cos 2\omega\right]\right\} + O\left[\left(\frac{r}{r_0}\right)^2\right] \quad (3.44)$$

2.3.3.3.7 The Change in Perigee Radius

The change in perigee radius, Δr_p, is obtained from the already established fact that $\Delta a = 0$, and the relation,

$$r_p = a(1-e)$$

from which, it follows that,

$$\Delta r_p = -a \Delta e \tag{3.45}$$

By inspection of the expression for Δe, as given by Eq. (3.34), and the S component of the disturbing acceleration, as given by Eq. (3.24), it is observed that Δr_p can be expressed in an alternate form as a function of the S-component evaluated at perigee,

$$S_p = 3Kr_p\left[AB\cos 2\omega - \tfrac{1}{2}(A^2-B^2)\sin 2\omega\right] \tag{3.46}$$

Hence, by (3.45), (3.34), and (3.46), one has that

$$\Delta r_p = \frac{5\pi e}{n^2}\left(\frac{a}{r_p}\right)\sqrt{1-e^2}\, S_p = \frac{5\pi e}{n^2}\sqrt{\frac{1+e}{1-e}}\, S_p \tag{3.47}$$

Since $\Delta a = 0$, the mean motion, n and the period P do not change to the accuracy of this simplified first order theory.

2.4 THE EFFECT OF SOLAR RADIATION PRESSURE ON THE ORBIT OF AN EARTH SATELLITE

2.4.1 Basic Review of the Problem

2.4.1.1 Definition of the Perturbing Force

The mechanical action of solar radiation pressure on reflective and absorbing bodies can be interpreted on the basis of either the electromagnetic or quantum theory. According to the Quantum theory, solar radiation can be interpreted as a flux of photons; each photon has an energy $h\nu$ and a momentum $\frac{h\nu}{c}$ where h is Planck's constant, ν is the frequency and c is the speed of light. If N is the number of photons which fall on a unit surface normal to the sun's ray in a unit time, and S is the corresponding energy, then $N = \frac{S}{h\nu}$, and the momentum imparted by the photons to a unit surface in a unit time is

$$F = N\left(\frac{h\nu}{c}\right) = \left(\frac{S}{h\nu}\right) * \left(\frac{h\nu}{c}\right) = \frac{S}{c}$$

in units of $\left[\frac{MASS}{LENGTH * (TIME)^2}\right]$ provided the surface absorbs all of the photons.

If the surface partially reflects the incident radiation, then the reflected photons which carry momentum in the opposite sense will impart to the surface an additional momentum $\alpha\frac{S}{c}$, where α is the reflection coefficient which depends on the reflection properties of the surface and which may vary between 0 (absolute black body and 1 (specular surface). Hence, the total radiation pressure on a unit surface (normal to the incident ray) per unit time, when $\alpha \neq 0$, will be $\frac{S}{c}$ $(1 + \alpha)$. In general, when the ray falls under an incident angle γ to the surface, the radiation pressure F is,

$$F = \frac{S}{c}(1+\alpha)\cos^2\gamma \left[\frac{MASS}{LENGTH * (TIME)^2}\right]$$

Let S_o be the power of solar radiation on a unit of the earth's surface per unit time, called the solar constant, and d_o the mean distance earth-sun. If d is the distance of the satellite from the sun, the corresponding solar constant S at this distance will be given by,

$$S = S_o\left(\frac{d_o}{d}\right)^2$$

Upon substitution of this expression for S, one has that,

$$F = \frac{S_o}{c}\left(1+\alpha\right)\left(\frac{d_o}{d}\right)^2 \cos^2\gamma \left[\frac{MASS}{LENGTH*(TIME)^2}\right]$$

The acceleration experienced by a body, of mass m and effective cross-sectional area A, under the influence of solar radiation pressure is determined as follows,

$$\mathcal{F} = \left(\frac{A}{m}\right)F = \left(\frac{A}{m}\right)\left(\frac{S_o}{c}\right)(1+\alpha)\left(\frac{d_o}{d}\right)^2 \cos^2\gamma \cdots \frac{LENGTH}{(TIME)^2}$$

The ratio $\frac{A}{m}$ is constant for a spherical satellite,
For a non-spherical satellite, both $\frac{A}{m}$ and γ vary with the orientations of the sun and of the satellite.

In vector notation, the perturbing acceleration $\bar{\mathcal{F}}$ can be defined as,

$$\bar{\mathcal{F}} = -\mathcal{F}\,\bar{D}$$

where $\bar{D}$ is a unit vector in the direction earth-sun, given by,

$$\begin{aligned}\bar{D} &= i\cos\Lambda_D + j\cos\epsilon\sin\Lambda_D + K\sin\epsilon\sin\Lambda_D \\ &= i\left(\cos^2\frac{\epsilon}{2} + \sin^2\frac{\epsilon}{2}\right)\cos\Lambda_D + j\left(\cos^2\frac{\epsilon}{2} - \sin^2\frac{\epsilon}{2}\right)\sin\Lambda_D \\ &\quad + K\sin\epsilon\sin\Lambda_D\end{aligned}$$

Λ_D is the true celestial longitude of the sun and ϵ is the obliquity.

The pointings in the inertial earth-equatorial frame of perigee $\bar{P}$, the direction $\bar{Q}$ normal to $\bar{P}$ and the unit normal $\bar{W}$ are defined as follows,

$$\begin{aligned}\bar{P} &= i\left(\cos\omega\cos\Omega - \cos i\sin\omega\sin\Omega\right) + j\left(\cos\omega\sin\Omega\right. \\ &\quad \left.+ \cos i\sin\omega\cos\Omega\right) + K\sin i\sin\omega \\ &= i\left[\cos^2\frac{i}{2}\cos(\omega+\Omega) + \sin^2\frac{i}{2}\cos(\omega-\Omega)\right] \\ &\quad + j\left[\cos^2\frac{i}{2}\sin(\omega+\Omega) - \sin^2\frac{i}{2}\sin(\omega-\Omega)\right] + K\sin i\sin\omega\end{aligned}$$

$$\bar{Q} = i\,(-\sin\omega\cos\Omega - \cos i\cos\omega\sin\Omega) + j(-\sin\omega\sin\Omega + \cos i\cos\omega\cos\Omega)$$
$$+ K\sin i\cos\omega$$
$$= i\left[-\cos^2\frac{i}{2}\sin(\omega+\Omega) - \sin^2\frac{i}{2}\sin(\omega-\Omega)\right]$$
$$+ J\left[+\cos^2\frac{i}{2}\cos(\omega+\Omega) - \sin^2\frac{i}{2}\cos(\omega-\Omega)\right] + K\sin i\cos\omega$$

$$\bar{W} = i\sin i\sin\Omega - j\sin i\cos\Omega + K\cos i$$

The direction cosines A*, B*, C* of $\overline{D}$ (earth-sun pointing) in the $\overline{P}$, $\overline{Q}$, $\overline{W}$ frame are obtained by dot products which, after some algebraic manipulation, yield,

$$A^* = \cos^2\frac{i}{2}\cos^2\frac{\epsilon}{2}\cos(\Lambda_D - \Omega - \omega) + \cos^2\frac{i}{2}\sin^2\frac{\epsilon}{2}\cos(\Lambda_D + \Omega + \omega)$$
$$+ \sin^2\frac{i}{2}\cos^2\frac{\epsilon}{2}\cos(\Lambda_D - \Omega + \omega) + \sin^2\frac{i}{2}\sin^2\frac{\epsilon}{2}\cos(\Lambda_D + \Omega - \omega)$$
$$+ \frac{1}{2}\sin i\sin\epsilon\left[\cos(\Lambda_D - \omega) - \cos(\Lambda_D + \omega)\right]$$

$$B^* = \cos^2\frac{i}{2}\cos^2\frac{\epsilon}{2}\sin(\Lambda_D - \Omega - \omega) - \cos^2\frac{i}{2}\sin^2\frac{\epsilon}{2}\sin(\Lambda_D + \Omega + \omega)$$
$$- \sin^2\frac{i}{2}\cos^2\frac{\epsilon}{2}\sin(\Lambda_D - \Omega + \omega) + \sin^2\frac{i}{2}\sin^2\frac{\epsilon}{2}\sin(\Lambda_D + \Omega - \omega)$$
$$+ \frac{1}{2}\sin i\sin\epsilon\left[\sin(\Lambda_D - \omega) + \sin(\Lambda_D + \omega)\right]$$

$$C^{*} = -\sin i \cos^{2}\frac{\varepsilon}{2} \sin(\Lambda_D - \Omega) + \sin i \sin^{2}\frac{\varepsilon}{2} \sin(\Lambda_D + \Omega)$$
$$+ \cos i \sin\varepsilon \sin\Lambda_D$$

Hence, the unit vector $\bar{D}$ will have the following form in the $\bar{P}$, $\bar{Q}$, $\bar{W}$ frame

$$\bar{D} = \bar{P}A^{*} + \bar{Q}B^{*} + \bar{W}C^{*}$$

and the perturbing vector acceleration $\bar{\mathcal{F}}$ in the same frame will have the following definition,

$$\bar{\mathcal{F}} = -\mathcal{F}\bar{D} = -\mathcal{F}(\bar{P}A^{*} + \bar{Q}B^{*} + \bar{W}C^{*})$$

Now, let $\bar{R}$ be a unit vector in the direction of the geocentric position vector of the satellite, and $\bar{S}$ a unit vector normal to $\bar{R}$ in the osculating plane. Then, in the $\bar{P}$, $\bar{Q}$, $\bar{W}$ frame, $\bar{R}$ and $\bar{S}$ are defined in terms of the true anomaly η as,

$$\bar{R} = \bar{P}\cos\eta + \bar{Q}\sin\eta$$

$$\bar{S} = -\bar{P}\sin\eta + \bar{Q}\cos\eta$$

Finally, the components of the perturbing acceleration $\bar{\mathcal{F}}$ in the direction of $\bar{R}$, $\bar{S}$, $\bar{W}$ can be determined as,

$$R = \bar{\mathcal{F}} \cdot \bar{R} = -\mathcal{F}(A^{*}\cos\eta + B^{*}\sin\eta)$$

$$S = \bar{\mathcal{F}} \cdot \bar{S} = -\mathcal{F}(-A^{*}\sin\eta + B^{*}\cos\eta)$$

$$W = \bar{\mathcal{F}} \cdot \bar{W} = -\mathcal{F}C^{*}$$

2.4.1.2 The Effect of the Perturbing Force on Orbit Decay

An earth satellite is, among other things, subjected to both the gravitational force of the sun and to the solar radiation pressure force. These two forces act in preceisely opposite directions. Since the earth experiences nearly the same solar gravitational acceleration as does the satellite, the net effect of the geocentric force, due to solar gravitation during one revolution of the satellite, is very small whereas, in the case of radiation pressure effect, the resultant acceleration of the satellite is considerably higher than the corresponding acceleration of the earth and is strongly dependent on the area-to-mass ratio of the satellite. Therefore, the radiation pressure effect has to be taken into account.

The effect of solar radiation pressure becomes significant at orbital altitudes above 500 NM, and is particularly emphasized for balloon-type satellites for which the area-to-mass ratio is large. There is a marked difference between the solar radiation pressure force and the solar gravitational force in that the former becomes a discontinuous function of time when the satellite enters the earth's shadow. If the satellite is continuously in sunlight, the force is continuous. In this case, the short-periodic perturbation effects could be neglected. However, there is the possibility that such effects will be cumulative. Therefore, the short-periodic terms must be retained in the solutions. There are, however, no secular orbital variations due to direct solar radiation pressure though the amplitudes of some long period variations may become significantly large and may represent the principal contribution to orbit decay. Finally, it is noted that all elements, except the semi-major axis, contain long period terms.

While the effect of solar radiation pressure on ordinary satellite is very small, it may produce significant changes in the perigee height of satellites with high area-to-mass ratios. For certain resonant conditions, this effect accumulates monotonically and drastically affects the satellite's lifetime. (The change in perigee height, due to the influence of solar radation pressure was up 3.7 miles per day for the 100 foot Echo balloon, and up to 0.7 miles per day for the 12 foot Beacon satellite). In general, during a complete orbital period, solar radiation pressure causes a first order perturbation of all six orbital elements. However, when the entire orbit is in sunlight, solar radiation pressure has no effects on the semi-major axis or on the orbital period. When the orbit is not entirely in sunlight, the semi-major axis, as well, is affected by short period variations, therefore, this element is not subject to significant perturbations.

For certain combinations of orbital altitudes and inclinations, the effect of solar radiation pressure builds up monotonically, seriously affecting the orbit lifetime. There are in all 15 possibilities when resonance may take place. However, most interesting resonance occurs when the perigee of the satellite moves in step with the sun. In this case, oblateness keeps approximately constant the angle between the perigee pointing and the projection of the earth-sun line onto the orbital plane is approximately constant due to the earth's oblateness.

Hence, solar radiation pressure can increase or decrease the eccentricity monotonically. The critical argument for this type of resonance, in the terminology of celestial mechanics, is $(\Omega + \omega - \Lambda_s)$; Λ_s is the celestial longitude of the earth-sun line, $(\Lambda_s - \Omega)$ is the sun's longitude with respect to the line of nodes, and ω is the perigee longitude with respect to the same line. The condition for this resonance is: $(\dot{\Omega} + \dot{\omega} - \dot{\Lambda}_s) = 0$. When resonance occurs, the eccentricity is the most important orbital element, since any change in it affects the perigee radius which, in turn, influences the satellite's lifetime.

It is interesting that a circular orbit (in a first order theory) remains circular under the influence of the gravitational attraction of a third body, but tends to become ecliptic under the influence of solar radiation pressure. This can be explained by the fact that the gravitational attraction of the sun acts on both the earth and the satellite but, due to the small area-to-mass ratio of the earth, solar radiation pressure affects only the satellite significantly.

2.4.2 Review of the Available Literature

2.4.2.1 General Comments on the Papers Reviewed

The literature on the effect of solar radiation pressure on the orbit of an earth satellite is very limited because, until recently, it was considered that this effect was negligibly small as compared to the influence of earth oblateness and the effects of luni-solar gravitation. The few available papers to date were prompted by the need of explaining the discrepancies between theory and observations of satellites with high area-to-mass ratios. Papers on the subject were written by Musen (Reference 4.1), Musen-Bryant-Bailie (Reference 4.2), Parkinson-Jones-Shapiro (Reference 4.3), Cook (Reference 4.4), Kozai (Reference 4.5), Wyatt (Reference 4.6), and Geyling (Reference 4.7).

The work by Musen appears to excell that of the others in that it includes resonance for the case when the perigee moves in step with the sun. However, he neglects the effect of the earth's shadow. The work by Parkinson and associates is limited to the discussion of nearly circular orbits, the amplitude of the perigee height oscillation for a special case of resonance and, in particular, to the displacement of the geometric center of such orbits. Parkinson, however, includes the effect of the earth's shadow. Cook's paper presents an analytical technique for the evaluation of the perturbations in the osculating elements, but does not indulge in a discussion or an assessment of the physical and geometrical aspects of the problem. This is true also of the work by Kozai. Both Cook and Kozai consider the effect of the earth's shadow. Wyatt restricts himself to the investigation of the solar radiation pressure effects on the short-term secular variations in the orbital period. Geyling's treatment of the problem is based on Hamiltonian mechanics and the variations in satellite position referred to a time dependent moving frame whose origin always coincides with the satellite's position in unperturbed motion. The treatment, which is very involved, does not provide a clear geometrical interpretation of the problem. Also, Geyling investigates only the special case of circular orbits.

The theories expounded in all of the papers are of the first order. Some authors (Musen-Bryant-Bailie) neglect the effect of the earth's shadow. They justify this by the fact that the earth's shadow causes changes in perturbation amplitude, without altering the nature of the perturbation. It is unfortunate that most authors do not specify clearly what assumptions they have made in regard to factors such as: the nature of the perturbing function, re-radiation from the earth, whether the radiation is totally absorbed or partially reflected, and whether the radiation flux is assumed to be constant at all times.

2.4.2.2 Methods and Techniques

Musen derives the expressions for the rates of change in the osculating elements, caused by solar radiation pressure, by the method of variation of vector elements. He introduces the vector element ($e\ \overline{P}$), where $\overline{P}$ is the perigee pointing, to determine the perturbations within the orbital plane. Cook and Kozai use Lagrange's planetary equations, which define the time rates of change in the osculating elements in terms of the components of the disturbing acceleration. The same method is used by Wyatt to define the time rate of the semi-major axis, disregarding the rates of the remaining elements, as his paper is limited to the investigation of the short term secular variations in the orbital period only. Geyling uses the Hamiltonian approach to dynamic problems, a time dependent disturbing function and a time dependent moving coordinate frame centered at the satellite's position in unperturbed motion. Only variations in satellite position with respect to this moving frame are considered. Parkinson's paper is not based on any method in particular, as it is concerned only with the displacement of the geometric center of nearly circular orbits and the amplitude of the perigee height oscillation.

2.4.2.3 Integration Procedures

Musen eliminates the true anomaly, on the right hand sides of the expressions defining the time rates of the osculating elements, by expanding the dyadic products of the vector elements variations in Fourier series with respect to the true anomaly and retaining only the constant terms in this development. Since he neglects the effect of the earth's shadow, the rates of change of the osculating elements are integrated directly with respect to time over a complete revolution of the satellite, provided that there are not sharp resonance conditions. Parkinson integrates the time rate of the displacement of the geometric center of nearly circular orbits with respect to time. He is not otherwise concerned with the changes in the osculating elements. Cook and Wyatt eliminate the time argument in favor of the true anomaly η and integrate between the limits η_1 and η_2; η_1 is the value of the true anomaly where the satellite leaves the earth's shadow and η_2 where it enters the shadow. Cook does not present, however, a technique for the determination of these limits. Wyatt developed an approximating technique for their determination. Kozai performs the integration with respect to the eccentric anomaly E between the limits E_1 and E_2, and recommends numerical methods for the determination of these limits.

2.4.2.4 Critical Evaluation of the Papers Reviewed

2.4.2.4.1 The Method Based on General Perturbations

The theory is based on the principals of general perturbations and the integration of Lagrange's planetary equations with respect to either the eccentric or the true anomaly.

2.4.2.4.1.1 The Work of Y. Kozai (Reference 4.5)

Assumptions: The parallax of the sun is negligible; the solar flux is constant along the satellite's orbit if there is no shadow; re-radiation from the surface of the earth can be neglected.

Completeness: Complete first order theory with the effect of the earth's shadow included. Expressions for all six orbital elements are presented. Both short period and long period terms are combined in the solutions. The reference frame is the inertial earth-equatorial system.

Evaluation: Kozai presents equations without derivations. The nature of the perturbing function is not discussed and many factors inherent to it remain unexplained. There is no comment as to whether the radiation pressure acceleration is constant at all times or varies with the orientation of the satellite with respect to the solar pointing. The solar flux is assumed constant along the orbit if there is no shadow, but it is not specified whether the radiation is totally absorbed by the satellite or either wholly or partially reflected. All these factors should be clearly defined, and the perturbing function should include them as parameters in order to make the analysis complete and meaningful.

2.4.2.4.1.2 The Work of G. E. Cook (Reference 4.4)

Assumptions: The force produced on the satellite by solar radiation pressure is independent of its distance from the sun; the magnitude of the force, while the satellite is in sunlight, is constant for spherical satellites, whereas for non-spherical satellites a suitable average value may be used.

Completeness: A complete first order theory, including the effects of the earth's shadow. Solutions are given for all orbital elements with the exception of the mean anomaly. They include both short period and long period terms. No technique is presented for the evaluation of the limits of integration.

Evaluation: The approach is an extension of the analysis on luni-solar perturbations. The integration is performed in terms of the true anomaly, but the limits are left undetermined. There is no discussion of the problem or an assessment of the solutions. Cook does not indluge in any discussion about the nature of the perturbing force either; only a few assumptions to this effect are made.

2.4.2.4.1.3 The Work of S. P. Wyatt (Reference 4.6)

Assumptions: The radial acceleration of the perturbing force is

$$f = \left(\frac{A}{m} L_{\odot}\right) / (4 \pi r_{\odot}^{2} c)$$

where $L_{\odot}$ is the total power output of the sun, $r_{\odot}$ = 1 a.u., c is the speed of light, A the average cross-sectional area of the satellite, and m is its mass; the orientation of the vector $\bar{f}$ is fixed relative to the satellite's orbit during one revolution; the magnitude of $\bar{f}$ is approximately constant; re-radiation from the earth's surface may be neglected.

Completeness: Incomplete first order theory, as only short term secular variations in the orbital period are considered. The effect of the earth's shadow is included.

Evaluation: The analysis is incomplete and restricted in scope. The frame of reference is the orbital plane of the satellite with the X-axis in the direction of the intersection of the reference plane with a perpendicular plane which contains the sun. Special cases of orbital orientation and shape are discussed. A quasi-general solution for the limits of integration is derived, by expanding the determining equation in powers of eccentricity, which is inefficient because of slow convergence. It appears that Wyatt's paper is primarily concerned with the interference of the nature of the atmosphere.

2.4.2.4.1.4 The Work of P. Musen (Reference 4.1)

Assumptions: The perturbing force for non-spherical satellites is not constant; the effect of the earth's shadow may be neglected.

Completeness: A complete first order theory in the long periodic terms only, since the affect of the earth's shadow is not considered. The effect of a special case of resonance, when the perigee moves in step with the sun, is investigated in detail.

Evaluation: The analysis and development of equations are based on vector elements variations. The equations for the scalar osculating elements are deduced from the equations for the vectorial elements. The basic vector equation is

$$GM \frac{d(e\bar{P})}{dt} = \mathbf{r}_0 \cdot \bar{F}$$

where $\bar{P}$ is a unit vector in the perigee direction, $\bar{F}$ the vector acceleration of the satellite under the influence of the radiation pressure and Γ is Herrick's function $\Gamma = 2[\bar{r}][\bar{v}] - [\bar{v}][\bar{r}] - (\bar{r} \cdot \bar{v}) I$, where

$\bar{r}$, $\bar{v}$ are the position and velocity vestors, and I is the planar unit matrix. The long period part in Γ is separated from the short periodic one by expansion into Fourier series and retaining only the constant terms in the development. Since the effect of the earth's shadow is neglected, the theory, although very interesting, is not sufficiently rigorous for practical applications.

2.4.2.5 Selection of Paper for Detailed Development

From the critical review of the available papers on the subject of the perturbative effects of solar radiation pressure on the orbit of an earth satellite, it was concluded that the paper by Y. Kozai, "Effects of Solar Radiation Pressure on the Motion of an Artificial Satellite," Smithsonian Institute Special Report No. 56, January 30, 1961, is the best for analytical development for the following reasons:

1. It presents a complete first order theory.
2. The analysis is rigorous and includes the effects of the earth's shadow.
3. The techniques used are simple and straightforward.
4. It suggests a method for the determination of the shadow boundaries.

2.4.3 Analytical Development of Y. Kozai's Approach

2.4.3.1 The Perturbing Acceleration

The perturbing acceleration due to solar radiation pressure acts in the opposite direction of the earth-sun pointing.** Denoting the perturbing acceleration vector by $\bar{\mathcal{F}}$, its magnitude by $\mathcal{F}$, and the unit vector in the earth-sun direction by $\bar{D}$, it follows that,

$$\bar{\mathcal{F}} = -\mathcal{F}\bar{D} \qquad \frac{LENGTH}{(TIME)^2} \tag{4.1}$$

** Actually satellite-sun pointing; however, the two pointings are almost coincident.

where, as derived in Section 2. 4. 1. 1,

$$\mathcal{F} = \left(\frac{A}{m}\right)\left(\frac{S_o}{c}\right)(1+\alpha)\left(\frac{d_o}{d}\cos\gamma\right)^2 \qquad \frac{LENGTH}{(TIME)^2} \tag{4.2}$$

in which A is the effective cross-sectional area of the satellite, m its mass, S_o the solar constant (that is, the power of solar radiation on a unit of the earth's surface per unit time), α is the reflection coefficient (0 for absolute black body and 1 for specular surfaces), c is the speed of light, d_o the earth-sun distance, d the distance from the satellite to the sun, and γ is the incidence angle of the sun's rays to the surface. The ratio (A/m) is constant for spherical satellites and so is the incidence angle γ. For non-spherical satellites, both (A/m) and γ vary with the orientations of the satellite and the sun.

The direction cosines A^*, B^*, C^* of the unit vector $\bar{D}$, with respect to the $\bar{P}$, $\bar{Q}$, $\bar{W}$ orbital frame of the satellite (where $\bar{P}$ is the perigee pointing), were already derived in Section 2. 4. 1. 1, so that it can now be defined as,

$$\bar{D} = \bar{P}A^* + \bar{Q}B^* + \bar{W}C^* \tag{4.3}$$

where,

$$\begin{aligned} A^* = {} & \cos^2\frac{i}{2}\cos^2\frac{\varepsilon}{2}\cos(\Lambda_0 - \Omega - \omega) + \cos^2\frac{i}{2}\sin^2\frac{\varepsilon}{2}\cos(\Lambda_0 + \Omega + \omega) \\ & + \sin^2\frac{i}{2}\cos^2\frac{\varepsilon}{2}\cos(\Lambda_0 - \Omega + \omega) + \sin^2\frac{i}{2}\sin^2\frac{\varepsilon}{2}\cos(\Lambda_0 + \Omega - \omega) \\ & + \frac{1}{2}\sin i \sin\varepsilon\left[\cos(\Lambda_0 - \omega) - \cos(\Lambda_0 + \omega)\right] \end{aligned} \tag{4.4}$$

$$\begin{aligned} B^* = {} & \cos^2\frac{i}{2}\cos^2\frac{\varepsilon}{2}\sin(\Lambda_0 - \Omega - \omega) - \cos^2\frac{i}{2}\sin^2\frac{\varepsilon}{2}\sin(\Lambda_0 + \Omega + \omega) \\ & - \sin^2\frac{i}{2}\cos^2\frac{\varepsilon}{2}\sin(\Lambda_0 - \Omega + \omega) + \sin^2\frac{i}{2}\sin^2\frac{\varepsilon}{2}\sin(\Lambda_0 + \Omega - \omega) \\ & + \frac{1}{2}\sin i \sin\varepsilon\left[\sin(\Lambda_0 - \omega) + \sin(\Lambda_0 + \omega)\right] \end{aligned} \tag{4.5}$$

$$C^* = -\sin i \cos^2 \frac{\varepsilon}{2} \sin(\Lambda_0 - \omega) + \sin i \sin^2 \frac{\varepsilon}{2} \sin(\Lambda_0 + \omega) + \cos i \sin \varepsilon \sin \Lambda_0 \qquad (4.6)$$

in which Λ_D is the true celestial longitude of the sun, ϵ is the obliquity, i, Ω are the inclination and the nodal longitude of the orbital plane of the satellite relative to the inertial earth-equatorial frame of reference, and ω is the argument of perigee of the satellite's orbit.

Note that in Kozai's notation,

$$\begin{aligned} A^* &= -S(0) = -S \\ B^* &= -T(0) = -T \\ C^* &= -W(0) = -W \end{aligned} \qquad (4.7)$$

where the "0" argument refers to the true anomaly.

In view of relation (4.3), eq. (4.1) can now be written as,

$$\bar{\mathcal{F}} = -\mathcal{F}(\bar{P}A^* + \bar{Q}B^* + \bar{W}C^*) \qquad (4.8)$$

Now, defining by $\bar{R}$, $\bar{S}$, $\bar{W}$, three unit vectors in the respective directions of the radius vector $\bar{r}$ of the satellite, the direction perpendicular to $\bar{r}$ in the osculating plane (such that $\bar{S} \cdot \bar{V} \leq 0$), and in the direction of the unit normal to the orbital plane, it follows that,

$$\begin{aligned} \bar{R} &= \bar{P} \cos \eta + \bar{Q} \sin \eta \\ \bar{S} &= -\bar{P} \sin \eta + \bar{Q} \sin \eta \end{aligned} \qquad (4.9)$$

where η is the true anomaly.

The components of the perturbing acceleration $\bar{\mathcal{F}}$ (as given by eq. 4.8), in the respective directions of $\bar{R}$, $\bar{S}$, $\bar{W}$, are obtained by forming the dot products of $\bar{\mathcal{F}}$ by $\bar{R}$, $\bar{S}$, $\bar{W}$,

$$
\begin{aligned}
R &= -\mathcal{F}(A^* \cos\eta + B^* \sin\eta) \\
S &= -\mathcal{F}(-A^* \sin\eta + B^* \cos\eta) \\
W &= -\mathcal{F} C^*
\end{aligned}
\tag{4.10}
$$

In Kozai's notation,

$$
\begin{aligned}
R &= a^3 n^2 \frac{\mathcal{F}}{\mu}(S \cos\eta + T \sin\eta) = a^3 n^2 F\, S(\eta) \\
S &= a^3 n^2 \frac{\mathcal{F}}{\mu}(-S \sin\eta + T \cos\eta) = a^3 n^2 F\, T(\eta) \\
W &= a^3 n^2 \frac{\mathcal{F}}{\mu} W \qquad\qquad = a^3 n^2 F\, W(0)
\end{aligned}
\tag{4.11}
$$

2.4.3.2 Lagrange's Planetary Equations

$$\frac{da}{dt} = 2\frac{a^2}{h}\left[R e \sin\eta + S(1 + e\cos\eta)\right] \tag{4.12}$$

$$\frac{de}{dt} = \frac{p}{h}\left[R \sin\eta + S(\cos\eta + \cos E)\right] \tag{4.13}$$

$$\frac{di}{dt} = \frac{r}{h} W \cos u^* \tag{4.14}$$

$$\frac{d\Omega}{dt} = \frac{r}{h} W \frac{\sin u^*}{\sin i} \tag{4.15}$$

$$\frac{d\omega}{dt} = \frac{p}{he}\left[-R\cos\eta + S\left(1 + \frac{1}{1 + e\cos\eta}\right)\sin\eta\right] - \cos i\,\frac{d\Omega}{dt} \tag{4.16}$$

$$\frac{dM}{dt} = n - \sqrt{1-e^2}\left[2\frac{r}{h}R + \left(\frac{d\omega}{dt} + \cos i \frac{d\Omega}{dt}\right)\right] \tag{4.17}$$

where h is the angular momentum per unit mass, η the true anomaly, and u^* the argument of latitude, $u^* = \eta + \omega$.

2.4.3.3 Integration of the Time Rates of the Osculating Elements

In order to make possible the integration in closed form of the time rates of the osculating elements, Kozai assumes the direction and the nagnitude of the perturbing acceleration, as well as the orbital osculating elements, fixed over a revolution of the satellite. Further, he assumes that the solar flux is constant along the orbit of the satellite if there is no shadow, and that there is no re-radiation from the surface of the earth.

The perturbations of the first order over a revolution are derived in closed form by eliminating the time argument and the true anomaly η in favor of the eccentric anomaly, E, using the relations,

$$\begin{aligned} dt &= \left(\frac{1-\cos E}{n}\right) dE \\ \sin\eta &= \frac{\sqrt{1-e^2}\sin E}{1-e\cos E} \\ \cos\eta &= \frac{\cos E - e}{1-e\cos E} \\ 1+e\cos\eta &= \frac{1-e^2}{1-e\cos E} \end{aligned} \tag{4.18}$$

and integrating between the limits E_1 and E_2, where E_1 is the eccentric anomaly of the point of the satellite exit from the shadow and E_2 that of the point of the satellite entry into the shadow.

2.4.3.3.1 The Change in the Semi-Major Axis

$$\frac{da}{dt} = 2\frac{a^2}{h}\left[R e \sin\eta + S(1+e\cos\eta)\right] \tag{4.19}$$

By relations (4. 10),

$$e(R\sin\eta + S\cos\eta) = -\mathcal{T}B^{*}e$$

$$S = -\mathcal{T}(-A^{*}\sin\eta + B^{*}\cos\eta)$$

Therefore,

$$\frac{da}{dt} = -\frac{2}{n\sqrt{1-e^{2}}}\,\mathcal{T}\left[-A^{*}\sin\eta + B^{*}(\cos\eta + e)\right]$$

Now, applying relations (4. 18) yields,

$$\frac{da}{dE} = -\frac{2\mathcal{T}}{n^{2}}\left(-A^{*}\sin E + B^{*}\sqrt{1-e^{2}}\,\cos E\right)$$

Finally, the perturbation in the semi-major axis, Δa, is obtained by integrating with respect to E between the limits E_1 and E_2,

$$\Delta a = -\frac{2\mathcal{T}}{n^{2}}\left| A^{*}\cos E + B^{*}\sqrt{1-e^{2}}\,\cos E \right|_{E_1}^{E_2} \tag{4.20}$$

where A* and B* are given by (4. 4) and 4. 5).

2. 4. 3. 3. 2 The Change in Eccentricity

$$\frac{de}{dt} = \frac{p}{h}\left[R\sin\eta + S(\cos\eta + \cos E)\right] \tag{4.21}$$

By relations (4. 10)

$$R \sin \eta + S \cos \eta = -\mathcal{J} B^*$$

$$S = -\mathcal{J}(-A^* \sin \eta + B^* \cos \eta)$$

Therefore,

$$\frac{de}{dt} = -\mathcal{J} \frac{\sqrt{1-e^2}}{an}\left[-A^* \sin \eta \cos E + B^*(1 + \cos \eta \cos E)\right]$$

Now, elimination of dt and η through relations (4. 18) yields,

$$\frac{de}{dE} = -\mathcal{J} \frac{\sqrt{1-e^2}}{an^2}\left[-\frac{1}{2} A^* \sqrt{1-e^2} \sin 2E + B^*\left(\frac{3}{2} - 2e \cos E + \frac{1}{2} \cos 2E\right)\right]$$

Finally, integration with respect to E between the limits E_1 and E_2 produces the perturbation, Δe, in the eccentricity,

$$\Delta e = -\mathcal{J} \frac{\sqrt{1-e^2}}{an^2}\left|\frac{1}{4} A^* \sqrt{1-e^2} \cos 2E + B^*\left(\frac{1}{4} \sin 2E - 2e \sin E + \frac{3}{2} E\right)\right|_{E_1}^{E_2} \quad (4.22)$$

where, again, A^* and B^* are given by relations (4. 4) and (4. 5).

2. 4. 3. 3. 3 The Change in Inclination

$$\frac{di}{dt} = \frac{r}{h} W \cos u^* = -\mathcal{J} C^* \frac{\sqrt{1-e^2}}{an} \frac{\cos(\eta + \omega)}{1 + e \cos \eta} \quad (4.23)$$

By relations (4. 18),

$$\cos\omega\left(\frac{\cos\eta}{1+e\cos\eta}\right)-\sin\omega\left(\frac{\sin\eta}{1+e\cos\eta}\right)=\cos\omega\left(\frac{\cos E-e}{1-e^2}\right)-\sin\omega\;\frac{\sin E}{\sqrt{1-e^2}}$$

and

$$dt=\left(\frac{1-e\cos E}{n}\right)dE$$

Now, substitution of these relations reduces eq. (4.23) to

$$\frac{di}{dE}=-\frac{\mathcal{J}C^*}{an^2\sqrt{1-e^2}}\left[\cos\omega(\cos E-e)-\sqrt{1-e^2}\sin\omega\sin E\right](1-e\cos E)$$

$$=-\frac{\mathcal{J}C^*}{an^2\sqrt{1-e^2}}\left[\cos\omega\left((1+e^2)\cos E-\frac{e}{2}\cos 2E-\frac{3}{2}e\right)\right.$$

$$\left.-\sqrt{1-e^2}\sin\omega\left(\sin E-\frac{e}{2}\sin 2E\right)\right]$$

The perturbation in inclination, Δi, is now obtained by integrating with respect to E between the limits E_1 and E_2,

$$\Delta i=-\frac{\mathcal{J}C^*}{an^2\sqrt{1-e^2}}\left|\cos\omega\left((1+e^2)\sin E-\frac{e}{4}\sin 2E-\frac{3}{2}eE\right)\right. \tag{2.24}$$

$$\left.+\sqrt{1-e^2}\sin\omega\left(\cos E-\frac{e}{4}\cos 2E\right)\right|_{E_1}^{E_2}$$

where C* is given by relation (4.6).

2.4.3.3.4 The Change in Nodal Longitude

$$\frac{d\Omega}{dt}=\frac{r}{h}W\frac{\sin u^*}{\sin i}=-\mathcal{J}C^*\frac{\sqrt{1-e^2}}{an\sin i}\;\frac{\sin(\eta+\omega)}{1+e\cos\eta} \tag{4.25}$$

By relations (4.18),

$$\cos\omega\left(\frac{\sin\eta}{1+e\cos\eta}\right)+\sin\omega\left(\frac{\cos\eta}{1+e\cos\eta}\right)=\cos\omega\left(\frac{\sin E}{\sqrt{1-e^2}}\right)+\sin\omega\left(\frac{\cos E-e}{1-e^2}\right)$$

and

$$dt=\left(\frac{1-e\cos E}{n}\right)dE$$

Substitution of these relations reduces eq. (4.25) to

$$\frac{d\Omega}{dE}=-\frac{J\,C^*}{an^2\sin i\sqrt{1-e^2}}\left[\sin\omega\,(\cos E-e)+\sqrt{1-e^2}\cos\omega\sin E\right](1-e\cos E)$$

$$=-\frac{J\,C^*}{an^2\sin i\sqrt{1-e^2}}\left[\sin\omega\left((1+e^2)\cos E-\frac{e}{2}\cos 2E-\frac{3}{2}e\right)+\sqrt{1-e^2}\cos\omega\left(\sin E-\frac{e}{2}\sin 2E\right)\right]$$

The perturbation to nodal longitude, $\Delta\Omega$, will come out as the result of integration with respect to E between the limits E_1 and E_2,

$$\Delta\Omega=-\frac{J\,C^*}{an^2\sin i\sqrt{1-e^2}}\left|\sin\omega\left((1+e^2)\sin E-\frac{e}{4}\sin 2E-\frac{3}{2}eE\right)-\sqrt{1-e^2}\cos\omega\left(\cos E-\frac{e}{4}\cos 2E\right)\right|_{E_1}^{E_2} \tag{4.26}$$

2.4.3.3.5 The Change in the Argument of Perigee

$$\frac{d\omega}{dt}=\frac{p}{he}\left[-R\cos\eta+\left(1+\frac{1}{1+e\cos\eta}\right)S\sin\eta\right]-\cos i\,\frac{d\Omega}{dt} \tag{4.27}$$

By relations (4.10),

$$-R\cos\eta + S\sin\eta = \mathcal{J}A^*$$

$$S = -\mathcal{J}(-A^*\sin\eta + B^*\cos\eta)$$

Therefore,

$$\frac{d\omega}{dt} = \mathcal{J}\,\frac{\sqrt{1-e^2}}{ane}\left[A^*\left(1+\frac{\sin^2\eta}{1+e\cos\eta}\right) - B^*\,\frac{\sin\eta\cos\eta}{1+e\cos\eta}\right] - \cos i\,\frac{d\Omega}{dt}$$

But,

$$1+\frac{\sin^2\eta}{1+e\cos\eta} = \frac{1-e\cos E+\sin^2 E}{1-e\cos E} = \frac{e\cos E+\frac{1}{2}\cos 2E-\frac{3}{2}}{1-e\cos E}$$

$$\frac{\sin\eta\cos\eta}{1+e\cos\eta} = \frac{\sin E(\cos E-e)}{\sqrt{1-e^2}(1-e\cos E)} = -\frac{e\sin E-\frac{1}{2}\sin 2E}{\sqrt{1-e^2}\,(1-e\cos E)}$$

and

$$dt = (1-e\cos E)\,\frac{dE}{n}$$

Substitution of these relations produces,

$$\frac{d\omega}{dE} = -\frac{\mathcal{J}}{an^2e}\left[A^*\sqrt{1-e^2}\left(e\cos E+\frac{1}{2}\cos 2E-\frac{3}{2}\right) - B^*\left(e\sin E-\frac{1}{2}\sin 2E\right)\right] - \cos i\,d\Omega$$

Integration yields,

$$\Delta\omega = -\frac{\mathcal{J}}{an^2e}\left|A^*\sqrt{1-e^2}\left(e\sin E+\frac{1}{4}\sin 2E-\frac{3}{2}E\right) + B^*\left(e\cos E-\frac{1}{4}\cos 2E\right)\right|_{E_1}^{E_2} - \cos i\,\Delta\Omega$$

2.4.3.3.6 The Change in Mean Anomaly

$$\frac{dM}{dt} = n - \sqrt{1-e^2}\left[2\,\frac{r}{h}R + \left(\frac{d\omega}{dt} + \cos i\,\frac{d\Omega}{dt}\right)\right] \tag{4.29}$$

By relations (4.10),

$$R = -\mathcal{F}(A^*\cos\eta + B^*\sin\eta)$$

Therefore,

$$2\,\frac{r}{h}R\,dt = -2\mathcal{F}\frac{\sqrt{1-e^2}}{an}\left[A^*\frac{\cos\eta}{1+e\cos\eta} + B^*\frac{\sin\eta}{1+e\cos\eta}\right]dt$$

But, by relations (4.18)

$$\frac{\cos\eta}{1+e\cos\eta} = \frac{\cos E - e}{1-e^2}$$

$$\frac{\sin\eta}{1+e\cos\eta} = \frac{\sin E}{\sqrt{1-e^2}}$$

$$dt = (1-e\cos E)\,\frac{dE}{n}$$

Substitution of these relations yields,

$$2\,\frac{r}{h}R\,dt = -\frac{2\mathcal{F}}{an^2\sqrt{1-e^2}}\left[A^*\left((1+e^2)\cos E - \frac{e}{2}\cos 2E - \frac{3}{2}e\right) + B^*\sqrt{1-e^2}\left(\sin E - \frac{e}{2}\sin 2E\right)\right]$$

Integration produces

$$\int_{E_1}^{E_2} 2\,\frac{r}{h}R\,dt = -\frac{2\mathcal{F}}{an^2\sqrt{1-e^2}}\left|A^*\left((1+e^2)\sin E - \frac{e}{4}\sin 2E - \frac{3}{2}eE\right)\right.$$
$$\left. - B^*\sqrt{1-e^2}\left(\cos E - \frac{e}{4}\cos 2E\right)\right|_{E_1}^{E_2}$$

Now, substituting this expression in eq. (4.29) yields,

$$\Delta M = -\frac{3}{2}\int_0^{2\pi} \frac{\Delta a}{a}\, dM + \frac{2\mathcal{J}}{an^2}\Big| A^*\Big((1+e^2)\sin E - \frac{e}{4}\sin 2E - \frac{3}{2} e E\Big) - B^*\sqrt{1-e^2}\Big(\cos E - \frac{e}{4}\cos 2E\Big)\Big|_{E_1}^{E_2} - \sqrt{1-e^2}\,(\Delta\omega + \cos i\,\Delta\Omega) \tag{4.30}$$

2.4.3.4 The Changes in the Osculating Elements when the Satellite Does Not Enter the Earth's Shadow During One Revolution

When the satellite does not enter the shadow during one revolution, the limits of integration (E_1, E_2) become (0, 2π) and consequently all terms depending on trigonometric functions of E vanish. Therefore, relations (4.20), (4.22), (4.24), (4.26), (4.28), and (4.30) reduce to

$$\Delta a = 0 \tag{4.31}$$

$$\Delta e = -3\mathcal{J}B^*\pi\,\frac{\sqrt{1-e^2}}{an^2} \tag{4.32}$$

$$\Delta i = 3\mathcal{J}C^*\pi\,\frac{e\cos\omega}{an^2\sqrt{1-e^2}} \tag{4.33}$$

$$\Delta\Omega = 3\mathcal{J}C^*\pi\,\frac{e\sin\omega}{an^2\sin i\sqrt{1-e^2}} \tag{4.34}$$

$$\Delta\omega = 3\mathcal{J}A^*\pi\,\frac{\sqrt{1-e^2}}{an^2 e} - \cos i\,\Delta\Omega \tag{4.35}$$

$$\Delta M = -6\mathcal{J}A^*\pi\,\frac{e}{an^2} - \sqrt{1-e^2}\,(\Delta\omega + \cos i\,\Delta\Omega) \tag{4.36}$$

3.0 RECOMMENDED PROCEDURES

The material presented in the body of this report can be utilized in two distinctly different processes. First, the formulations can be employed to estimate the magnitudes of the perturbing influence of any particular force. For this application, either of the approaches for the earth's oblateness perturbation can be mechanized depending upon the type of data desired. However, the number of these applications is relatively small compared to those which exist for a technique capable of estimating the effects of all perturbing influences in an efficient manner. (This second process is extremely important since, for many cases, it affords the advantage of avoiding numerical integration in the construction of the motion of a spacecraft.)

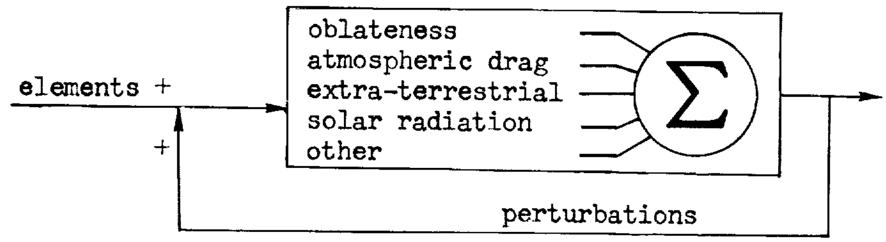
In this second application, however, similarities in the construction of the general perturbations solutions for the various effects dictate that a particular approach to the earth's perturbation, characterized in the text by Kozai's formulation, be employed. This opinion is based upon the relative simplicity for the higher order theories, and the fact that the perturbations which are evaluated are more compatible with the outputs of the other perturbations analyses.

The basic assumption employed in attempting to construct a semi-analytic model of the spacecraft's trajectory (an analytic theory will not be practical for the case where more perturbing influences are encountered. Thus, a combination of analytic and numerical techniques will be presented) is that the coupling of the perturbing effects is sufficiently small as to allow a particular element (e.g., a) to be written as

$$a(t) = a(0) + \int_0^t \dot{a}\, dt$$

$$\approx a(0) + \sum_{i=1}^{n} \sum_{j=1}^{m} \Delta a_j = a(t_n - t_{n-1}) + \sum_{j=1}^{m} \Delta a_j$$

where j denotes the type of perturbation (oblateness, drag, extra-terrestrial gravitation, or radiation pressure), n denotes the number of steps taken in the approximation of the integral, and the change Δ is evaluated over the time interval corresponding to the ith step. In this procedure, the perturbations in each element are evaluated for some specified period, the results fo all perturbing influences summed and the estimates for the elements for the next step predicted. This process is depicted in the following sketch:



In the strictest sense, the elements are being numerically integrated, though numerical extrapolation formula are not employed. However, it is important to note a major difference between the approach and that which is normally applied in the generation of a trajectory by direct integration of the accelerations. This difference is, that the step size can be extrememly large (relative to the purely numerical approach), since the primáry error in the process (the coupling between the perturbations) is small for most trajectories for relatively long periods of time and since the secondary errors due to roundoff and loss of numerical significance are reduced by increasing the step size. (These facts are the direct result of the analytic integration process utilized in the construction of the solution). But, because this approach is a form of numerical integration, a measure which can be utilized to judge whether or not the step size is too large (small) must be constructed. One such measure is the difference in the perturbations as evaluated from the elements resulting from the previous step and those obtained by utilizing elements of the form

$$a_i(t) = a(t_n - t_{n-1}) + \frac{1}{2} \sum_{\substack{j=1 \\ \neq i}}^{m} \Delta a_j$$

In this latter case, the perturbations are evaluated for the ith force (oblateness, drag ...) based on elements which have been modified to more closely represent the average elements in the absence of the ith influence (i.e., an attempt is made to introduce the coupling of the perturbing forces).

The computational methodology to effect this solution is presented in Figures 3.1 and 3.2 for the case in which position and velocity data are desired for input and output. This approach has been employed in numerous studies with considerable success. One such study was performed in the definition of orbits for the IMP Satellites (Ref. 5.1). In this study, it was desired to maximize the lunar gravitational perturbation on a highly elliptic earth satellite orbit to aid in the definition of the lunar potential function. However, a severe constraint was introduced (the satellite was required to have a one year lifetime) which required that the study be performed numerically since there was the tendency for this perturbation to reduce perigee altitude below safe limits. While the application to this problem was not without incident, the results so closely agreed with the numerically integrated trajectories that launch windows and preliminary trajectories could be generated from the simplified program logic. This fact drastically reduced the computational load associated with the development of precision trajectories.

Since this degree of precision was obtained in a case where the magnitudes of the perturbations were large, and since the formulation provides an extremely efficient means of generating a trajectory, the method presented also has application to completely self contained guidance systems. This application, however, does not appear extremely important in the light of current G&N systems approaches due to the extreme emphasis in such systems on minimizing the cost, size, complexity ... of the system. In the future, such an application will probably become feasible (application appears to be limited primarily by the lack of availability of a small low cost general purpose computer of sufficient capacity to perform this task in addition to the others required of it).

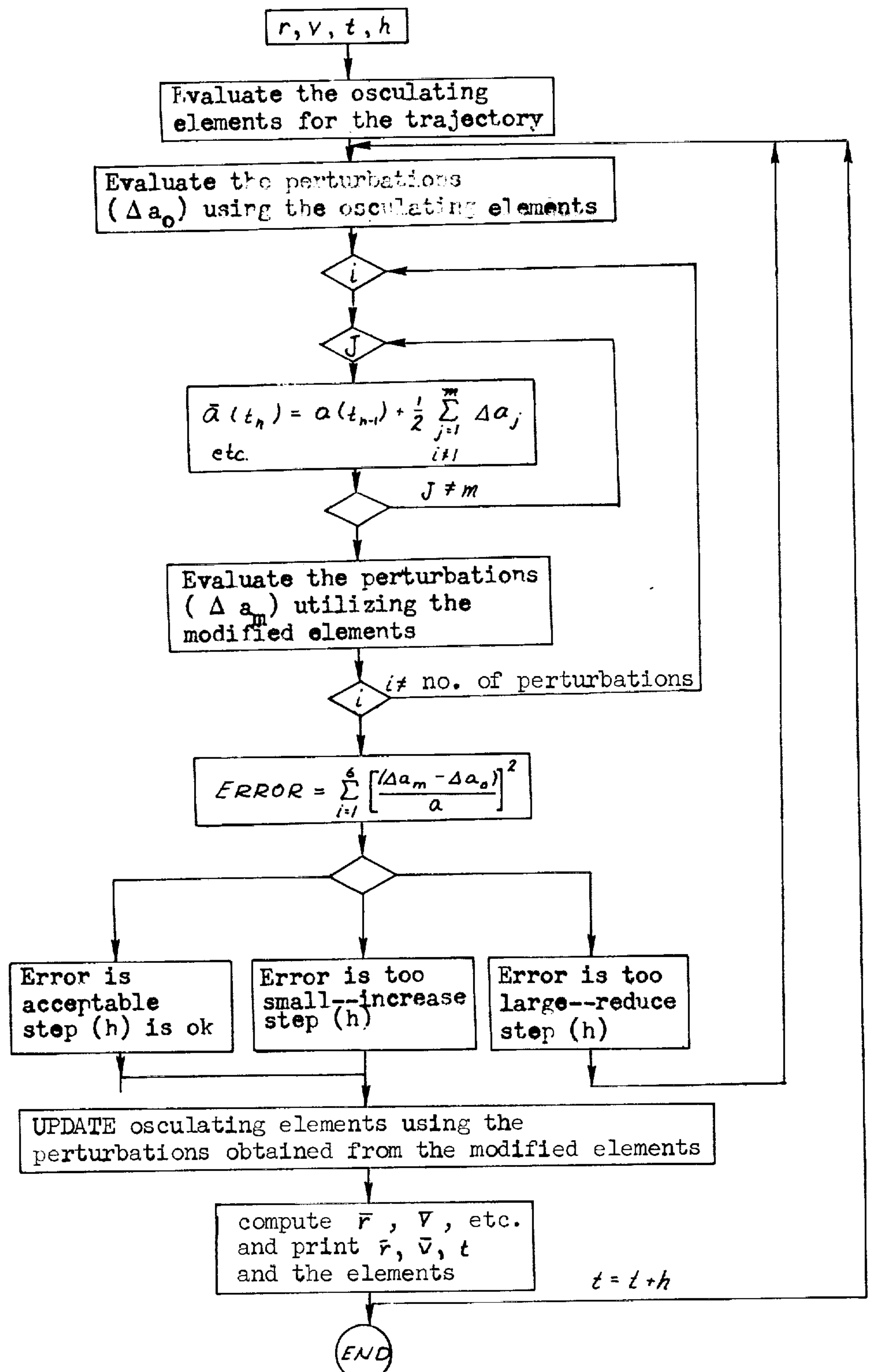
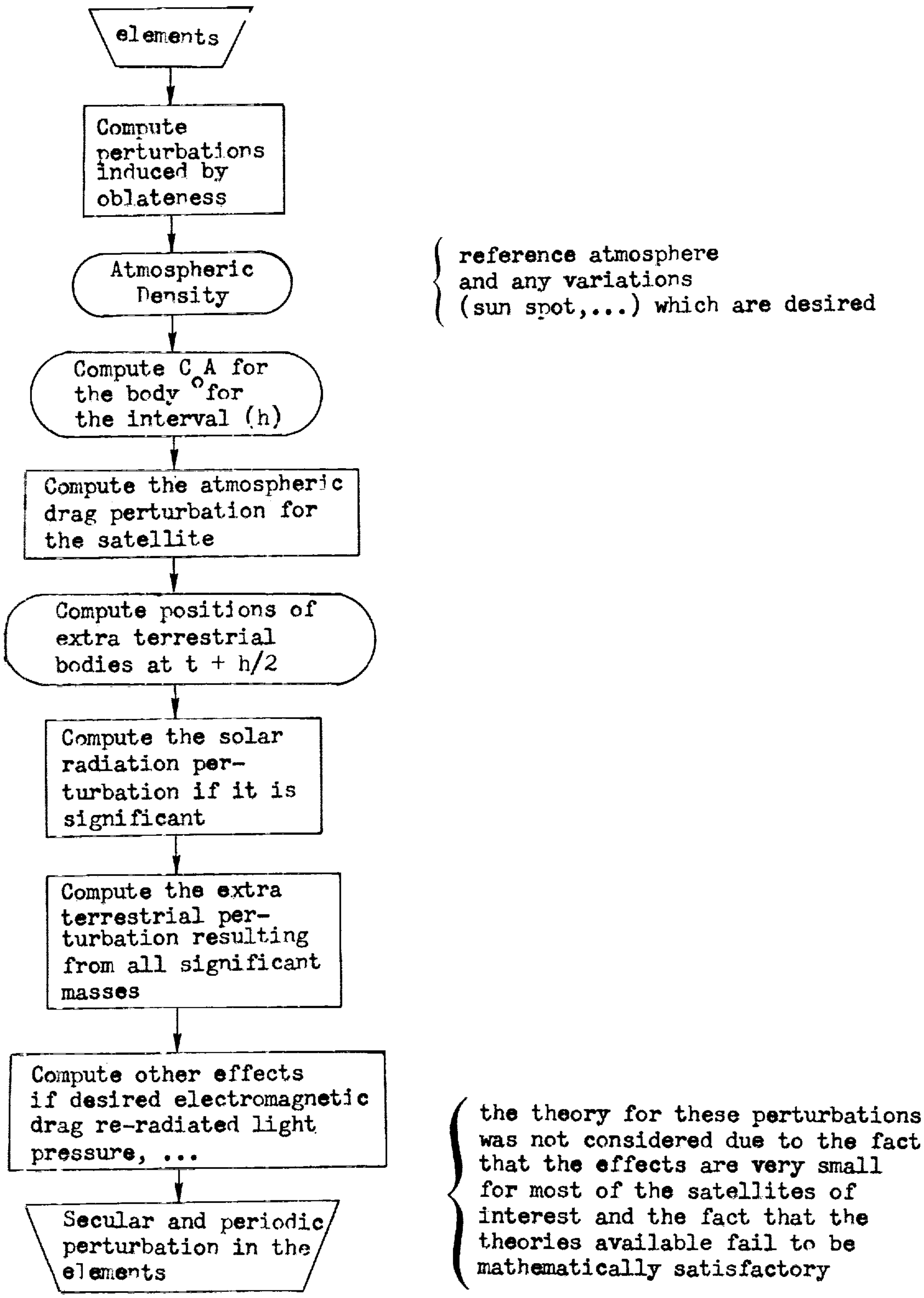


Figure 3.1 Overall Computational Logic


elements
Compute perturbations induced by oblateness
Atmospheric Density
reference atmosphere and any variations (sun spot,...) which are desired
Compute C_oA for the body for the interval (h)
Compute the atmospheric drag perturbation for the satellite
Compute positions of extra terrestrial bodies at t + h/2
Compute the solar radiation perturbation if it is significant
Compute the extra terrestrial perturbation resulting from all significant masses
Compute other effects if desired electromagnetic drag re-radiated light pressure, ...
the theory for these perturbations was not considered due to the fact that the effects are very small for most of the satellites of interest and the fact that the theories available fail to be mathematically satisfactory
Secular and periodic perturbation in the elements

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