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GUIDANCE, FLIGHT MECHANICS AND TRAJECTORY OPTIMIZATION

Volume VIII - Boost Guidance Equations

by G. E. Townsend, A. S. Abbott, and R. R. Palmer

Prepared by
NORTH AMERICAN AVIATION, INC.
Downey, Calif.
for George C. Marshall Space Flight Center

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FOREWORD

This report was prepared under contract NAS 8-11495 and is one of a series intended to illustrate analytical methods used in the fields of Guidance, Flight Mechanics, and Trajectory Optimization. Derivations, mechanizations and recommended procedures are given. Below is a complete list of the reports in the series.

Volume I	Coordinate Systems and Time Measure
Volume II	Observation Theory and Sensors
Volume III	The Two Body Problem
Volume IV	The Calculus of Variations and Modern Applications
Volume V	State Determination and/or Estimation
Volume VI	The N-Body Problem and Special Perturbation Techniques
Volume VII	The Pontryagin Maximum Principle
Volume VIII	Boost Guidance Equations
Volume IX	General Perturbations Theory
Volume X	Dynamic Programming
Volume XI	Guidance Equations for Orbital Operations
Volume XII	Relative Motion, Guidance Equations for Terminal Rendezvous
Volume XIII	Numerical Optimization Methods
Volume XIV	Entry Guidance Equations
Volume XV	Application of Optimization Techniques
Volume XVI	Mission Constraints and Trajectory Interfaces
Volume XVII	Guidance System Performance Analysis

The work was conducted under the direction of C. D. Baker, J. W. Winch, and D. P. Chandler, Aero-Astro Dynamics Laboratory, George C. Marshall Space Flight Center. The North American program was conducted under the direction of H. A. McCarty and G. E. Townsend.

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a_T	Thrust per Unit Mass (Acceleration)
c^*	Matrix of Partial Derivatives of the Required Velocity with Respect to the Position
f_i, g_i	Mathematical Functions of the Time-to-Go and the Mass Rate
F	Thrust or Thrust per Unit Mass
g	Gravitational Attraction
H	The Hamiltonian of the System
J	Comparison (loss) Function
m	Instantaneous Mass
r	Radial Distance from Center-of-Force
R,S,W	Coordinates defined along the Radius, Circumferentially and in the Direction of the Angular Momentum of the Motion
t	Time
T, T_{go}	Time-to-Go
u	The Control to be Extended in Attaining the Desired State
V	Velocity Magnitude
X,Y,Z	Coordinates in the Inertial Reference System
$\hat{x},\hat{y},\hat{z}$ / $\underline{1}_x,\underline{1}_y,\underline{1}_z$	Unit Vector Notation
X	The State of the System (Position and Velocity)
α, β	Out-of-Plane and In-Plane (Relative to S) Steering Angles
θ	Steering Angle Relative to the Inertial $\hat{y}$
μ, K^2	Universal Gravitational Constant Times the Mass of the Central Body
λ_i	Lagrange Multipliers
$\Lambda(t,t_0)$	Matrix of Adjoint Parameters
$\Phi(t,t_0)$	The State Transition Matrix

Subscripts

$\_$	Vector
avg	Average
b	Burn
e	Exhaust
f	Final
g	to be gained
o	Initial
s,r	Circumferential, Radial
R	Required

Superscripts

$\rightarrow$	Vector
$\wedge$	Unit Vector
$\cdot$	$\frac{d}{dt}$
T	Transpose

1.0 STATEMENT OF THE PROBLEM

This monograph will develop near optimum steering equations for boost vehicles. However, a very severe constraint will be imposed which will remove this problem from the spectrum of problems normally formulated in a rigorous manner and solved numerically on a large digital computer (see monographs on optimization formulation techniques, SID 65-1200-4 and SID 65-1200-7). This constraint is that the solution must be possible in less than real time* on a small digital computer of the type employed in the guidance system of boost vehicles.

Three separate approaches to boost guidance will be developed and discussed. These are:

1) Path Adaptive (iterative) Guidance
2) Explicit Guidance Employing Guidance Polynomials
3) Perturbation Guidance

These discussions will have as objectives the demonstration of differences in the formulation of the guidance problem, the development of a means of assessing the mechanization requirements** for each, and the presentation of information necessary to assess the potential accuracy, the flexibility and the limitations of these three forms of guidance.

Path Adaptive (iterative) Guidance as it will be discussed refers to the use of an approximate solution to the equations of motion to iteratively (that is, the solution will be repeated at points along the resultant trajectory) define the optimum steering logic. The assumptions necessary to generate the approximate solution in regards to the nature of the thrust as a function of time, algebraic and trigonometric approximations, gravitational approximations, and time-to-go approximations will all be discussed. The attention will then shift to the mechanization of the resultant material to demonstrate the performance of the technique and indicate sections of the logic which could be improved to produce more optimum performance. Path Adaptive Guidance has been discussed in the open literature by several authors (References 1.1, 1.2, 1.3, 1.4, 1.5, 1.6 and others). However, an approach will be taken here which parallels that of two, I. E. Smith and G. W. Cherry (References 1.1, 1.2, 1.3, 1.4). Minor differences will, of course, be apparent if the material is compared to the references; however, reasons for the differences will normally be given in way of justification.

The second section of the monograph presents a discussion of another approach to the development of a path adaptive guidance system. The objective of this section is the formulation of the guidance problem in terms

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*Real time is time as observed by a stationary clock. The requirement that the solution be possible in less than real time assures that subsequent time will be available for corrective action if it is required.
**No attempt will be made to assure that the same guidance computer can be employed in each of the three approaches. Rather, the requirements will be used to indicate the type of guidance computer which is acceptable.

of polynomials involving the instantaneous state of the system. The major motivation behind this approach is the fact that a steering logic which is somewhat more efficient than that mentioned in the previous paragraph could be obtained at the expense of some additional preflight computations. However, other motivation and objectives and techniques employed in formulating the guidance equation will be presented in the development.

The perturbation guidance discussions differ from those mentioned in the previous paragraphs in that this type of guidance tacitly assumes that the mission is completely defined before launch, that a reference nominal trajectory is available, and that deviations from the reference trajectory will be small so that linear theory can be employed to generate the steering commands to return the vehicle to the nominal trajectory. This approach is extremely simple since most of the difficult computations can be made before launch on a large ground based computer. Further, the resultant trajectory will be more nearly optimum than those yielded in the other approaches to guidance since the zeroth order steering data (provided on tape) will be the result of a simulation of the optimum boost problem on an "exact" model of the earth and vehicle. The method has one major disadvantage, however, in that it requires large amounts of precomputed data. These data, in effect, limit the flexibility of the system once it is launched.

The discussions of perturbation guidance are divided into two principal parts:

1) Velocity-to-be-gained (or required velocity)

2) Linearized Perturbation Guidance

In the first of these approaches, the velocity required to assure a position constraint on the terminal state is the controlled parameter. These discussions present the development of quantities to be utilized as error signals and of the optimal use of these data to generate the required steering commands. The works of Sarture (Reference 3.3) and others (References 3.4 and 3.5) are reviewed to define the complete analytic framework of the problem. In the second approach, methods for employing instantaneous measurements of the state deviation to generate guidance commands intended to minimize some measure of the terminal error are discussed. This portion of the monograph will concern itself with the material of the type presented in References 3.19 and 3.20.

The first and third of these approaches to the guidance problem have been, or are being, successfully applied to boost vehicles. The second, due principally to numerical problems implicit in the preflight simulation and construction of the coefficients of the polynomial is not presently being considered for application. Thus, major emphasis will be placed on the iterative and perturbation guidance approaches. These discussions will present detailed derivations of the equations which will be mechanized and will attempt to establish the basis for all of the assumptions which have been made.

2.0 STATE-OF-THE-ART

2.1 PATH ADAPTIVE GUIDANCE (ITERATIVE)

2.1.1 Introduction

In the other sections of this monograph, a series of methods are discussed which are capable of generating steering commands as a function of time and the state at that time. These methods were developed upon the assumption that the computational capabilities of the guidance computer are very limited and thus employ large quantities of precomputed data on the one hand or guidance polynomials which are difficult to compute on the other. This section presents an approach to the problem which is capable of overcoming the major objections to such guidance schemes and which is still sufficiently simple as to avoid the tremendous computational requirements of a rigorous variational calculus formulation.

The mechanism of this formulation is an analytic solution to the optimum programming problem for the special case of a constant flow rate under the assumption that an average gravity vector can be defined. While the results of this process are reasonable, provision has been made to allow for the repetitive solution of this problem using the progression of measured states to achieve a high degree of accuracy, in the large, without requiring each of the independent computations to be precise.

The method of analysis is similar to that first prepared by I. E. Smith in Reference 1.1; however, minor modifications have been made in the procedures to effect a higher degree of similarity between the results of this analysis and the results of a precise optimum transfer problem. To aid in the appreciation of the assumptions employed and of limitations imposed on the analysis, frequent reference will be made to relate approaches to this guidance concept. The purpose of these references will be to contrast the accuracy and mechanization differences as well as the relative efficiencies of the approaches being considered.

2.1.2 Out-of-Plane Guidance

The general boost-to-orbit guidance problem is generally preceded by one in which the launch time is selected such that the displacement from the desired plane (i.e., the plane containing the terminal state) is reasonably small. This step is accomplished in order to minimize the amount of work expended to turn the plane of the trajectory at the time the transfer and target planes intersect. Thus, the differential equation for the motion of a vehicle in this perpendicular (to the target plane) direction can be separated from the remaining equations with small error provided a first-order correction is applied. This observation is illustrated in the sketches and formulation which follow.

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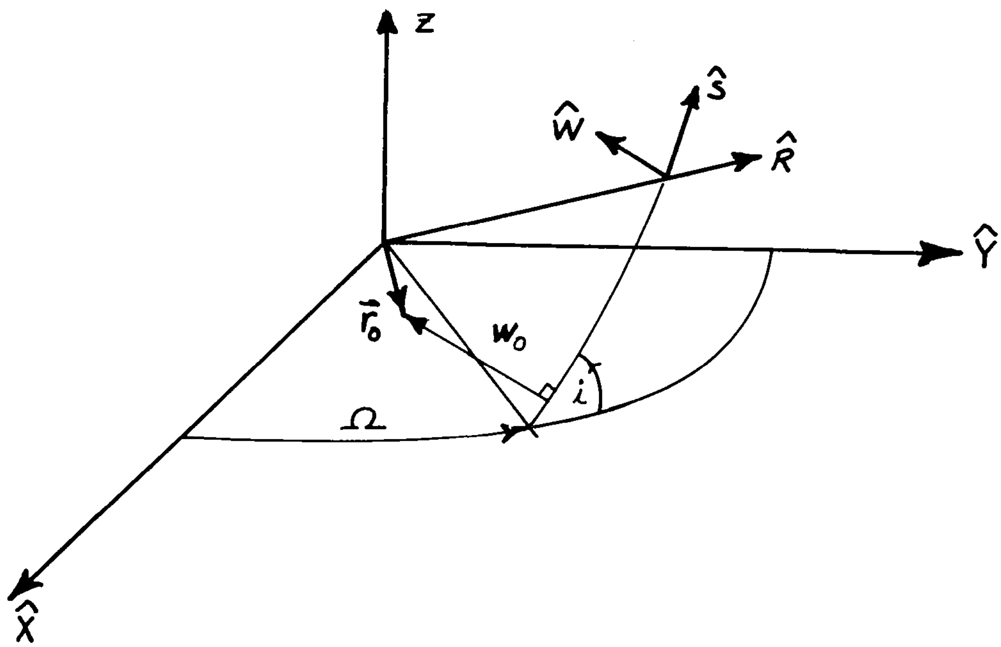
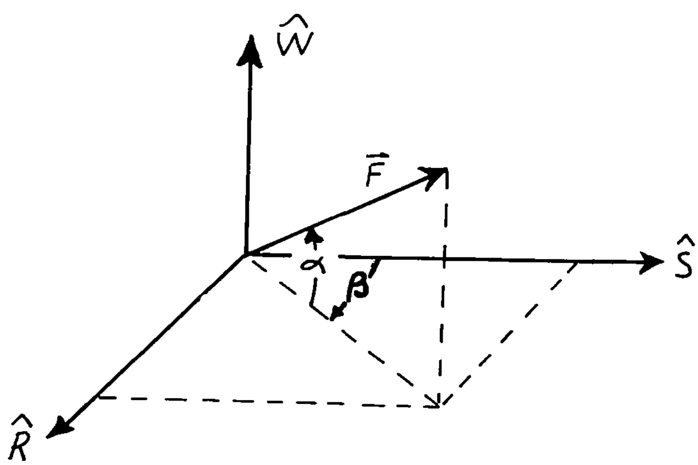
First, consider the definition of the coordinate system



where $\hat{R}$, $\hat{S}$, $\hat{W}$ are unit vectors with origin at the aim point ($\hat{R}$ along the desired radius vector; $\hat{W}$ along the desired angular momentum vector); second, define two orientation angles for the thrust vector in this coordinate system



where $\vec{F}$ is the applied force vector per unit mass whose magnitude is F and whose orientation is defined with respect to the target plane by the angle α . (This choice of reference axis was made to separate the out-of-plane dynamics to as great a degree as possible from the in-plane dynamics. This selection was based upon the consideration of the fact that α (t) will be quite small for most trajectories since the plane of motion can be controlled to a degree by launch time selection and since the terminal value will in most cases be very nearly zero in order to assure that the terminal velocity perpendicular to the target plane will be zero. Under these definitions, the equations of motion normal to the target plane are

$$\ddot{W} = F \sin\alpha - \frac{\mu}{r^2}\hat{r}\cdot\hat{W} \tag{2.1.1}$$

Now, the thrust attitude angle, α , will be defined such that the change in velocity parallel to the desired plane resulting from out of the plane thrust is as small as possible; i.e., the loss function

$$J = \int F(1-\cos\alpha)\,dt \tag{2.1.2}$$

is as small as possible.

In contrast to a solution of these equations which strives for maximum accuracy, the guidance problem in general strives for both relatively high accuracy and computational simplicity. Thus, the first of a series of approximations will be introduced to allow analytic solution of the problem by uncoupling the out-of-plane dynamics. It is assumed that the integral of the corrective term in the equation of motion (that containing gravity) can be expressed as

$$\frac{1}{T}\int_0^T -\frac{\mu}{r^2}\hat{r}\cdot\hat{W}\,dt = \vec{g}_{avg}\cdot\hat{W} \equiv \bar{G}. \tag{2.1.3}$$

Attempts to evaluate this time average acceleration will be postponed until the present discussions can be concluded. It is noted, however, that since the integral itself is small, relatively crude approximations (for example, truncated series) for both w and r as functions of time could allow the dot product $\vec{g}\cdot\hat{W}$ to be evaluated independent of the analysis of the in-plane motion with adequate precision. This procedure doesn't appear desirable though, due to the fact that subsequent steps (in the analysis of in-plane motion) will demonstrate the necessity of generating an accurate estimate of the time average gravitational acceleration. This fact means that $\bar{G}$ can be estimated from the equation $\bar{G} = \vec{g}_{avg}\cdot\hat{W}$ with more than adequate precision.

Now, since an average value of the gravity correction, $\bar{G}$, is assumed known, the optimum thrust commands for $\alpha = \alpha(t)$ can be determined by considering the optimization of

$$I(\alpha) = \int_0^T F(1-\cos\alpha)\,dt$$

subject to the integral constraints

$$J_1(\alpha) = -W(T) \quad + \quad W(0) + \dot{W}(0)T + \frac{\bar{G}T^2}{2} + \int_0^T \left[\int_0^T F \sin\alpha \, ds \right] dt \quad (2.1.4)$$

$$J_2(\alpha) = -\dot{W}(T) \quad + \quad \bar{G}T + \dot{W}(0) + \int_0^T F \sin\alpha \, dt \qquad (2.1.5)$$

But this statement is identical to that of the class of isoperimetric problems of the calculus of variations; i.e., if the augmented integral I* is defined to be

$$I^* = I(\alpha) + \lambda_1 J_1 + \lambda_2 J_2$$
$$= \int_0^T f^* dt$$

then, the Euler-Lagrange equations for the system become

$$\frac{\partial f^*}{\partial \alpha} - \frac{d}{dt}\left(\frac{\partial f^*}{\partial \dot{\alpha}} \right) = 0$$

But, I* (minus the terms which are not functions of α) is

$$I^* = \int_0^T F(1-\cos\alpha) + \lambda_1 \int_0^T \left[\int_0^T F \sin\alpha \, ds \right] dt + \lambda_2 \int_0^T F \sin\alpha \, dt \quad (2.1.6)$$

This equation can be simplified by integrating the second term by parts as

$$\int_0^T \int_0^t F \sin\alpha \, ds\, dt = T \int_0^T F \sin\alpha \, dt - \int_0^T t F \sin\alpha \, dt \quad .$$

This fact allows I* to be written as

$$I^* = \int_0^T \left[F(1-\cos\alpha) + (\lambda_1 T + \lambda_2)(F \sin\alpha) - \lambda_1 t F \sin\alpha \right] dt \quad (2.1.7)$$

or f* as

$$f^* = F(1-\cos\alpha) + F\left[\lambda_1 (T-t) + \lambda_2\right] \sin\alpha \quad . \qquad (2.1.8)$$

Thus, since f* is independent of $\dot{\alpha}$, the Euler-Lagrange equations are

$$\frac{\partial f^*}{\partial \alpha} \equiv 0 = F \sin \alpha + F\left[\lambda_1 (T-t) + \lambda_2\right] \cos \alpha$$

and since $F \neq 0$,

$$\begin{aligned} \tan \alpha &= -\lambda_1 (T-t) - \lambda_2 \\ &= -(\lambda_1 T + \lambda_2) + \lambda_1 t \quad . \\ &\equiv A + Bt \end{aligned} \tag{2.1.7}$$

or

$$\sin \alpha = \frac{A + Bt}{\sqrt{1 + (A + Bt)^2}}$$

independent of the nature of F.

The values of A and B can be obtained by integrating the equations of motion (i.e., the constraint equations) and substituting the boundary conditions for the terminal and initial states. This task can be performed for the general case, however, the resulting expressions are nonlinear in the constants A and B. This fact means that relatively elaborate procedures are required to evaluate the constants and suggests that a simpler solution would generally be preferred for the on-board guidance problem. Thus, an attempt will be made to approximate sin α for the case where the change in α is relatively small (relative to one radian) during the burn of any single stage. For this case

$$\tan \alpha = \tan (\alpha^* + \Delta\alpha)$$

and

$$\Delta\alpha \approx \frac{A + Bt - \tan \alpha^*}{\sec^2 \alpha^*} \tag{2.1.10}$$

where for the moment α^* is an unknown constant reference. Now, since $\Delta\alpha$ is known, the sine of α can be written as

$$\begin{aligned} \sin \alpha &= \sin (\alpha^* + \Delta\alpha) \\ &\approx \sin \alpha^* + \Delta\alpha \cos \alpha^* \\ &\approx \sin \alpha^* + \frac{(A + Bt - \tan \alpha^*) \cos \alpha^*}{\sec^2 \alpha^*} \\ &\equiv C + Dt \end{aligned} \tag{2.1.11}$$

which again is linear in time. This fact means that a linear sine steering program will be near optimum provided that the assumption $(\Delta\alpha)^2 << 1$ is not violated. For this reason, the degree-of-optimization will be compromized slightly to effect major simplifications in the guidance process and the constants C and D will be defined by fitting the boundary conditions to the equations of motion. Note that the angle α^* which was introduced for the purpose of expansion is not required nor can it be from this information (2 equations and 3 unknowns).

Consider the case where the motion of the vehicle is produced by a constant flow rate motor (this type of operation is characteristic of nearly all large liquid motors and most large solids); i.e.,

$$F = \frac{F(0)}{1-\dot{m}(t+t_b)} \equiv \frac{F_0}{(1-\dot{m}t_b)-\dot{m}t} \tag{2.1.12}$$

where t_b denotes the burning time which has ellapsed since this particular stage was ignited and where both F and $\dot{m}$ have been normalized by the initial mass of the stage [i.e., F = thrust/m(o); $\dot{m}$=mass rate/m(o)] Under this assumption, the linear sine steering program will produce the following results for the constraint equations. The terminal velocity is

$$\begin{aligned}
\dot{w}(T) &= \int_0^T \left[\frac{F_0(C+Dt)}{(1-\dot{m}t_b)-\dot{m}t} + \bar{G}\right] dt + \dot{w}(0) \\
&= \dot{w}(0) + \bar{G}T + F_0 C\int_0^T \frac{dt}{(1-\dot{m}t_b)-\dot{m}t} + F_0 D\int_0^T \frac{t\,dt}{(1-\dot{m}t_b)-\dot{m}t} \qquad (2.1.13) \\
&= \dot{w}(0) + \bar{G}T + F_0 C f_1 + F_0 D f_2
\end{aligned}$$

where

$$f_1 = \frac{-1}{\dot{m}} \ln\left[1 - \frac{\dot{m}T}{1-\dot{m}t_b}\right]$$

$$\begin{aligned}
f_2 &= \frac{-T}{\dot{m}} - \frac{(1-\dot{m}t_b)}{\dot{m}^2} \ln\left[1 - \frac{\dot{m}T}{1-\dot{m}t_b}\right] \\
&\equiv \frac{-T}{\dot{m}} + \frac{1-\dot{m}t_b}{\dot{m}} f_1
\end{aligned}$$

And, the terminal position is

$$w(T) = \int_0^T \int_0^t \left[\frac{F_o (C + DS)}{(1 - \dot{m} t_b) - \dot{m} S} + \bar{G} \right] ds\, dt + \dot{w}(0) T + w(0)$$

$$= w(0) + \dot{w}(0) T + \frac{\bar{G}}{2} T^2 + F_o C \int_0^T \int_0^t \frac{ds\, dt}{(1 - \dot{m} t_b) - \dot{m} s} + F_o D \int_0^T \int_0^t \frac{s\, ds\, dt}{(1 - \dot{m} t_b) - \dot{m} S} \tag{2.1.14}$$

$$= w(0) + \dot{w}(0) T + \frac{\bar{G}}{2} T^2 + F_o C g_1 + F_o D g_2$$

where

$$g_1 = \int_0^T f_1(t) dt = \int_0^T - \frac{1}{\dot{m}} \ln \left[1 - \frac{\dot{m} t}{1 - \dot{m} t_b} \right] dt$$

$$= \frac{1}{\dot{m}} \left(\frac{1 - \dot{m} t_b}{\dot{m}} \right) \left\{ \left[1 - \frac{\dot{m} T}{1 - \dot{m} t_b} \right] \left[\ln \left(1 - \frac{\dot{m} T}{1 - \dot{m} t_b} \right) - 1 \right] + 1 \right\}$$

$$g_2 = \int_0^T f_2(t) dt = \int_0^T \left[- \frac{t}{\dot{m}} + \frac{1 - \dot{m} t_b}{\dot{m}} f_1 \right] dt$$

$$\equiv - \frac{T^2}{2 \dot{m}} + \frac{1 - \dot{m} t_b}{\dot{m}} g_1$$

Thus, since these equations are linear in the unknowns C and D, the solution to the system

$$\left\{ \begin{array}{l} \dot{w}(T) - \dot{w}(0) - G T \\ w(T) - w(0) - \dot{w}(0) T - \dfrac{\bar{G} T^2}{2} \end{array} \right\} = F_o \begin{bmatrix} f_1 & f_2 \\ g_1 & g_2 \end{bmatrix} \left\{ \begin{array}{c} C \\ D \end{array} \right\}$$

can be evaluated as follows:

$$\left\{ \begin{array}{c} C \\ D \end{array} \right\} = \frac{1}{F_o \Delta} \begin{bmatrix} g_2 & -f_2 \\ -g_1 & f_1 \end{bmatrix} \left\{ \begin{array}{l} \dot{w}(T) - \dot{w}(0) - \bar{G} T \\ w(T) - w(0) - \dot{w}(0) T - \dfrac{\bar{G} T^2}{2} \end{array} \right\} \tag{2.1.15}$$

where

$$\Delta = f_1 g_2 - f_2 g_1$$

There is one chance for error in this solution resulting from the fact that $\Delta \to 0$ as $T \to 0$. This fact requires that the limits for both C and D be established in order to assure that the solution is always determinate. This step is not readily accomplished. However, by noting that in the limit W, $\dot{W}$, $\ddot{W}$ and G approach zero (the terminal plane is the desired plane), it is possible to conclude that C and D should also approach zero in the limit.

This solution completely solves the problem under the assumption that $\sin\alpha = C+Dt$. Further, though there is no assurance that the approximations which allowed this solution will also be reasonable, it is noted that successive re-evaluations of C and D as the epoch corresponding to zero time is moved along the trajectory (in terms of w and $\dot{w}$) will assure that the process will converge for all of the cases of interest. However, the degree-of-optimality for cases involving values for which the sine of α cannot be approximated as specified is questionable).

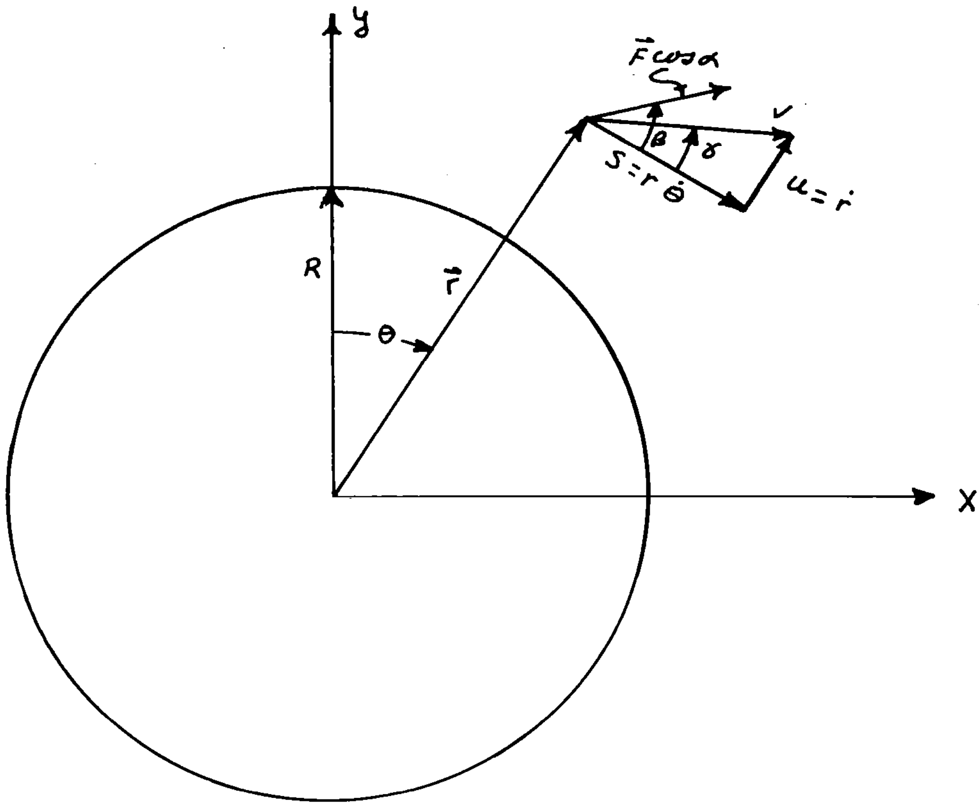
A slight variation of this approach was employed by Cherry (Reference 1.3, 1.4). However, rather than assuming the nature of $\vec{r}(t)$ for the purpose of computing the gravity correction, Cherry noted that the linear tangent steering which results for $\bar{G} = 0$ or a constant is approximately equivalent to linear sine steering. He then assumed a "near optimum" steering law

$$\sin\alpha = \overset{*}{C} + \overset{*}{D}t + \frac{\mu \vec{r}\cdot\hat{W}}{r^3 F(t)} .$$

This assumption causes the gravity contribution to be cancelled from the equation of motion and allows C* and D* to be evaluated in a manner similar to that outlined on the previous pages. However, if a reasonable estimate of $\bar{G}$ can be made, the approach outlined previously will always provide a better estimate of the optimal $\alpha = \alpha(t)$ with only slight additional complexity.

2.1.3 In-Plane Guidance

The discussions of the previous section provided the basis for analyzing the motion normal to the desired target plane (the plane of $\vec{r}_f$, $\vec{v}_f$). Thus, at this point the analysis will be restricted to the motion in, or parallel to, the desired plane. Consider the sketch and nomenclature presented below:



The equation of motion in the instantaneous coordinate frame $[\vec{r},\ \hat{W} \times \hat{r}]$

$$\dot{s} = -\frac{su}{r} + F\cos\alpha\cos\beta \tag{2.1.16a}$$

$$\dot{u} = \frac{s^2}{r} - g_o\left(\frac{R}{r}\right)^2 + F\cos\alpha\sin\beta \tag{2.1.16b}$$

where

$$\dot{\theta} = \frac{s}{r} \tag{2.1.16c}$$

$$\dot{r} = u \tag{2.1.16d}$$

$$F = \frac{F_0}{1 - \dot{m}(t + t_b)}$$

and where the approximate form of the optimum program for α ($\dot{m}$ = constant) was derived in the previous section.

If these equations [denoted by $f[x(t), u(t), t] - \dot{x} = 0$] are to be integrated along a path which yields a minimum value for some scalar measure of performance (for example, propellant expended) defined by the equation

$$J = \varphi[x(T), T] + \int_{t_0}^{T} \psi[x(t), u(t), t]\, dt \tag{2.1.17}$$

where u (t) represents the control vector, then the differential equations defining the optimal path (control) can be generated by adjoining the system of differential equations to the performance index

$$J = \varphi[x(T), T] + \int_{t_0}^{T} \Big[\psi[x(t), u(t), t] + \lambda^T(t)\big\{f[x(t), u(t), t] - \dot{x}\big\}\Big] dt$$

$$\equiv \varphi[x(T), T] + \int_{t_0}^{T} (H - \lambda^T \dot{x})\, dt \tag{2.1.18}$$

where H denotes the Hamiltonian. But the integral of this equation can be evaluated by parts

$$\int_{t_0}^{T} (H - \lambda^T \dot{x})\, dt = \lambda^T x \Big|_{t_0}^{T} + \int_{t_0}^{T} (H + \dot{\lambda}^T x)\, dt$$

so that

$$J = \varphi[x(T), T] + \lambda^T(T)\, x(T) - \lambda^T(t_0)\, x(t_0) + \int_{t_0}^{T} \Big\{H[x(t), u(t), t] + \dot{\lambda}^T(t)\, x(t)\Big\} dt \tag{2.1.19}$$

Now J is to attain an extremum along the path which is produced by a special control u = u(t). This condition is obtained by forming the first variation of J.

$$\delta J = \left[\left(\frac{\partial \varphi}{\partial x} - \lambda^T\right) \delta x\right]_T + \lambda^T \delta x \Big|_{t_0} + \int_{t_0}^{T} \left[\left(\frac{\partial H}{\partial x} + \dot{\lambda}^T\right) \delta x + \frac{\partial H}{\partial u}\, \delta u\right] dt$$

At this point in the solution, the multipliers λ are arbitrary. Thus, they can be selected in such a manner as to simplify the problem. Note in the previous equation that variations in the control and in the present state can be specified arbitrarily. However, the variations in the state (X) at times in the interval $[t_o, T]$ and at T are dependent [i.e., no a' priori information is known. $\delta X(t)$]. Thus, the multipliers will be selected such that the coefficient of $\delta X(t)$ is zero for all $t_o \leq t \leq T$.

$$\dot{\lambda}^T = -\frac{\partial H}{\partial x} = \frac{\partial \psi}{\partial x} - \lambda^T \frac{\partial f}{\partial x} \tag{2.1.20}$$

with boundary conditions (to simplify the bracketed term outside the integral)

$$\lambda_i^T(T) = \left.\frac{\partial \varphi}{\partial x_i}\right|_T \tag{2.1.21}$$

for all unconstrained components of the state at t = T and $\delta X(T) = 0$ for the constrained components of the state at t = T. Under these conditions, δJ reduces to .

$$\delta J = \lambda^T(t_o)\,\delta x(t_o) + \int_{t_o}^{T} \frac{\partial H}{\partial u}\,\delta u\,dt \tag{2.1.22}$$

Thus, if $\delta J = 0$ for arbitrary $\delta X(t_o)$ and $\delta u(t)$, then

$$\frac{\partial H}{\partial u} = 0 \tag{2.1.23}$$

for all $t_o \leq t \leq T$.

Now consider the special case of interest in the guidance problem where minimization of fuel expended for the case of a constant flow rate is the objective. For this case, the problem reduces to the brachistochrone problem (i.e., the comparison function is simply the terminal time). Thus, the Hamiltonian is a function of the end conditions alone. This fact allows H to be written as

$$H = \lambda_1\left(F\cos\alpha\cos\beta - \frac{su}{r}\right) + \lambda_2\left[F\cos\alpha\sin\beta - g_o\left(\frac{R_o}{r}\right)^2 + \frac{s^2}{r}\right] + \lambda_3\left(\frac{s}{r}\right) + \lambda_4(u) \tag{2.1.24}$$

Now, since F and α are known functions of time (for assumptions made earlier), the only control possible exists due to the angle β. Differentiating the Hamiltonian to obtain the equations defining the optimal policy for β yields

$$\frac{\partial H}{\partial \beta} = 0 = -\lambda_1 F\cos\alpha\sin\beta + \lambda_2 F\cos\alpha\cos\beta$$

or

$$\tan \beta = \frac{\lambda_2}{\lambda_1} \tag{2.1.25}$$

where λ_2 and λ_1 are defined by integrating

$$\dot{\lambda}_1 = \lambda_1 \frac{u}{r} - \lambda_2 \frac{2s}{r} - \lambda_3 \frac{1}{r} \tag{2.1.26a}$$

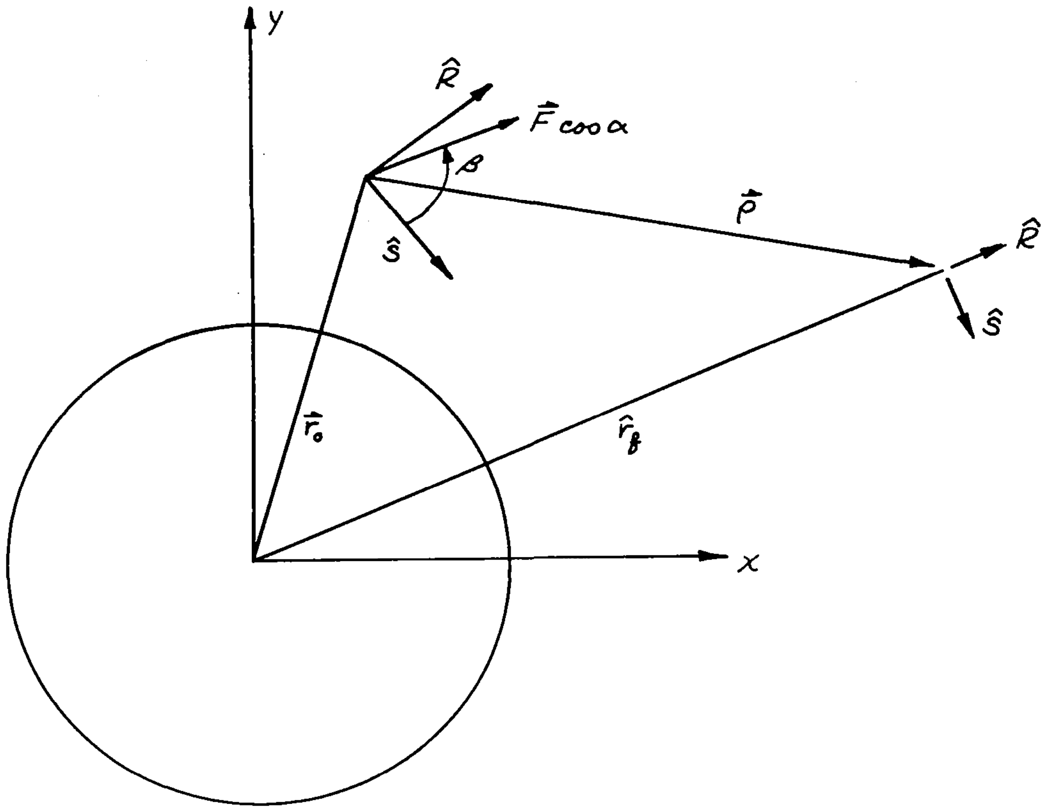
$$\dot{\lambda}_2 = \lambda_1 \frac{s}{r} - \lambda_4 \tag{2.1.26b}$$

$$\dot{\lambda}_3 = 0 \tag{2.1.26c}$$

$$\dot{\lambda}_4 = -\lambda_1 \frac{su}{r^2} - \lambda_2 \left(- \frac{s^2}{r^2} + 2 g_0 \frac{R^2}{r^3} \right) + \lambda_3 \frac{s}{r^2} \tag{2.1.26d}$$

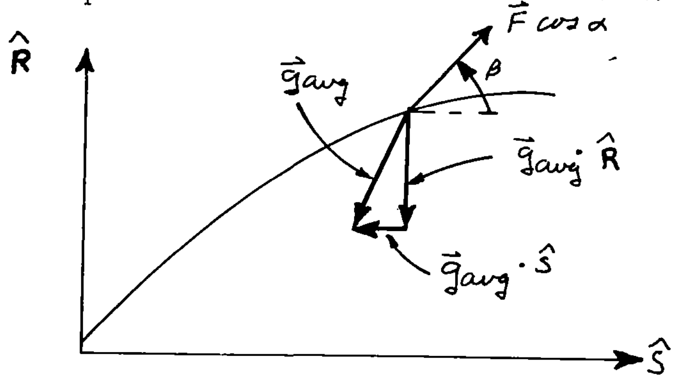
The problem encountered at this point is one of computational complexity in spite of assumptions made separating the out-of-plane and planar dynamics. This is not to say that solution is impossible; rather, solution seems unnecessarily complex for application to a small on-board guidance computer. Thus, attention will turn to further simplification of these equations which will allow an analytic solution to be developed. Subsequent discussions will then attempt to assure that the accuracy obtained from a mechanization of the analytic solution will approach that obtained from a more rigorous formulation.

The first observation is that the range angles subtended during powered flight for systems in which the thrust acceleration is of the order of the gravitational acceleration are generally small. This fact means that the direction-of-gravity is nearly constant and suggests that a "flat earth" model can be employed to advantage. Consider the following sketch showing the initial position vector, the target position vector and the set of unit vectors (R, S) which could be utilized for the purpose of constructing a flat earth model (this set conforms to the notation of the out-of-plane guidance discussion).



In this coördinate system ($\hat{R}$, $\hat{S}$), the direction-of-gravity is unknown as a function of time. However, as was the case in the analysis of the out-of-plane motion, an average gravitational acceleration can be assumed. Unfortunately, the solution of the problem is relatively sensitive to errors in this assumption. Thus, care must be exercised to assure that both the magnitude and direction of this acceleration are selected in a rational manner. The estimate recommended for the generation of this model is, therefore, the time average. Subsequent discussions will define the procedure to be followed in the generation of this time average gravity vector. Thus, the present discussion can assume that the time average has been developed and is known.

The first step in the derivation of analytic steering equations is now the reformulation of the equations of motion and the Hamiltonian. Consider the following sketch:



Under these assumptions, the approximate equations of motion are,

$$\dot{V}_s = F\cos\alpha\cos\beta + \vec{g}_{avg}\cdot\hat{S} \tag{2.1.27a}$$

$$\dot{V}_r = F\cos\alpha\sin\beta + \vec{g}_{avg}\cdot\hat{R} \tag{2.1.27b}$$

$$\dot{s} = V_s \tag{2.1.27c}$$

$$\dot{r} = V_R \tag{2.1.27d}$$

So that the Hamiltonian for this simple case is

$$H = \lambda_{V_s}(F\cos\alpha\cos\beta + \vec{g}_{avg}\cdot\hat{S}) + \lambda_{V_r}(F\cos\alpha\sin\beta + \vec{g}_{avg}\cdot\hat{R}) + \lambda_s(V_s) + \lambda_r(V_R) \tag{2.1.28}$$

and the differential equations for the time varying Lagrange multipliers (obtained from

$$\dot{\lambda}^T = -\frac{\partial H}{\partial X}$$

are

$$\dot{\lambda}_{V_s} = -\lambda_s \qquad \dot{\lambda}_s = 0$$

$$\dot{\lambda}_{V_r} = -\lambda_r \qquad \dot{\lambda}_r = 0$$

But these equations are uncoupled and may thus be integrated directly to yield

$$\lambda_s = C_1$$

$$\lambda_r = C_2$$

$$\lambda_{V_s} = -C_1 t + C_3$$

$$\lambda_{V_r} = -C_2 t + C_4$$

So that the form of the steering program discussed earlier becomes

$$\tan\beta = \frac{\lambda_{V_R}}{\lambda_{V_s}} = \frac{-C_2 t + C_4}{-C_1 t + C_3} \tag{2.1.29}$$

(This law is commonly referred to as the bilinear steering law.) The four constants of this law must be evaluated from the boundary conditions imposed on the problem. Before performing this evaluation, however, it is recalled that the boundary conditions for the λ can be selected arbitrarily if the terminal state is not constrained. [(This fact affords the possibility of simplifying the form of the steering law without attempting to solve the equations of motion.)] For example,

$$\lambda^T(T) = \left.\frac{\partial \varphi}{\partial x}\right|_{t=T}$$

Thus, if

$$\varphi \neq \varphi\,[S(T)] \quad , \quad \lambda_S = C_1 = 0$$

Under this condition, the bilinear steering law reduces to the linear tangent steering law. Note that relaxing only one of the constraints on the terminal state eliminated two of the four constants of integration (the two new constants appear as ratios). This fact and observations of the small sensitivity of the resultant trajectories to the constraints on the longitudinal position component (noted in References 1.1, 1.2) have lead several investigators to the assumption that the performance afforded by the bilinear law can be adequately approximated using a linear tangent steering concept. The mathematical justification for this assumption will be presented in the following paragraphs.

The equations of motion

$$\ddot{S} = F \cos\alpha \cos\beta + \vec{g}_{avg} \cdot \hat{S}$$

$$\ddot{r} = F \cos\alpha \sin\beta + \vec{g}_{avg} \cdot \hat{R}$$

can be made explicit functions of time to aid the integration process provided a series of small angle approximations can be made to facilitate solution. These approximations will produce a slight inefficiency (in-so-far as propellant expenditure is concerned); however, since the launch time is selected so that the initial displacement from the desired plane is small, since most of the boost trajectories for orbital, lunar and interplanetary flights are characterized by small values of angle between the velocity and local horizontal vectors at injection, and since to attain this state (in a near optimum fashion for trajectories composed of an atmospheric arc and a guided arc) requires a reasonably small attitude angle when initiating the guidance, the assumptions are believed reasonable.

The first simplification results from the assumptions that the change in the attitude of the thrust vector for control of the motion normal to the desired plane is small during any single burning process, and that maximum amplitude of the angle itself is sufficiently small as to allow the cosine of α to be represented as

$$\cos\alpha \approx 1 - \tfrac{1}{2}(C + Dt)^2 \approx \left(1 - \frac{C^2}{2}\right) - CDt \qquad (2.1.30)$$

where the constants C and D are known from the analysis of the out-of-plane control problem.

The second simplification results from the fact that

$$\tan\beta = \frac{-C_2 t + C_4}{-C_1 t + C_3}$$

can be approximated as

$$\begin{aligned} \tan(\beta^* + \Delta\beta) &\approx \frac{\tan\beta^* + \Delta\beta}{1 - \Delta\beta\tan\beta^*} \\ &\approx (\tan\beta^* + \Delta\beta)(1 + \Delta\beta\tan\beta^*) \\ &\approx \tan\beta^* + \Delta\beta(1 + \tan^2\beta^*) \\ &= \tan\beta^* + \Delta\beta(\sec^2\beta^*) \end{aligned} \qquad (2.1.31)$$

provided the change in the thrust attitude during the period of interest (i.e., one stage burn) is small. Thus, it is possible to represent $\Delta\beta$ as

$$\begin{aligned} \Delta\beta &= \frac{1}{\sec^2\beta^*}\left[-\tan\beta^* + \frac{-C_2 t + C_4}{-C_1 t + C_3}\right] \\ &= \frac{-C_2^* t + C_4^*}{-C_1^* t + C_3^*} \end{aligned} \qquad (2.1.32)$$

where

$$-C_2^* = C_1\tan\beta^* - C_2$$

$$-C_1^* = C_1\sec^2\beta^*$$

$$C_4^* = C_4 - C_3\tan\beta^*$$

$$C_3^* = C_3\sec^2\beta^*$$

(Note that the reference angle, β^*, is unknown at this point. Subsequent steps will, however, define all required information). Now, since $\Delta\beta$ is assumed small

$$\begin{aligned} \sin\beta &= \sin(\beta^* + \Delta\beta) \\ &\approx \sin\beta^* + \Delta\beta\cos\beta^* \\ &= \sin\beta^* + \frac{-c_2^* t + c_4^*}{-c_1^* t + c_2^*}\cos\beta^* \end{aligned} \qquad (2.1.33)$$

and

$$\begin{aligned}\cos\beta &= \cos(\beta^* + \Delta\beta) \\ &\approx \cos\beta^* - \Delta\beta \sin\beta^* \\ &= \cos\beta^* - \frac{-C_2^* t + C_4^*}{-C_1^* t + C_3^*} \sin\beta^*\end{aligned} \tag{2.1.34}$$

The third and final approximation is now made by noting that since the range angle (from initiation of guided flight to injection) is small and since most of the trajectories desired are characterized by relatively small values of the flight path angle [$\sin^{-1}$ $(\hat{r} \cdot \hat{v})$] at injection into orbit, the product $\Delta\beta$ sin β * is much smaller than $\cos\beta^*$. Thus, to the first order $\cos\beta = \cos\beta^*$. This observation means that the equations for motion in the horizontal direction (S) are nearly independent of the constants of the steering program, or conversely that this component of the injection position and velocity can be matched to a good degree without enforcing a constraint on the program. Finally, since this is the case, the steering program for β can be modified to reflect the relaxation of the constraint on S(T) without major error. As was demonstrated earlier, this relaxation is accomplished by simply equating C_1 to zero. The result of this step is now

$$\sin\beta \approx \sin\beta^* + \left[\frac{-C_2 t}{C_3^*} + \frac{C_4^*}{C_3^*}\right] \cos\beta^* \equiv K_1 - K_2 t \tag{2.1.35}$$

Thus, the problem of defining the constants C_2/C_3^*, C_4/C_3^* and β * reduces to any equivalent problem of determining K_1 and K_2.

The composite constants K1 and K2 can now be evaluated by integrating the equation for position in the $\hat{R}$ direction and substituting the boundary conditions. This task, while straightforward, is somewhat involved due to the number of terms required. For this reason, a shorthand notation will be adopted for convenience.

$$f_1 = \int_0^T \frac{dt}{(1-\dot{m}t_b) - \dot{m}t} = -\frac{1}{\dot{m}} \ln\left[1 - \frac{\dot{m}T}{1-\dot{m}t_b}\right]$$

$$\begin{aligned}f_2 = \int_0^T \frac{t\,dt}{(1-\dot{m}t_b) - \dot{m}t} &= \frac{1}{\dot{m}^2}\left\{-\dot{m}T - (1-\dot{m}t_b)\ln\left[(1-\dot{m}t_b) - \dot{m}T\right]\right\} \\ &= -\frac{T}{\dot{m}} + \frac{1-\dot{m}t_b}{\dot{m}} f_1\end{aligned}$$

$$f_3 = \int_0^T \frac{t^2 dt}{(1-\dot{m}t_b)-\dot{m}t} = -\frac{T^2}{2\dot{m}} - \frac{(1-\dot{m}t_b)T}{\dot{m}^2} - \frac{(1-\dot{m}t_b)^2}{\dot{m}^3}\left\{ \ln\left[1 - \frac{\dot{m}T}{1-\dot{m}t_b}\right]\right\}$$

$$= -\frac{T^2}{2\dot{m}} - \frac{(1-\dot{m}t_b)T}{\dot{m}^2} + \frac{(1-\dot{m}t_b)^2 f_1}{\dot{m}^2}$$

$$g_1 = \int_0^T f_1(t)dt = -\frac{1}{\dot{m}}\int_0^T \ln\left(1 - \frac{\dot{m}T}{1-\dot{m}t_b}\right)dt$$

$$= \frac{1}{\dot{m}}\left(\frac{1-\dot{m}t_b}{\dot{m}}\right)\left\{\left[1 - \frac{\dot{m}T}{1-\dot{m}t_b}\right]\left[\ln\left(1 - \frac{\dot{m}T}{1-\dot{m}t_b}\right) - 1\right] + 1\right\}$$

$$= \frac{1-\dot{m}t_b}{\dot{m}^2}\left\{\left[1 - \frac{\dot{m}T}{1-\dot{m}t_b}\right]\left[-\dot{m}f_1 - 1\right] + 1\right\}$$

$$= \frac{1-\dot{m}t_b}{\dot{m}^2}\left\{\left[1 - \frac{\dot{m}T}{1-\dot{m}t_b}\right]\left[-\dot{m}f_1\right] + \frac{\dot{m}T}{1-\dot{m}t_b}\right\}$$

$$g_2 = \int_0^T f_2(t)\,dt$$

$$= -\frac{T^2}{2\dot{m}} + \frac{1-\dot{m}t_b}{\dot{m}} g_1$$

$$g_3 = \int_0^T f_3(t)\,dt$$

$$= -\frac{T^3}{6\dot{m}} - \frac{1-\dot{m}t_b}{2\dot{m}^2}T^2 + \frac{(1-\dot{m}t_b)^2}{\dot{m}^2} g_1$$

Thus,

$$\ddot{r} = \frac{F_o}{(1-\dot{m}t_b)-\dot{m}t}\left[\left(1-\frac{c^2}{2}\right)-CDt\right](K_1-K_2t)+\vec{g}_{avg}\cdot\hat{R} \qquad (2.1.36)$$

yields,

$$\Delta\dot{r} \equiv \dot{r}(T)-\dot{r}(0)-g_{avg}\cdot\hat{R}\,T = F_o\left\{CDK_2f_3-\left[\left(1-\frac{c^2}{2}\right)K_2+K_1CD\right]f_2 + \left(1-\frac{c^2}{2}\right)K_1f_1\right\} \qquad (2.1.37)$$

and,

$$\Delta r \equiv r(T)-r(0)-\dot{r}(0)T-\frac{1}{2}\vec{g}_{avg}\cdot\hat{R}\,T^2 = F_o\left\{CDK_2g_3-\left[\left(1-\frac{c^2}{2}\right)K_2+K_1CD\right]g_2 + \left(1-\frac{c^2}{2}\right)K_1g_1\right\} \qquad (2.1.38)$$

But these results can be expressed in matrix notation as

$$\begin{Bmatrix}\dfrac{\Delta\dot{r}}{F_o}\\[2ex] \dfrac{\Delta r}{F_o}\end{Bmatrix} = \begin{bmatrix} f_3 & f_2 & f_1\\ g_3 & g_2 & g_1\end{bmatrix}\begin{bmatrix} CD & 0\\ -\left(1-\frac{c^2}{2}\right) & -CD\\ 0 & \left(1-\frac{c^2}{2}\right)\end{bmatrix}\begin{Bmatrix}K_2\\ K_1\end{Bmatrix}$$

$$= \left[\begin{array}{c|c} CDf_3-\left(1-\frac{c^2}{2}\right)f_2 & -CDf_2+\left(1-\frac{c^2}{2}\right)f_1\\ \hline CDg_3-\left(1-\frac{c^2}{2}\right)g_2 & -CDg_2+\left(1-\frac{c^2}{2}\right)g_1\end{array}\right]\begin{Bmatrix}K_2\\ K_1\end{Bmatrix} \qquad (2.1.39)$$

$$\equiv \begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{Bmatrix}K_2\\ K_1\end{Bmatrix}$$

Thus, the solution for the constants is

$$\begin{Bmatrix}K_2\\ K_1\end{Bmatrix} = \frac{1}{\Delta}\begin{bmatrix}A_{22} & -A_{12}\\ -A_{21} & A_{11}\end{bmatrix}\begin{Bmatrix}\frac{\Delta\dot{r}}{F_o}\\ \frac{\Delta r}{F_o}\end{Bmatrix} \qquad (2.1.40)$$

$$\Delta = A_{11}A_{22} - A_{12}A_{21}$$

This solution completes the problem of determining the steering equations for any single stage of a rocket vehicle under the assumptions and limitations outlined in the preceding pages. Before turning attention to the problems of generating accurate estimates of the time-to-go and the average gravity (required as a part of the mechanization), however, it is noted that a major simplification of the problem can be effected with slight error for most cases simply by assuming that since α (t) is normally small

$$\cos \alpha \approx 1$$

i.e.,

$$C = D = 0$$

This substitution will reduce the solution just outlined to one which is identical to that for the out-of-plane guidance. This fact can in turn be utilized in a guidance mechanization to reduce the storage requirements for the computer by allowing the same block of logic to be utilized for two applications.

As a final comment by way of contrast, it is noted that Cherry (References 1.3, 1.4) approached the problem of in-plane guidance in a completely different manner. Once again he noted that a modified steering logic could be prepared which would uncouple the equations of motion and allow him to treat the problem of in-plane motion as two problems of two degrees-of-freedom. This observation permits the same logic used for determining the steering constants for the out-of-plane motion to be used for the remaining problems (without assuming the flat earth) and assures that the desired terminal state can be approximated to the desired level providing that provision is made to control the mass flow rate. The guidance law, however, is not as optimal (in regards to expended propellant) as that presented earlier because of the fact that no mathematical justification (based on optimization) for the approximate steering logic can be prepared. Before discussing Cherry's formulation, it is noted, however, that his formulation is less complex than that presented earlier due to the fact that the gravity estimate has been eliminated (this step results in a less optimum solution). The equation employed by Cherry for motion along the instantaneous radius is

$$\ddot{r} = \frac{-\mu}{r^2} + \frac{(V \cos \gamma)^2}{r} + F \cos \alpha \sin \beta$$

where

$$\gamma = \sin^{-1} \frac{\bar{r} \cdot \bar{V}}{rV}$$

$$\beta = \sin^{-1} \frac{\bar{r} \cdot \bar{F}}{rF}$$

Thus, if

$$\frac{-\mu}{r^2} + \frac{(V \cos \gamma)^2}{r}$$

is smaller than the thrust acceleration (the term will normally vary between $-\frac{1}{2}$g and +2g for most orbital, lunar, and interplanetary trajectories of common interest depending on the time along the trajectory and the mission), then, a perturbed linear sine (Tangent) steering program will be near optimum (provided again that $\beta^2 \ll 1$ and $\alpha^2 \ll 1$). The law employed is

$$\sin \beta = \frac{\mu}{r^2} - \frac{(V \cos \gamma)^2}{r} + C_r + D_r t$$

which yields

$$\ddot{r} = C_r F + D_r F t .$$

Thus, the constants C_r and D_r can be readily evaluated from the present and desired radii and radial rates. The fifth component of the terminal state is controlled by matching the magnitude of the angular momentum at injection by controlling the time-to-go. (This final step will be discussed in subsequent paragraphs.)

2.1.4 Time-to-go

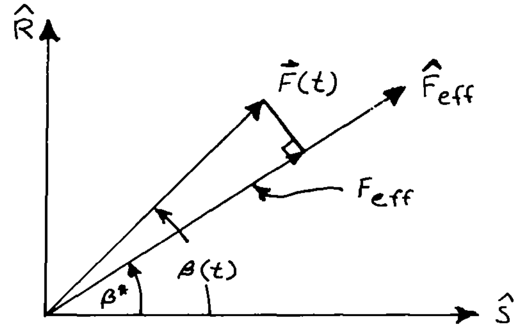
Implicit in the discussions which precede this section has been the assumption that an estimate of the time-to-go was available. Therefore, at this point in the process it is essential that several means of generating this estimate be considered.

The first procedure which is of interest was presented in Reference 1.1 and 1.2 in a slightly modified form. This approach notes that the desired terminal velocity can be matched as follows:

$$\vec{V}_f = \vec{V}_0 + \int_0^T \vec{F}\,dt + \vec{g}_{avg}\,T$$

$$\int_0^T \vec{F}\,dt = \vec{V}_f - \vec{V}_0 - \vec{g}_{avg}\,T \tag{2.1.41}$$

where $\vec{g}_{avg}$ is the time averaged gravity vector (Section 2.1.5) and T is the unknown time-to-go. But $\vec{F}$ is not constant in direction, rather it is being turned to effect the steering which is desired. Thus, there is a steering loss which will result. This loss must be estimated to assure that a realistic time-to-go is generated. Consider the following vector diagram and define the effective thrust as the component of the thrust vector in the direction required to match the terminal velocity under the assumption that there is no steering



To simplify the analysis, the assumption is made that the steering loss is dominated by the in-plane steering; in general, this assumption is well founded; however, the out-of-plane steering losses can be included without major revision. Under this assumption,

$$\left|\vec{F}_{eff}\right| = F\cos(\beta - \beta^*) \tag{2.1.42}$$

Thus, the first step in the solution process is the definition of the angle β^*. Consider the equation

$$\Delta\vec{V}_{eff} = \vec{V}_f - \vec{V}_0 - \vec{g}_{avg}\,T$$

and the corresponding components in the $\hat{R}$, $\hat{S}$, $\hat{W}$ coordinate system

$$\Delta V_{eff} \cos\alpha^* \sin\beta^* = V_{r_f} - V_{r_o} - \vec{g}_{avg} \cdot \hat{R}$$

$$\Delta V_{eff} \cos\alpha^* \cos\beta^* = V_{s_f} - V_{s_o} - \vec{g}_{avg} \cdot \hat{S}$$

Thus

$$\tan\beta^* = \frac{V_{r_f} - V_{r_o} - \vec{g}_{avg} \cdot \hat{R}\, T}{V_{s_f} - V_{s_o} - \vec{g}_{avg} \cdot \hat{S}\, T} \qquad (2.1.43)$$

Similarly, the equivalent out-of-plane steering angle (α *) can be defined, though it is not required at this time, as

$$\sin\alpha^* = \frac{1}{\Delta V_{eff}} \left[V_{w_f} - V_{w_o} - \vec{g}_{avg} \cdot \hat{W} \right] \qquad (2.1.44)$$

The second step in the process is the expansion of the function (β - β *)

$$\cos(\beta - \beta^*) = \cos\beta \cos\beta^* + \sin\beta \sin\beta^*$$

$$= \cos\beta^* \left[\cos\beta + \sin\beta \tan\beta^* \right] \qquad (2.1.45)$$

where

$$\cos^2\beta^* = \frac{1}{1 + \tan^2\beta^*}$$

$$\sin\beta = K_1 - K_2 t$$

$$\cos\beta = \sqrt{1 - (K_1 - K_2 t)^2}$$

$$\approx 1 - \frac{1}{2}(K_1 - K_2 t)^2 \qquad (2.1.46$$

$$= (1 - \frac{K_1^2}{2}) + K_1 K_2 t - \frac{K_2^2}{2} t^2$$

$$\equiv a + bt + ct^2$$

Finally, the change in velocity in the direction defined by the velocity constraint ($\Delta \vec{V}_{eff}$) can be estimated.

$$\Delta \vec{V}_{eff} = \int_0^T F \cos(\beta - \beta^*)\, dt \;\; \Delta \hat{V}_{eff}$$

$$\left| \Delta V_{eff} \right| = \int_0^T \frac{F_0 \cos\beta^*}{(1 - \dot{m} t_b) - \dot{m} t} \left[a + bt + ct^2 + \tan\beta^* (K_1 - K_2 t) \right] dt$$

$$\left|\Delta V_{eff}\right| \equiv \int_0^T \frac{F_o \cos\beta^*}{(1-\dot{m}t_b)-\dot{m}t}(d+et+ct^2)\,dt \tag{2.1.47}$$

where

$$d = a + K_1 \tan\beta^*$$
$$= \left(1-\frac{K_1^2}{2}\right) + K_1 \tan\beta^*$$
$$e = b - K_2 \tan\beta^*$$
$$= K_1 K_2 - K_2 \tan\beta^*$$
$$c = -\frac{K_2^2}{2}$$

Thus

$$\begin{aligned}\Delta V_{eff} &= F_o \cos\beta^* \Bigg\{ -\frac{d}{\dot{m}} \ln\left(1-\frac{\dot{m}T}{1-\dot{m}t_b}\right) \\ &\quad + e\left[-\frac{T}{\dot{m}} - \frac{1-\dot{m}t_b}{\dot{m}^2}\ln\left(1-\frac{\dot{m}T}{1-\dot{m}t_b}\right)\right] \\ &\quad + c\left|-\frac{T^2}{2\dot{m}} - \frac{(1-\dot{m}t_b)T}{\dot{m}^2} - \frac{(1-\dot{m}t_b)^2}{\dot{m}^3}\ln\left(\frac{1-\dot{m}T}{1-\dot{m}t_b}\right)\right|\Bigg\} \\ &= \Delta V_{IDEAL} f - gT - hT^2\end{aligned} \tag{2.1.48}$$

where

$$\Delta V_{IDEAL} = -\frac{F}{\dot{m}} \ln\left(1-\frac{\dot{m}T}{1-\dot{m}t_b}\right)$$

$$f = \cos\beta^*\left[d + e\frac{(1-\dot{m}t_b)}{\dot{m}} + c\left(\frac{1-\dot{m}t_b}{\dot{m}}\right)^2\right]$$

$$g = F_o\cos\beta^*\left[\frac{e}{\dot{m}} + \frac{c(1-\dot{m}t_b)}{\dot{m}^2}\right]$$

$$h = F_o\cos\beta^*\left[\frac{c}{2\dot{m}}\right]$$

Now these two non-linear equations for ΔV_{eff} [(2.1.41) and (2.1.48)] involving the unknown Time-to-go and several functions thereof (g_{avg}, K_1, K_2) must be solved. This solution can be accomplished iteratively by first noting that the steering losses and the gravitational losses, while significant, are not dominant. Thus, a first estimate of the time-to-go can be obtained by assuming

$$\Delta V^{*}_{eff} = \left[(\bar{V}_f - \bar{V}_o)\cdot(\bar{V}_f - \bar{V}_o)\right]^{1/2}$$

$$= -\frac{F_o}{\dot{m}} \ln\left[1 - \frac{\dot{m}T}{1-\dot{m}t_b}\right] \qquad (2.1.49)$$

i.e.,

$$T_{estimate} = \frac{1-\dot{m}t_b}{\dot{m}}\left[1 - e^{-\dot{m}\Delta V^{*}_{eff}/F_o}\right] \qquad (2.1.50)$$

(Subsequent passes through this set of equations for later estimates can use the previous solution minus the elapsed time since that solution, i.e.,

$$T_{estimate} \approx T_{n-1} - \Delta t \;)$$

Now, under the assumption that this estimate is sufficiently accurate, the iterative nature of the solution can be eliminated by expanding the terms on the right hand sides of equations (2.1.41) and (2.1.48) in Taylor series, collecting terms to form a polynomial in the correction to the estimated time-to-go, and solving. First, define the true time-to-go in terms of the estimate and the correction

$$T = T_{estimate} + \Delta T$$

(where ΔT will in general be small since the corrections are small). Next, expand the terms in the respective equations through the second order for accuracy

$$\frac{\partial}{\partial T}\left\{\frac{F_o}{\dot{m}} \ln\left[(1-\dot{m}t_b) - \dot{m}T\right]\right\}\Delta T = \frac{F_o\,\Delta T}{(1-\dot{m}t_b)-\dot{m}T}$$

$$\frac{\partial^2}{\partial T^2}\left\{\;\right\}\Delta T^2 = \frac{F_o\,\dot{m}\,\Delta T^2}{\left[(1-\dot{m}t_b)-\dot{m}T\right]^2}$$

$$\frac{\partial}{\partial T}\left\{\left[\vec{V}_f - \vec{V}_o - \vec{g}T\right]\cdot\left[\vec{V}_f - \vec{V}_o - \vec{g}T\right]\right\}^{1/2}\Delta T = -\frac{1}{\Delta V}\left[(\vec{V}_f - \vec{V}_o - \vec{g}T)\cdot\vec{g}\right]\Delta T$$

$$\frac{\partial^2}{\partial T^2}\left\{\;\right\}\Delta T^2 = \frac{1}{\Delta V^2}\left[\Delta V\,\vec{g}\cdot\vec{g} - \frac{\left[(\vec{V}_f - \vec{V}_o - \vec{g}T)\cdot\vec{g}\right]}{\Delta V}\right]\Delta T^2$$

and construct the equation

$$f(T_o)\,\Delta V_{IDEAL}(T_o) - g(T_o)\,T_o - h(T_o)\,T_o^2 + f(T_o)\left[\frac{F_o\,\Delta T}{(1-\dot{m}t_b)-\dot{m}T_o}\right]$$

$$-g(T_o)\,\Delta T - 2h(T_o)\,T_o\,\Delta T + \frac{1}{2}\left\{\frac{f(T_o)\,F_o\,\dot{m}\,\Delta T^2}{\left[(1-\dot{m}t_b)-\dot{m}T_o\right]^2}\right.$$

$$\left. - 2h(T_o)\,\Delta T^2\right\} = \left|\bar{V}_f - \bar{V}_o - \bar{g}\,T_o\right| \tag{2.1.51}$$

$$-\frac{\Delta T}{\Delta V}\left[(\bar{V}_f - \bar{V}_o - \bar{g}\,T_o)\cdot\bar{g}\right] + \frac{\Delta T^2}{2\,\Delta V^2}\left\{\Delta V g^2 - \frac{\left[(\bar{V}_f - \bar{V}_o - \bar{g}\,T)\cdot\bar{g}\right]}{\Delta V}\right\}$$

Now collecting the coefficients of the powers of ΔT, it is possible to write an equation of the form

$$A\,\Delta T^2 + B\,\Delta T + C = 0 \tag{2.1.52}$$

where

$$C = f\,\Delta V_{IDEAL} - gT - hT^2 - \Delta V_R$$

$$B = f\,\frac{F}{a_1} + \frac{a_2}{\Delta V_R} - g - 2hT$$

$$A = \frac{1}{2}\,\frac{fF\dot{m}}{a_1^2} - \frac{1}{2}\,\frac{g^2}{\Delta V_R} + \frac{1}{2}\,\frac{a_2^2}{\Delta V_R^3} - h$$

$$\Delta V_R = \left|\vec{V}_f - \vec{V}_o - \vec{g}\,T\right|$$

$$a_1 = (1-\dot{m}t_b) - \dot{m}T$$

$$a_2 = \left[\bar{V}_f - \bar{V}_o - \bar{g}\,T\right]\cdot g$$

The solution is now

$$\Delta T = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \tag{2.1.53}$$

where the sign ambiguity can be resolved by selecting the root which most closely agrees in an absolute sense with the linear estimate of the correction

$$\Delta T_{LINEAR} = -\frac{C}{B}$$

This procedure will enable the iteration process to be abandoned and will generally produce the desired accuracy. Note is made that a process similar to the one just outlined was discussed in References 1.1 and 1.2. However, the approach presented here is considered superior since uniformity in the order of the expansions used in the reference was not maintained, and since numerical experimentation has disclosed that many combinations of initial and terminal conditions require significant steering (i.e., the steering losses are significant so that the time-to-go estimates while reasonable are not accurate). Though it is noted that the desired terminal state can be achieved in spite of the resulting inaccuracy by the simple expedient of holding the steering angles at their maximum value ($\pm$) until the estimated time-to-go becomes reasonably accurate, the "fix" procedure will result in a much less optimum steering program.

At this point, attention is directed to the attitude angles α^* and β^*. These angles have not been employed except in the discussion of the steering losses in the material which preceded these discussions (rather, combinations of constants were employed); however, it was noted that such data could be employed by defining α and β in terms of a reference value and a perturbation as

$$\alpha = \alpha^* + \Delta \alpha$$
$$\beta = \beta^* + \Delta \beta$$

This approach was taken in References 1.1 and 1.2. By way of contrast, the results do not appear simpler nor do they appear to be more readily mechanized. Rather, as was noted in the discussion of the simplified in-plane guidance equations, the effect is to distort the similarity in the in-plane and out-of-plane control problems. Further, no improvement in the degree-of-optimality is achieved, since in both cases the limiting assumption is that the change in the attitude during a given burn is small. In way of defense of the reference, it is conceded that the approach which he outlines will be subject to different numerical problems than the one outlined here (i.e., the problems of round-off and loss of numerical significance may be less severe when the angles α (β) and $\Delta\alpha$ ($\Delta\beta$) are of different magnitudes). It is also conceded that his representation of the coupling between the in-plane and out-of-plane dynamics is more general. (The two should agree through the third-order terms in the series for cos α which should be more than adequate if the launch time is selected such that the desired plane-of-motion is nearly attained).

After several values of time-to-go have been predicted in this manner, a correction cycle can be superimposed on the result to provide improved predictions early in the flight by employing the memory afforded by the preceding times. This fact will assure improved accuracy in the generation of the steering command and better efficiency in the utilization of the propellant available. However, at this time no experimentation has been conducted to indicate the best form for this correction cycle.

Cherry in References 1.3, 1.4 approaches the problem of finding the time-to-go in a different manner. His approach is based on the assumptions that the rate of change of the angular momentum per unit mass of the vehicle is the moment of the applied force per unit mass (i.e., gravity is assumed to be along the instantaneous radius vector) and that the steering angles α and β are $<< 1$. Thus

$$\dot{h}(t) = \left| \vec{r}(t) \times \vec{F}(t) \right|$$

$$\approx r(t)\, F(t) \cos\beta(t)$$

$$\approx r(t)\, F(t) \left[1 - \frac{\beta^2(t)}{2} \right]$$

Now if an expansion for r = r (t) is assumed, this equation can be integrated from 0 to T and the result equated to the desired change in angular momentum. The problem with this approach is that any expansion for r = r (t) which matches the two ends of the arc will involve the time-to-go (see the discussions of average gravity). Thus, it is necessary to utilize a less precise representation in order to evaluate T and then to resolve the problem utilizing a more precise series. Assume

$$r = r_0 + \dot{r}_0 t + \frac{1}{2}\left(\frac{-\mu}{r_0^2} + \vec{F} \cdot \hat{r}_0 \right) t^2$$

$$\equiv r_0 + \dot{r}_0 t + \ddot{r}_0 \frac{t^2}{2}$$

and

$$\cos\beta = 1$$

then

$$\Delta h = \int_0^T \left(r_0 + \dot{r}_0 t + \ddot{r}_0 \frac{t^2}{2} \right) \frac{F_0}{(1 - t_b \dot{m}) - \dot{m} t}\, dt$$

$$\frac{\Delta h}{F_0} = -\frac{\ddot{r}_0 T^2}{4\dot{m}} - \frac{\left(\dot{m} \dot{r}_0 + (1 - t_b \dot{m}) \frac{\ddot{r}_0}{2} \right) T}{\dot{m}^2} - \frac{\dot{m}^2 r_0 + \dot{m} \dot{r}_0 (1 - t_b \dot{m}) + \ddot{r}_0 \frac{(1 - t_b \dot{m})^2}{2}}{\dot{m}^3} \ln\left[1 - \frac{\dot{m} T}{1 - \dot{m} t_b} \right]$$

This equation can now be solved iteratively by assuming that $(\Delta r / r_0)^2 << 1$ to obtain a first estimate for T

$$T \approx \frac{1}{\dot{m}} \left[1 - e^{-\frac{\dot{m} \Delta h}{r_0}} \right]$$

and then by modifying the process to utilize the complete equation

$$T = \frac{1-\dot{m}t_b}{\dot{m}}\left[1 - \exp\frac{\left\{\frac{\Delta h\,\dot{m}^3}{F_0} + \frac{\dot{m}^2\ddot{r}_0 T^2}{4} + \left[\dot{m}\,\dot{r}_0 + (1-\dot{m}t_b)\,\frac{\ddot{r}_0}{2}\right]\dot{m}T\right\}}{\dot{m}^2 r_0 + \dot{m}\dot{r}_0(1-\dot{m}t_b) + \frac{\ddot{r}_0}{2}(1-\dot{m}t_b)^2}\right]$$

This latter equation can be iterated (solved by the method of false position) until convergence to a desired degree is obtained. Further, if it is desired, the final value of T can be employed to evaluate the constants of the linear tangent steering program for β and the correction to T resulting from steering can be produced by integrating

$$\Delta h = \int_0^T r(t,T)\frac{F_0}{\left[(1-\dot{m}t_b) - \dot{m}t\right]}\left[1 - \frac{\beta^2(t,T)}{2}\right]dt$$

and resolving for T. However, this form of the correction cycle is considered too involved to be practical. Rather, a numerical correction cycle is considered preferable if Cherry's method is to be employed and if a suitable numerical technique can be devised. However, since more assumptions are required to obtain reliable estimates from this formulation, and since there could be trouble in the iterative solution process, the former method is preferred as an approach to determining the time-to-go.

2.1.5 Determination of the Time Average Gravity Vector

As the final step in the derivation of analytic steering equations for a single stage vehicle, it is necessary to construct a simplified model of the equations of motion. This process in turn is accomplished by considering the definition of the time average gravity vector.

$$\bar{g}_{avg} \quad \frac{\int_0^T \bar{g}\,dt}{\int_0^T dt} \tag{2.1.54}$$

where

$$\bar{g} = -\frac{\mu\bar{r}}{r^3}$$

This process requires that $\bar{g}$ be expressed as an explicit function of time. Consider the case where g will be approximated by an expansion of the following form

$$\bar{g} = \bar{g}_0 + \frac{d}{dt}\bar{g}\Big|_0 t + \bar{A}t^2 \tag{2.1.55}$$

where $\vec{A}$ is a vector constant used to assure that $\vec{g}(T)=\vec{g}(\vec{r}_f)$. But,

$$\left.\frac{d\vec{g}}{dt}\right|_o = \left.\frac{\partial\vec{g}}{\partial\vec{r}}\frac{d\vec{r}}{dt}\right|_o = \left.\frac{\partial\vec{g}}{\partial\vec{r}}\right|_o \vec{V}_o \qquad (2.1.56)$$

Consider the X component of gravity

$$g_x = \frac{-\mu x}{r^3}$$

Differentiation yields

$$\frac{\partial g_x}{\partial x} = \frac{-\mu}{r^3}\left[1 - 3\left(\frac{x}{r}\right)^2\right]$$

$$\frac{\partial g_x}{\partial y} = 3\,\frac{x}{r}\,\frac{y}{r}\,\frac{\mu}{r^3}$$

$$\frac{\partial g_x}{\partial z} = 3\,\frac{x}{r}\,\frac{z}{r}\,\frac{\mu}{r^3}$$

Similarly for the y and z components. The result is

$$\frac{\partial\bar{g}}{\partial\bar{r}} = -\frac{\mu}{r^3}\left(I - 3\begin{bmatrix} \frac{x}{r}\frac{x}{r} & \frac{x}{r}\frac{y}{r} & \frac{z}{r}\frac{z}{r} \\ \frac{y}{r}\frac{x}{r} & \frac{y}{r}\frac{y}{r} & \frac{y}{r}\frac{z}{r} \\ \frac{z}{r}\frac{x}{r} & \frac{z}{r}\frac{y}{r} & \frac{z}{r}\frac{z}{r} \end{bmatrix}\right) \qquad (2.1.57)$$

where I is a 3 x 3 identity matrix. Thus, since this matrix is known, the corresponding vector $d/dt\ (g)/_o$ is known. This fact allows the vector $\vec{A}\ T^2$ to be evaluated as

$$\vec{A}T^2 = (\vec{g}_f - \vec{g}_o) - \frac{d\vec{g}_o}{dt}T$$

Note that the product $\vec{A}\ \ T^2$ is well defined for all times $0 \leq T \leq T_{max}$.

Finally, the time average gravity vector can be computed by substitution of equations 2.1.55 into equation 2.1.54 and integrating.

$$\bar{g}_{avg} = \frac{1}{T}\left[\vec{g}_o T + \left.\frac{d\vec{g}}{dt}\right|_o \frac{T^2}{2} + \frac{\vec{A}\,T^3}{3}\right] \qquad (2.1.58)$$

$$= \vec{g}_o + \left.\frac{d\vec{g}}{dt}\right|_o \frac{T}{2} + \vec{A}\,\frac{T^2}{3}$$

2.1.6 Extension to More Than One-Stage Vehicles

The analysis of both the in-plane and out-of-plane guidance problems showed that the steering equations to be mechanized were independent of the time dependent nature of the thrust. This fact means that such variations as changes in the thrust due to progressive burn of a solid propellant, or discrete changes due to staging the vehicle do not alter the steering considered optimal. Thus, the procedures outlined on the previous pages can be employed without change for multi-staged vehicles of differing thrust levels, provided that means of estimating the burn time for each of the stages can be formulated.

To accomplish this expressed objective, it is assumed that the various stages have been designed in a near optimum manner, and that their selection for a particular mission was based on criteria of optimality which need not be considered here. It is also assumed that the propellant loadings for the various stages were selected based on rational logic. Under these assumptions, the optimal policy for utilizing the propellant is to expend all of the propellant in a particular stage (adjusted for loss and ullage) before igniting the next. This being the case, the maximum (average) burning times for each of the stages can be computed as

$$T_i = \frac{W_{p_i}}{\dot{W}_{avg_i}} \tag{2.1.59}$$

At this point in the process, the time-to-go is defined as

$$T_{go} = \sum_{i=1}^{n-1} T_i + T_n$$

where T_n denotes the unknown duration of burn of the last stage to be employed. Now, as before, an estimate of the quantity T_{go} is generated by solving equation (2.1.41) and the velocity constraint equation (2.1.48) dimultaneously. The equations for this solution are:

$$\Delta\vec{V}_{eff} = \vec{V}_f - \vec{V}_0 - \vec{g}_{avg}\left[\sum_{i=j}^{n-1} T_i + T_n\right] \tag{2.1.60}$$

and

$$\left|\Delta\vec{V}_{eff}\right| \equiv e\left\{\sum_{i=j}^{n-1}\left[\Delta V_i\right] + \frac{F_j}{\dot{m}_j}\ln(1-\dot{m}t_b)_j - \frac{F_n}{\dot{m}_n}\ln(1-\dot{m}T_n)\right\} - f\left[\sum_{i=j}^{n-1} T_i + T_n\right] \tag{2.1.61}$$

$$= e\left\{\sum_{i=j}^{n-1}\Delta V_i - \Delta V_e + \Delta V_n\right\} - f\left[\sum_{i=j}^{n-1} T_i + T_n\right]$$

where ΔV_i $(i=j...n-1)$ denotes the ideal increment in velocity corresponding to the maximum burning times for each of the stages below the one of unknown burning duration; where j denotes the number of the stage which is presently burning; where ΔV_c is the correction or adjustment in the ΔV obtainable from the jth stage to compensate for any ellapsed burning time for the stage; and where ΔV_n is the required increment in velocity for the nth stage. The procedure employed to solve this equation is as follows:

1) assume the number of stages n [this step will not be required once the first estimate of $T_n \leq T_{n_{max}}$ is obtained (unless subsequent computations of T_n are larger than $T_{n_{max}}$) since the last estimate of the number of stages can be employed].

2) solve equations (2.1.60) and (2.1.61) for T_n in the same fashion used in the single-stage formulation (in the single-stage case $V_{eff} = |\bar{V}_f - V_o| + V_c$).

3) check T_n against the maximum burning time available for the stage. If T_n is negative, reduce the number of stages by one and repeat the process. If T_n is greater than $T_{n_{max}}$, increment the number of stages by one and repeat the process. If $0 < T_n < T_{n_{max}}$, the number of stages and the time-to-go estimates have been defined and the steering constants can be computed.

The danger in this process is the possibility that the initial estimate of $\bar{g}_{avg}$ will become much less accurate as the burning times increase, due to the fact that the change in the radius was assumed to be reasonably small in order to yield a solution for the average gravity. This problem can be alleviated by defining an average gravity for each stage. However, such a step requires that intermediate radii be established (iteratively) for the terminal position for each of the stages. One means of accomplishing this objective would be to employ the estimation of $\bar{g}_{avg}$ presented in equation (2.1.58) to define a trajectory $[\bar{r} - \bar{r}\,(t)]$; and segmentation of this trajectory at times corresponding to the stage burn times to provide a means of generating improved estimates of the average gravity for each of the stages. (This process should converge quite rapidly, since gravity is not the dominant acceleration.) The term

$$\bar{g}_{avg} \sum_{i=j}^{n} T_i$$

can then be replaced by

$$\sum_{i=j}^{n} \bar{g}_{i_{avg}} T_i$$

No numerical experimentation has been performed for this case. Consequently, no estimate of the necessity of the iterative approach is readily available. Rather, it is noted that unless the problems are severe, the mechanization difficulties and the small improvement obtained for most trajectories of interest would dictate that the approach presented for single stage vehicles be employed.

2.1.7 Extension to Fixed Time Coasts

No assumption has been made in the previous analysis which would preclude calling the K^{th} stage a coast; i.e.,

$$T_{b_K} = 0$$

$$\Delta V_K = 0$$

$$T_K = T_{coast_K}$$

Thus, if the durations of the coasts are added to the total time in the computations of the gravitational loss, and if the velocity increments for the corresponding "stages" are equated to zero the result is

$$\Delta \vec{V}_{TOTAL} = \vec{V}_f - \vec{V}_o - \vec{g}_{avg} \left[\sum_{i=j}^{n-1} T_i + \sum_{i=1}^{m} T_{c_i} + T_n \right]$$
$$= c\left[\sum_{i=j}^{n-1} \Delta V_i - \Delta V_c + \Delta V_n \right] - f \left[\sum_{i=j}^{n-1} T_i + T_n \right]$$

The probability of error in the estimation of $\vec{g}_{avg}$ is now reasonably high. Thus, it is recommended that the iterative logic suggested in the extension to multi-stage vehicles be employed or that intermediate radius-velocity terminals ($\vec{r}$, $\vec{v}$ will define the coast arc), be established so that the total problem can be solved sequentially. This latter alternative is not unreasonable since the duration of the coast periods can be rationally selected only by performing a study of the effects of these durations on the requirements for the boost vehicle. Thus, if a fixed duration coast is to be commanded (based on simulations of the trajectories, etc.), it is reasonable to assume that the "terminals" can also be supplied. If so, each phase of the problem (K consecutive burns) can be considered as a separate problem.

The alternative to requiring intermediate terminals (thus losing flexibility in the targeting for the vehicle) is a more complete simulations capability for the guidance computer, for the crew and/or for the ground based tracking station.

2.1.8 Results of a Typical Simulation

The material of Sections 2.1.2 through 2.1.5 has been mechanized for numerical simulation to demonstrate the nature of the solution and the type of control derived and to define the general nature of the mechanization requirements. This simulation is illustrated in Figures 2.1.1, 2.1.2, 2.1.3, and 2.1.4. Figure 2.1.1 shows an overall logic for the guidance function and demonstrates the effects of a time delay in generating new guidance commands on the system mechanization. Figure 2.1.2 presents the first step in the process, that of the determination of time-to-go as a function of the error signals in position and velocity. Once the first estimate of time-to-go is made, estimates of the gravity vector and the steering constants generated utilizing the logic of Sections 2.1.5 and 2.1.4 as illustrated in Figures 2.1.3 and 2.1.4. At this point, the gravitational and steering losses are estimated and a correction to the estimated time-to-go (two corrections are made the first pass through the process). At this point, a corrected time-to-go can be predicted from previous memory, if available (this logic has not as yet been checked). The revised time estimate is then utilized to predict the guidance steering constants.

One point of particular merit should be noted here in regards to the computation of the steering constants. The possibility exists that the estimated time-to-go will be so poor that the steering constants C and K_1 (representing the sines of the angles α and β at the epoch t_o) will be larger than one. If this situation exists for either constant, the corresponding steering angle(s) should be set at $\pm 90°$ (depending on the sign of C and K_1) until the problem becomes better behaved.

Once the steering angles are defined, the motion of the vehicle is simulated for a small interval of time (the interval for the simulation is m times the step size of integration, where m is an input quantity intended to show the effects of appreciable computation times, i.e., time delays, on the resultant trajectory). The process is then repeated until the time-to-go reaches negligible proportions of until the magnitude of the velocity matches that which is required.

The results of two particular simulations are shown in Figures 2.1.5 and 2.1.6. These figures were generated for the problems:

$$\vec{r}_o = [\, 2. \times 10^7 \;\hat{R} - .5 \times 10^7 \;\hat{S} + .1 \times 10^6 \;\hat{W} \,] \text{ ft}$$

$$\vec{V}_o = [\, 10{,}000 \;\hat{R} + 15{,}000 \;\hat{S} + 1{,}000 \;\hat{W} \,] \text{ fps}$$

$$F_o/m_o = 35.$$

$$m/m_o = .003$$

$$\vec{r}_f = [\, 2.1 \times 10^7 \;\hat{R} + 0. \;\hat{S} + 0. \;\hat{W} \,] \text{ ft}$$

$$\vec{V}_f = [\, 0. \;\hat{R} + 25000. \;\hat{S} + 0. \;\hat{W} \,] \text{ fps}$$

and

$$\vec{r}_o = [\,2. \times 10^7 \; \hat{R} - .5 \times 10^7 \; \hat{S} + .1 \times 10^6 \; \hat{W}\,] \text{ ft}$$

$$\vec{V}_o = [\,8{,}000 \; \hat{R} + 15{,}000 \; \hat{S} + 1{,}000 \; \hat{W}\,] \text{ fps}$$

$$F_o/m_o = 35.$$

$$m/m_o = .003$$

$$\vec{r}_f = [\,2.1 \times 10^7 \; \hat{R} + 0. \; \hat{S} + 0. \; \hat{W}\,] \text{ ft}$$

$$\vec{V}_f = [\,0. \; \hat{R} + 25{,}000. \; \hat{S} + 0. \; \hat{W}\,] \text{ fps}$$

respectively. The figures show the manner in which the position relative to the desired injection point goes to zero and the nature of the approximations for the time-to-go and the steering constants. The figures also show the nature of the gravitation approximation by defining the constants of proportionality necessary to construct the "average" gravity vector from the instantaneous gravity vector and the terminal gravity vector. Note that for the major portion of the trajectory, the components of the average gravity in the g_o direction are nearly constant. (The sharp increase in this factor late in the flights is the result of indeterminancy in the solution for the proportionality constants as the elapsed time between t_o and t_f goes to zero, i.e., $\bar{g}_o \rightarrow \bar{g}_f$. No problem was encountered in the definition of the gravity vector itself). This fact implies that studies of the gravitational model employed in the analysis might reveal an extremely simple yet highly accurate empirical relationship which could be used to replace the present computation of the time average gravity vector.

Figure 2.1.7 demonstrates a measure of the off-optimum nature of the solution. Presented is a plot of the first estimate of the time-to-go for a series of problems identical to the samples which were discussed but with different components of velocity in the R direction at the time guidance is initiated. Also presented is a plot of the "true" time-to-go for the same runs. Note as the amount of steering increases (zero steering occurs for $\dot{r}_o \approx$ 9000 fps), the initial estimates become increasingly poor due to the assumptions made in developing the estimate. In all cases, however, the first estimate was better than that obtained without consideration of the steering losses. For the purposes of contrast, this latter estimate and the corresponding true time-to-go is also presented. Of particular interest is the fact that the initial guesses without the steering losses are valid (under an assumed accuracy constraint, e.g. 10%) over a smaller range and that more fuel (T_{true}) is required.

These results by no means exhaust the evaluation of the adaptive (iterative) guidance scheme. They are intended to show the nature of the solution, and to demonstrate possible revisions which will make the process more accurate, more efficient, and/or more flexible. Thus, continued effort in this area will be followed with great interest. However, the simulation does serve to indicate the approximate nature of the mechanization requirements (no attempt has been made to optimize the FORTRAN coding or core storage as would be the case in a true self-contained system). The resultant requirements are;

Time-to-go	1000 octal	(including steering losses)
Gravity	400 octal	
Guide	500 octal	
	1900 octal	

In addition, the following general purpose subroutines were mechanized: square root, natural logarithm, sine, cosine and dot product. The exact length of these routines is not known.

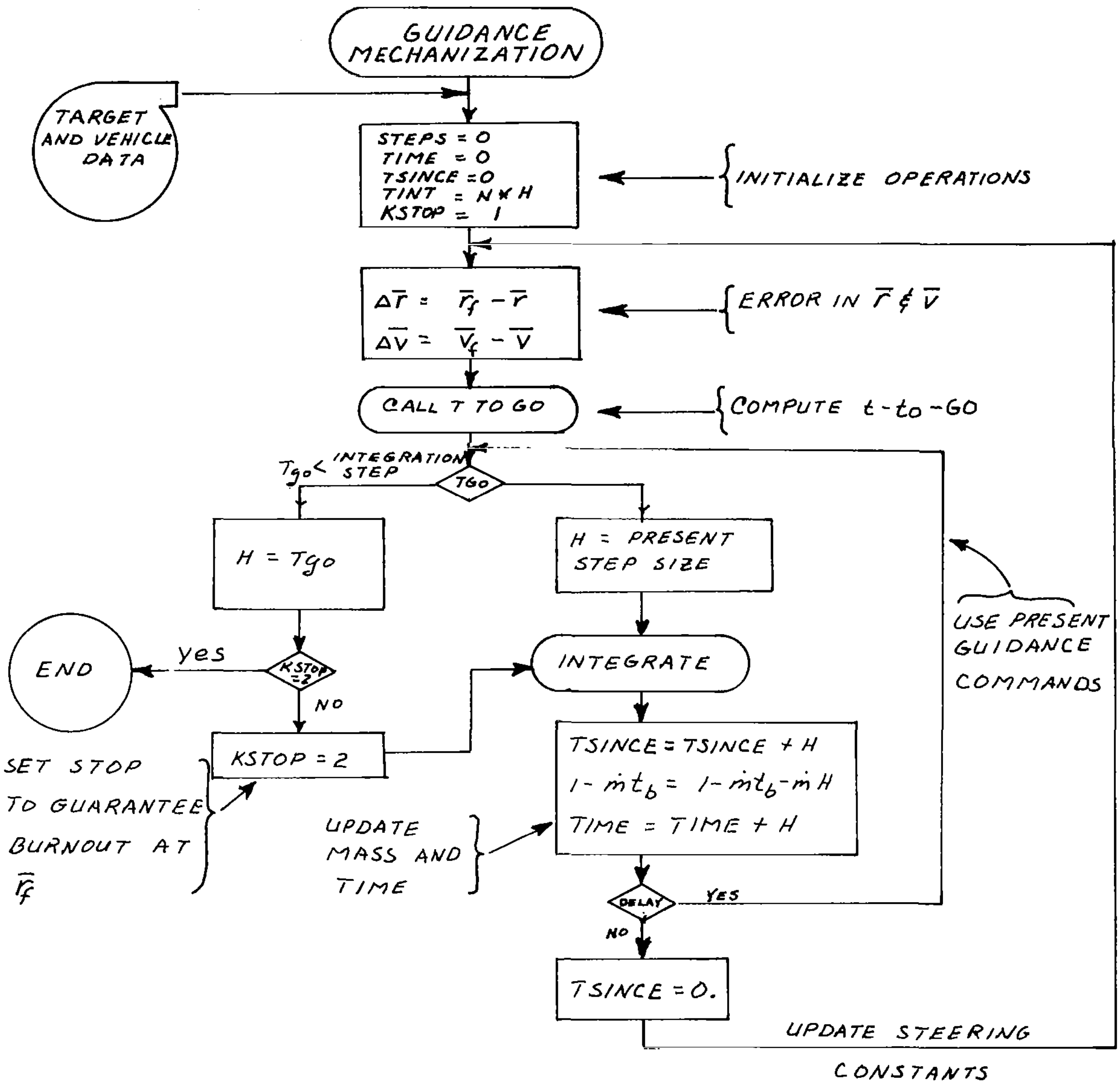


Figure 2.1.1

Iterative Guidance Mechanization

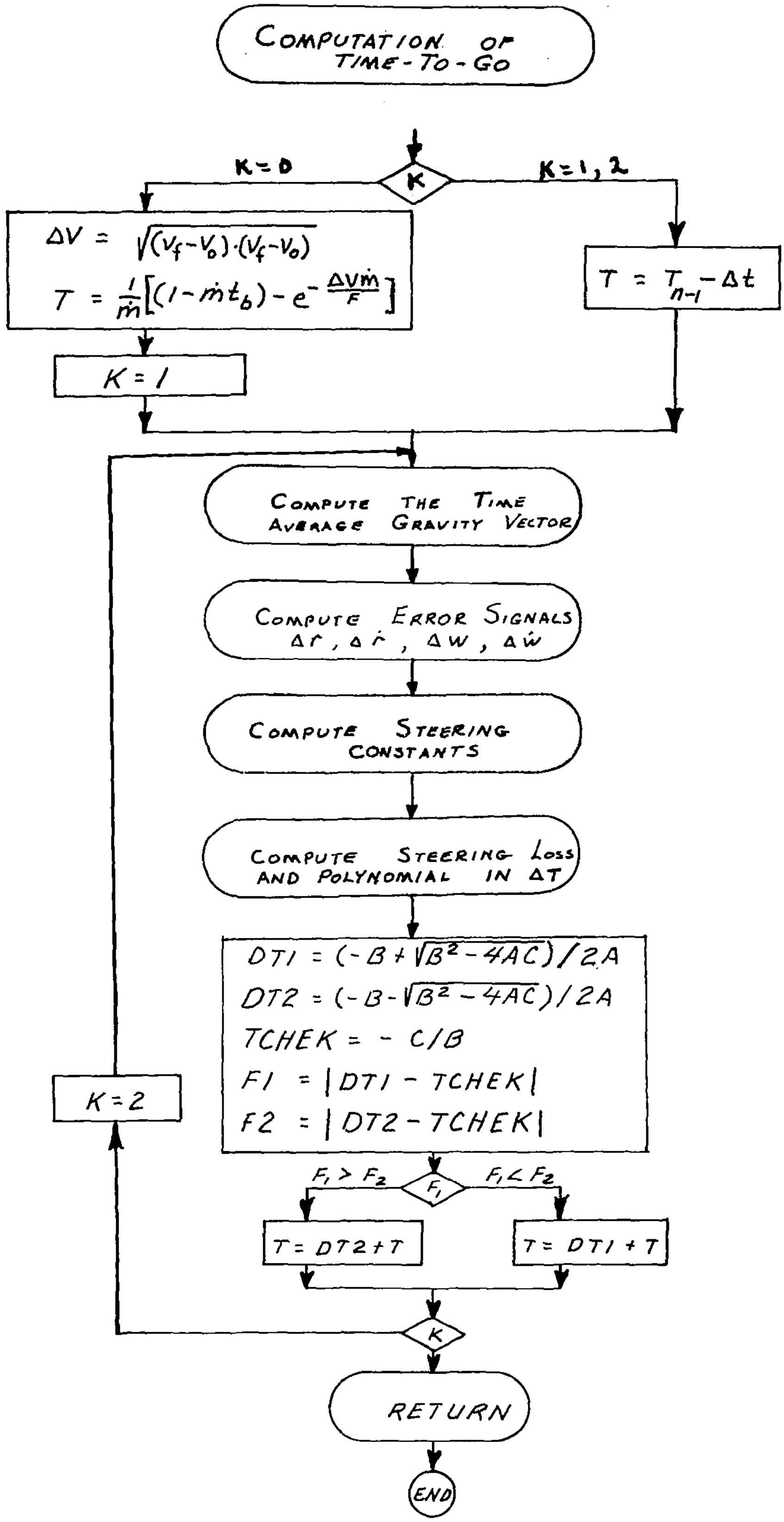


Figure 2.1.2
Time-To-Go Estimation

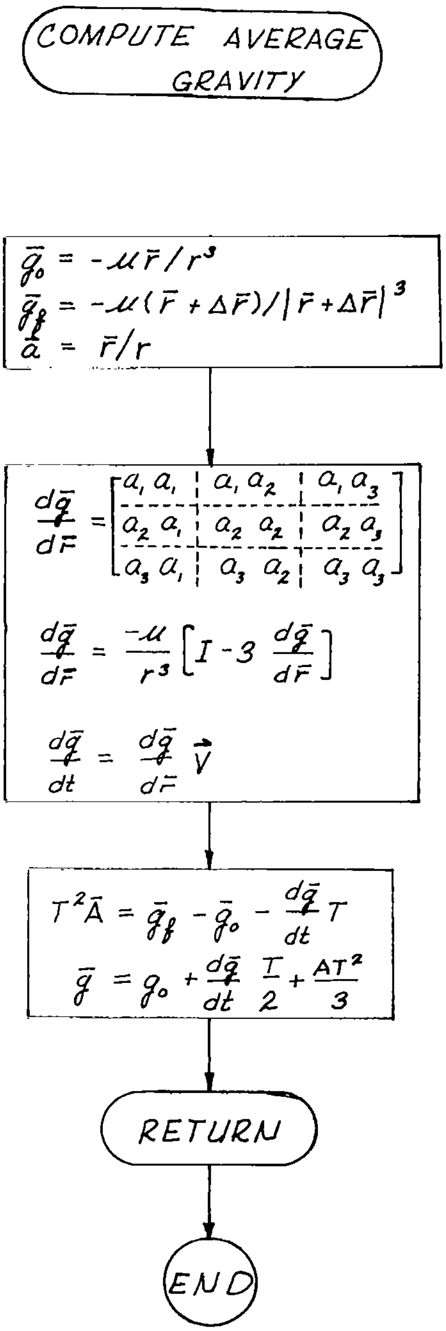


Figure 2.1.3
Average Gravity Computation

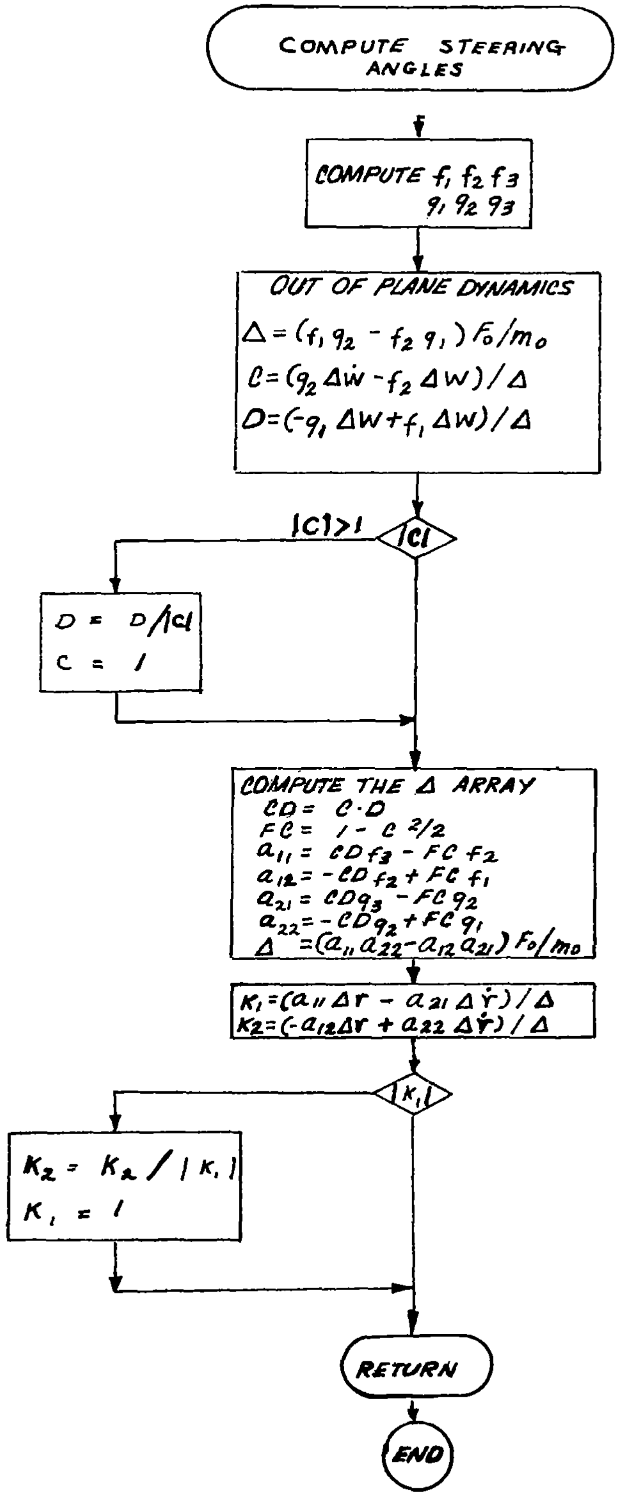


Figure 2.1.4
Steering Constant Computation

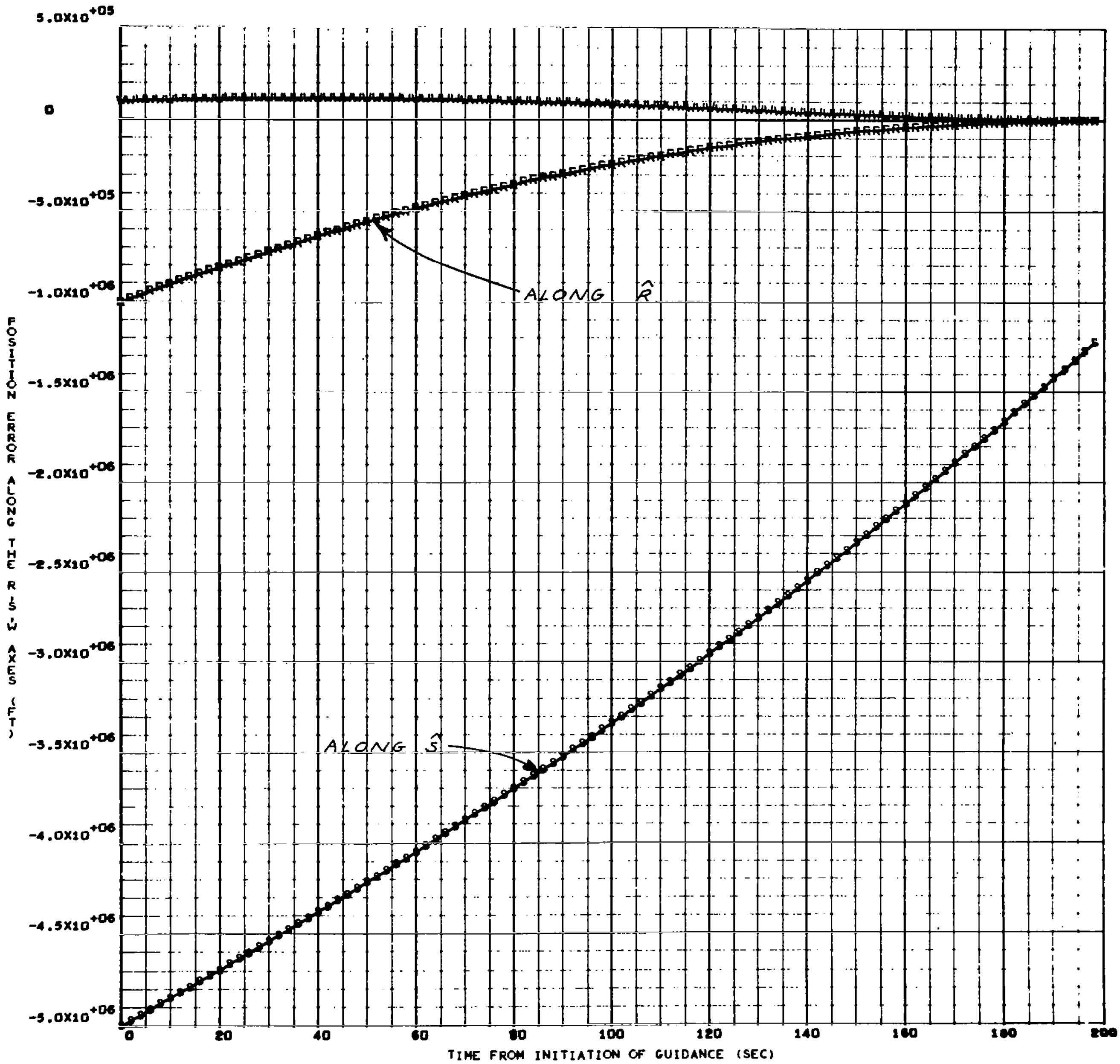


Figure 2.1.5a
Terminal Motion for Iterative Guidance

(Sample 1)

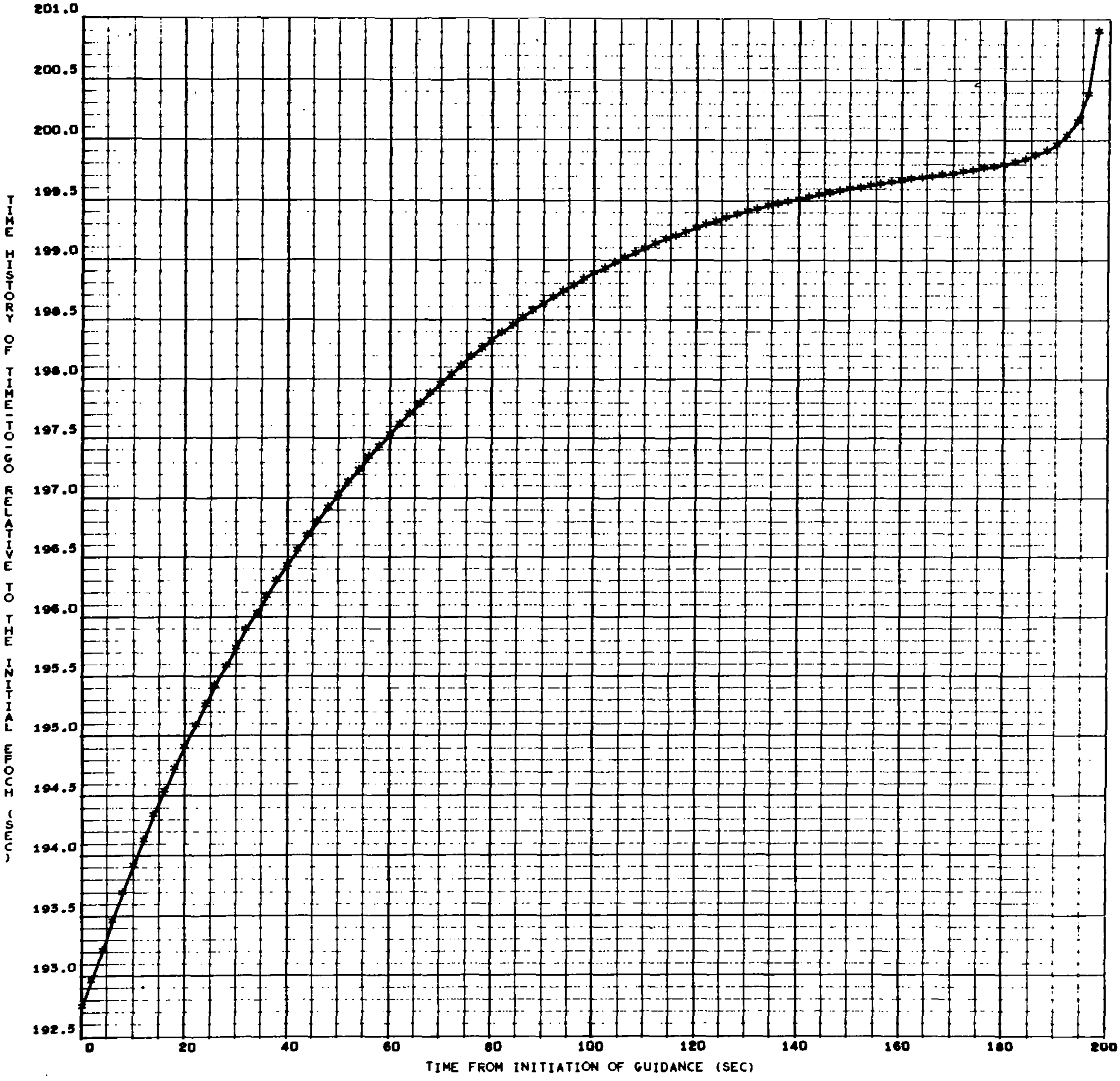


Figure 2.1.5b
Predicted Time of Flight for Iterative Guidance

(Sample 1)

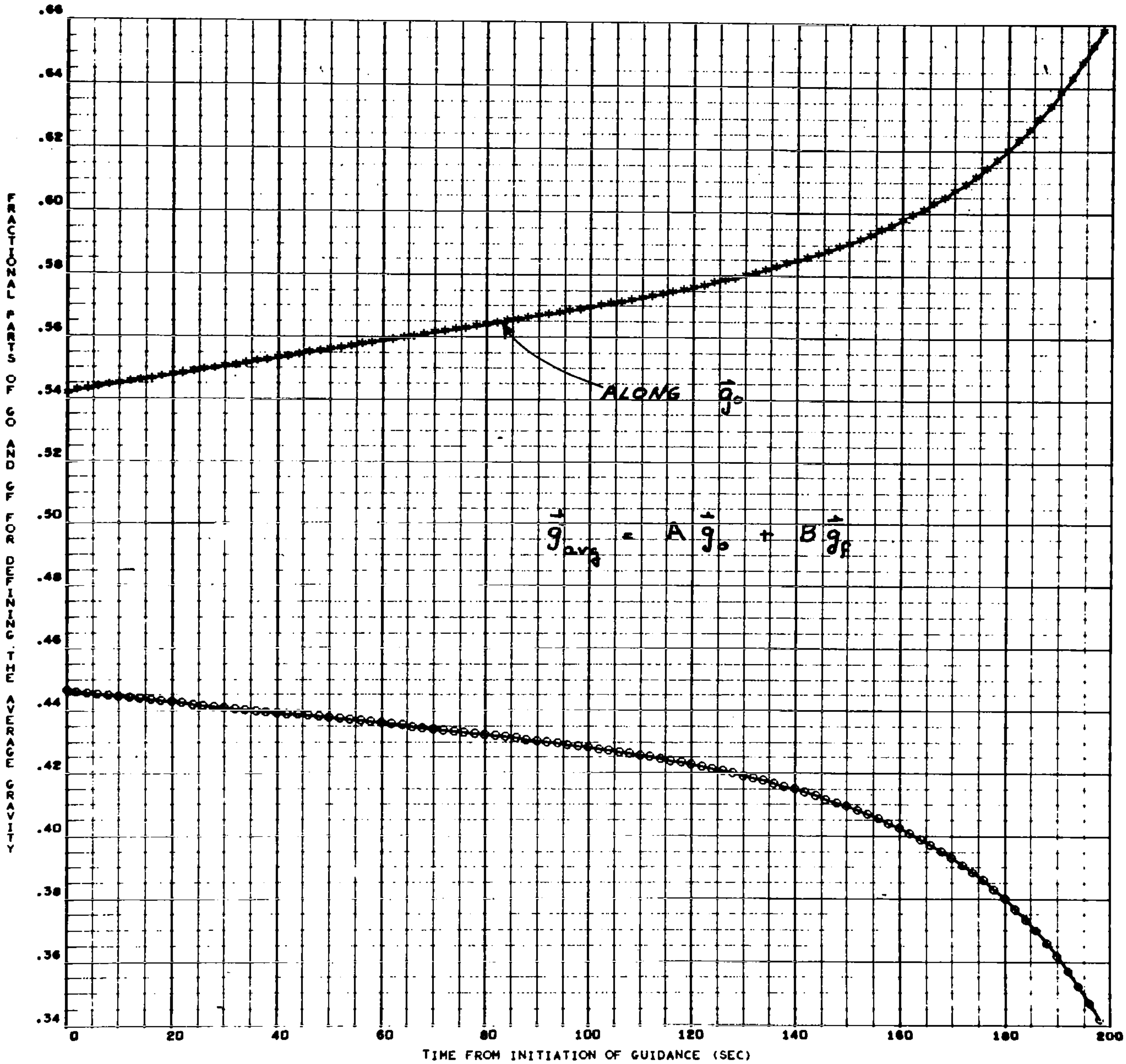


Figure 2.1.5c
Gravity Model for Iterative Guidance

(Sample 1)

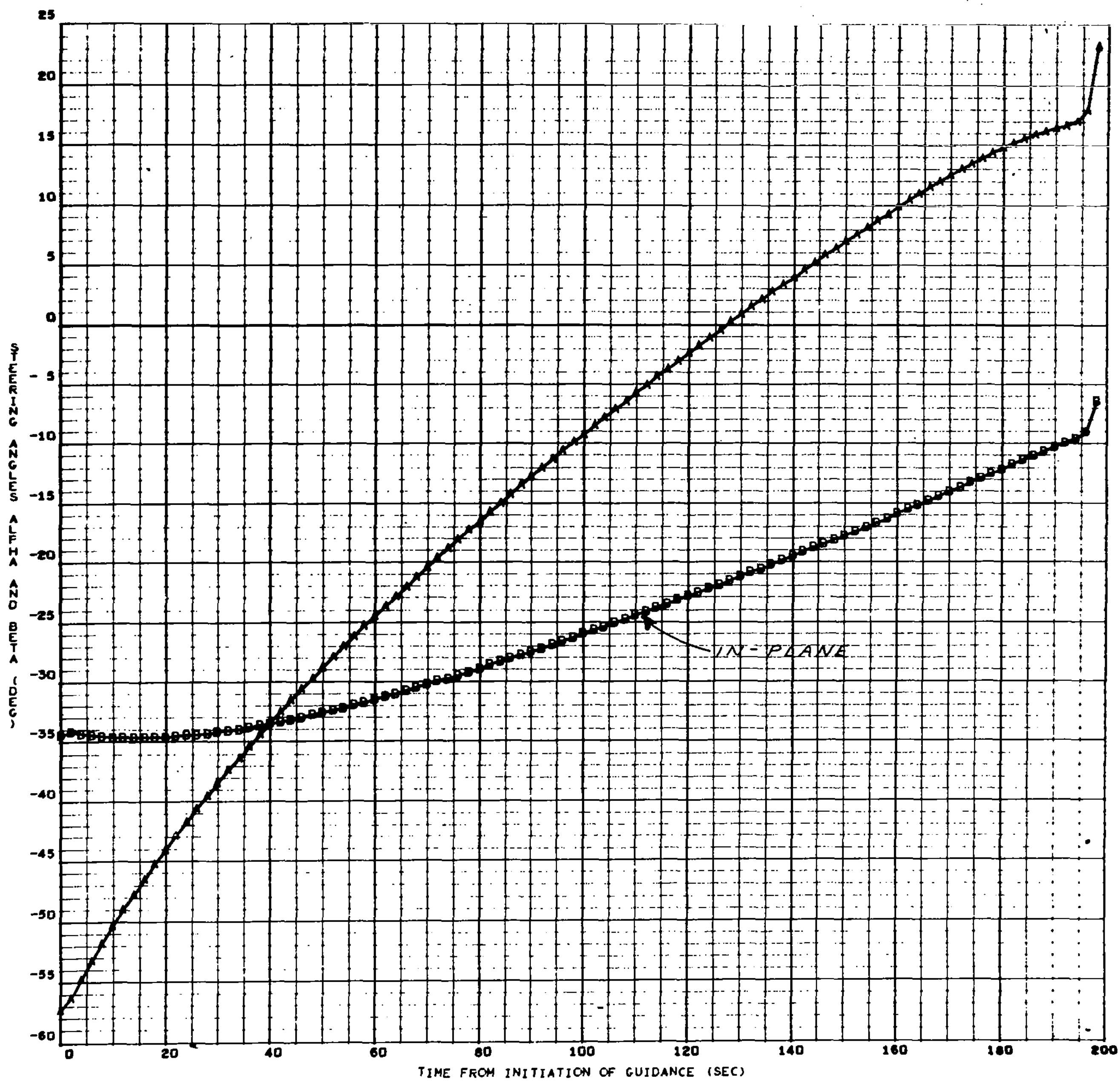


Figure 2.1.5d
Out-of-Plane and In-Plane Thrust Attitude
Angles for Iterative Guidance

(Sample 1)

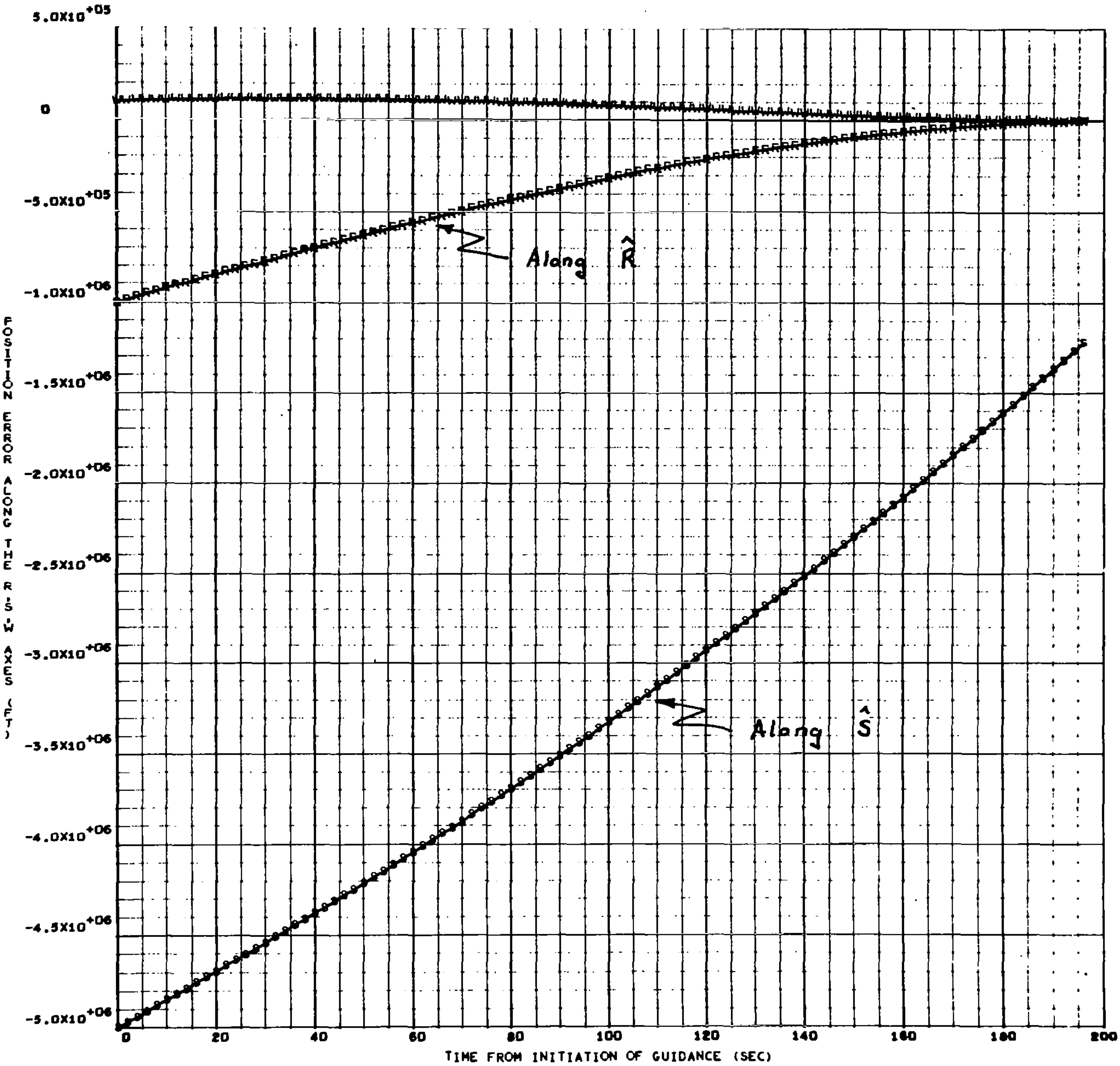


Figure 2.1.6a
Terminal Motion for Iterative Guidance

(Sample 2)

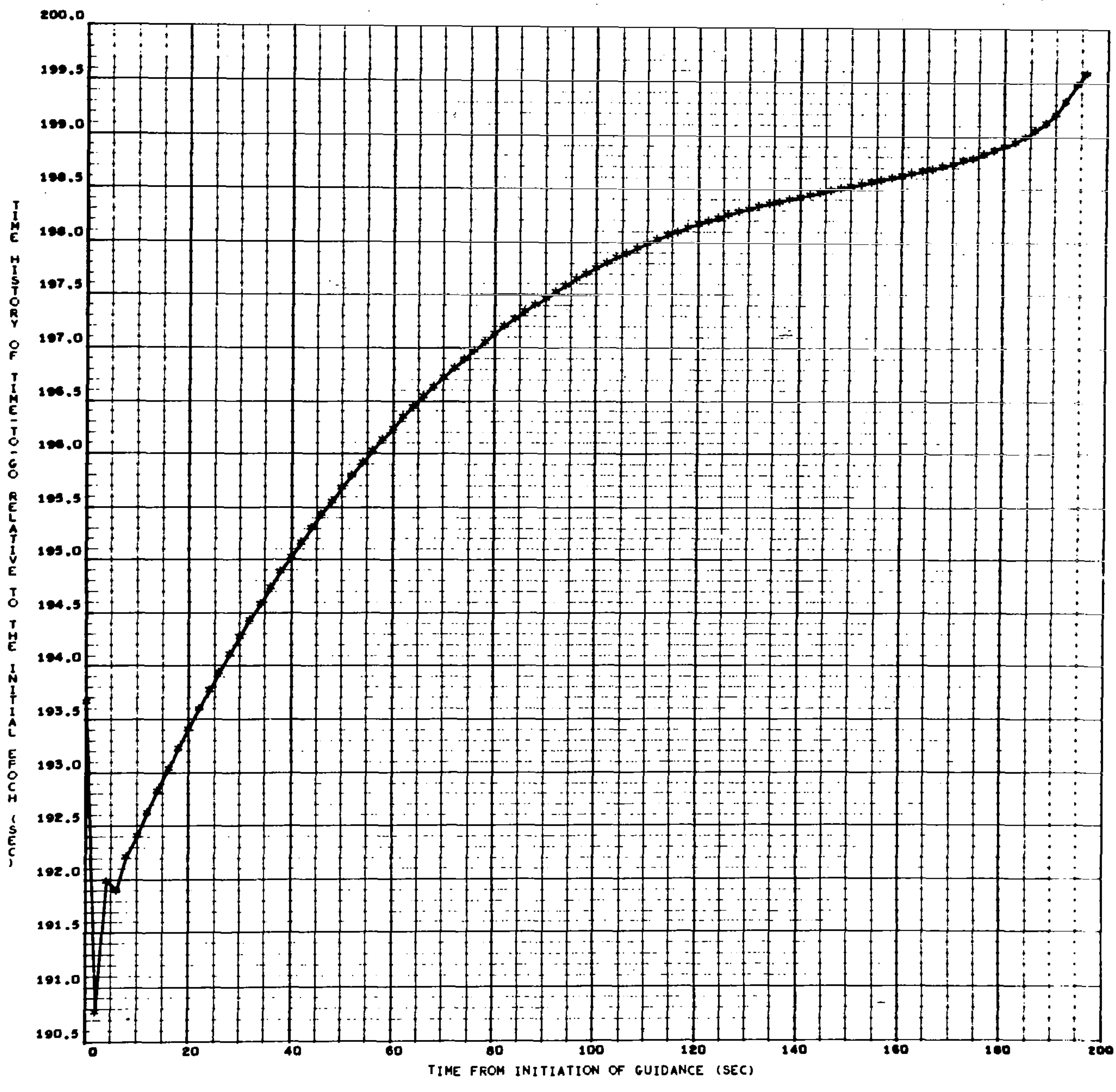


Figure 2.1.6b
Predicted Time of Flight for Iterative Guidance

(Sample 2)

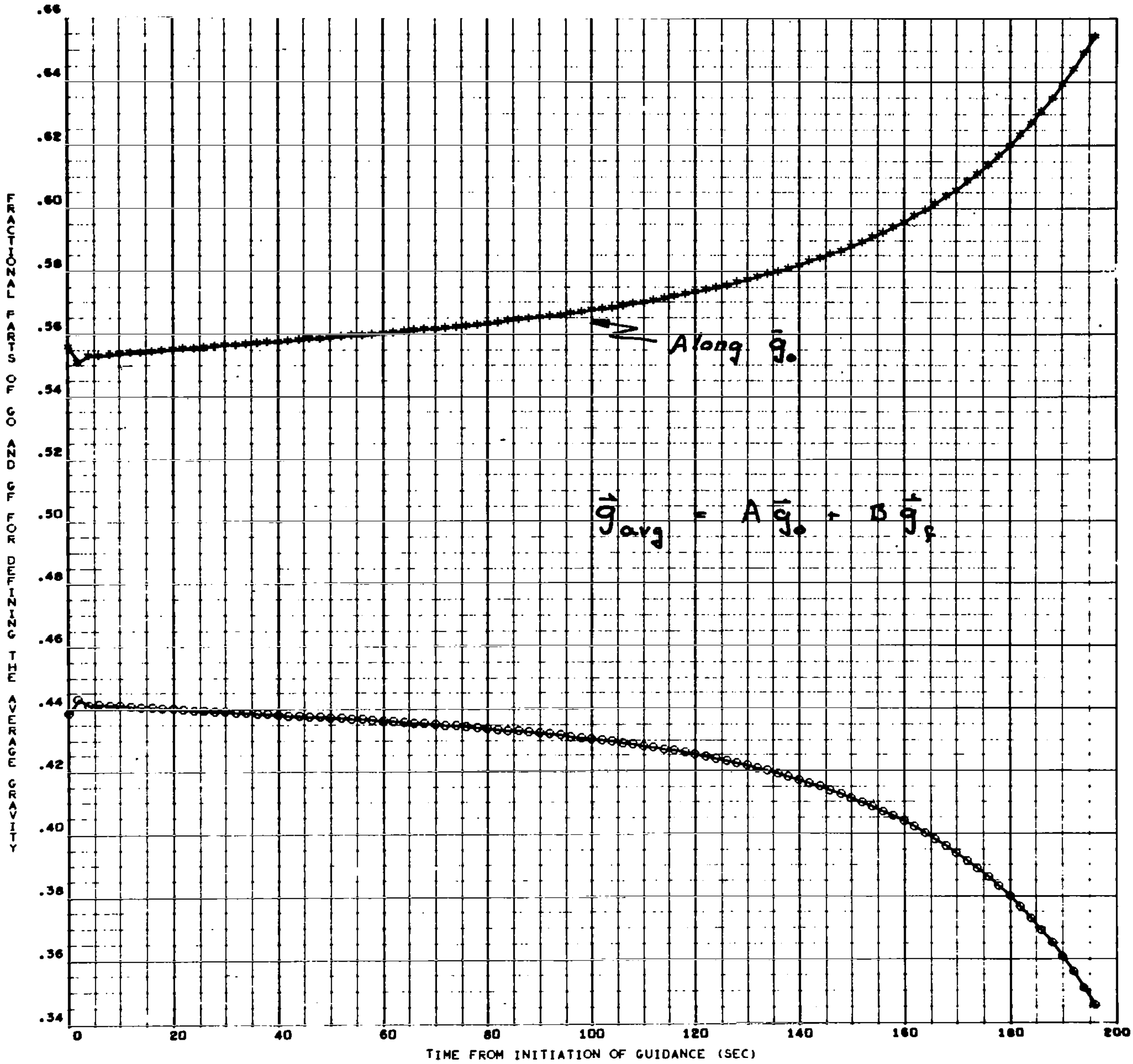


Figure 2.1.6c
Gravity Model for Iterative Guidance

(Sample 2)

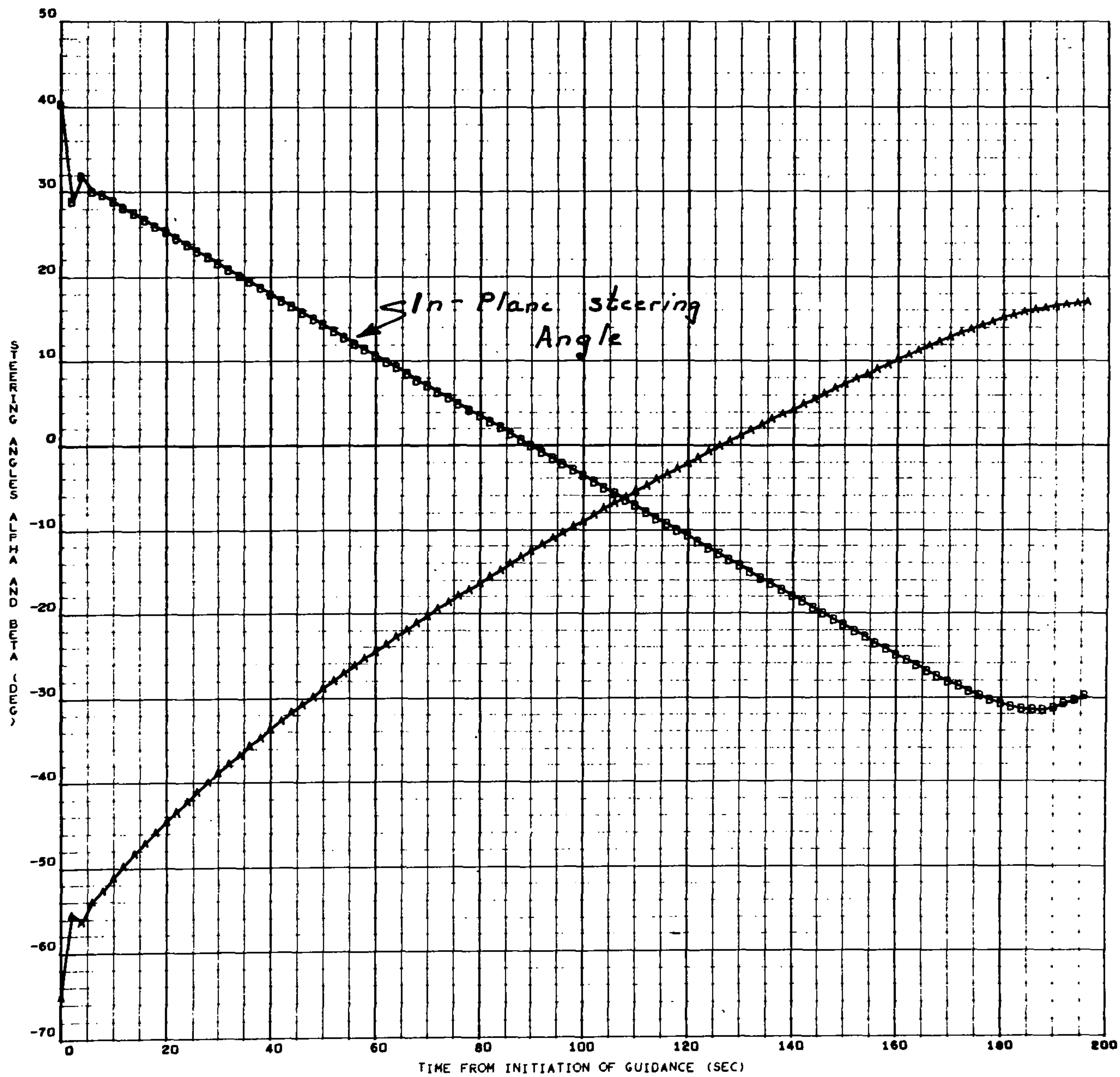
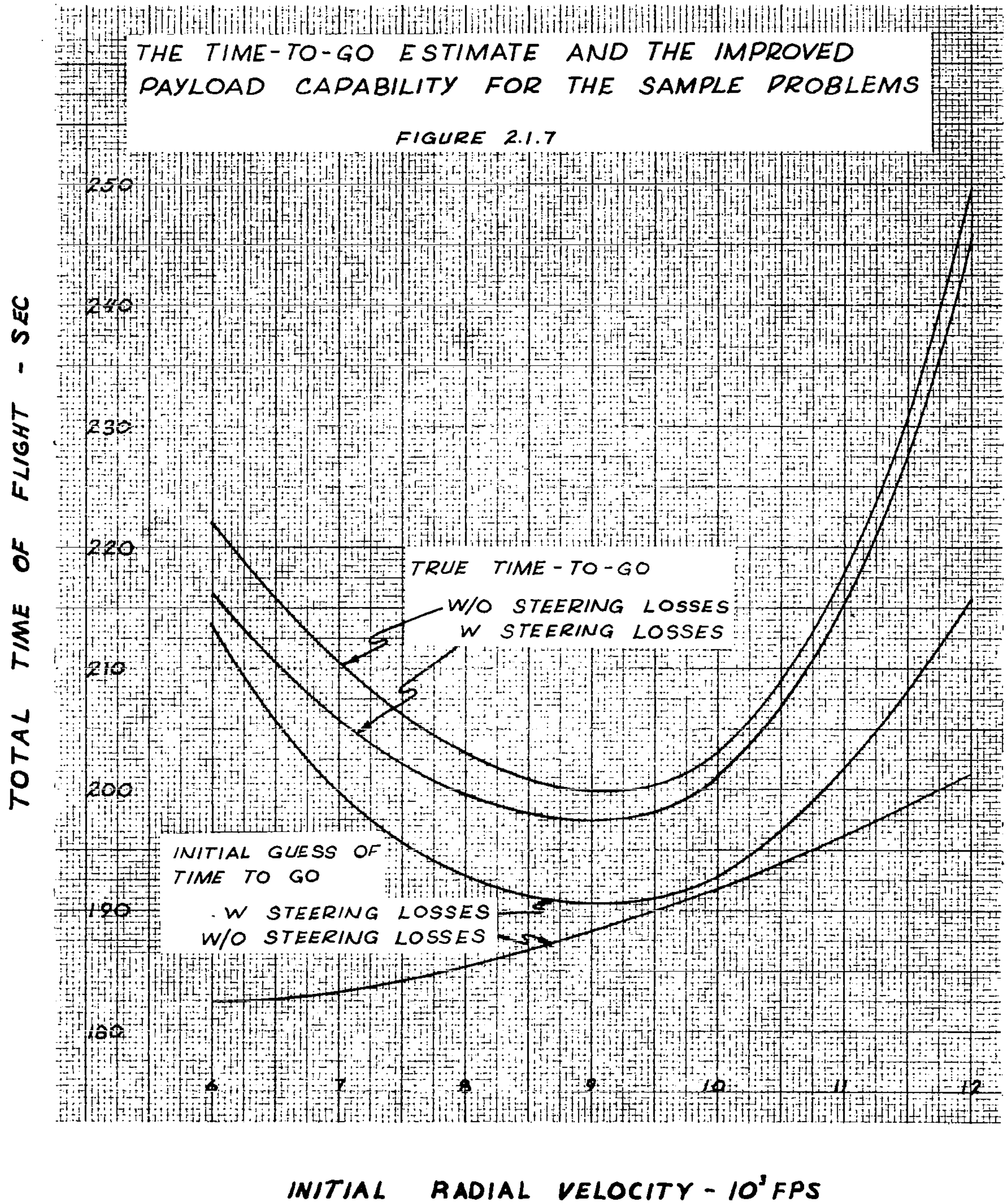


Figure 2.1.6d
Out-of-Plane and In-Plane Thrust
Attitude Angles for Iterative Guidance

(Sample 2)


THE TIME-TO-GO ESTIMATE AND THE IMPROVED PAYLOAD CAPABILITY FOR THE SAMPLE PROBLEMS
FIGURE 2.1.7
TOTAL TIME OF FLIGHT - SEC
250
240
230
220
210
200
190
180
TRUE TIME-TO-GO
W/O STEERING LOSSES
W STEERING LOSSES
INITIAL GUESS OF TIME TO GO
W STEERING LOSSES
W/O STEERING LOSSES
6
7
8
9
10
11
12
INITIAL RADIAL VELOCITY - 10^3 FPS

2.1.9 Extension to Constrain the Longitudinal Component of Position at Arrival

The discussions presented in the previous sections have been directed toward the formulation of the boost guidance problem in which only the radial and lateral rates and the terminal radius and lateral position were constrained. The present discussion is intended to introduce an approximate means of constraining the longitudinal displacement and rates.

Consider the equation for longitudinal motion

$$\ddot{S} = F \cos\alpha \cos\beta + \bar{g}_{avg}\, \hat{s}$$

where $\cos\alpha$ is assumed to obey equation (2.1.30)

$\sin\alpha$ is assumed to obey equation (2.1.35)

and consider the approximation of $\cos\beta$ by employing the identity

$$\cos\beta = \sqrt{1-\sin^2\beta}$$

$$\approx 1 - \frac{1}{2}\sin^2\beta \quad = 1 - \frac{1}{2}(K_1 - K_2 t)^2 \qquad \sin^4\beta \ll 1$$

This equation can be utilized to generate the solution for $\dot{S}(T)$ and $S(T)$; however, the solution will be non-linear in the constants K_1 and K_2. Further, there will be a set of four (4) boundary conditions which must be matched by selecting these two constants. Thus, in general, the equations will not yield a unique solution even under the assumption that they can be easily solved. This fact has lead to the partial reformulation of the problem around the lines suggested in equation (2.1.29). That is, additional constants will be introduced which will be selected so as to satisfy the terminal constraint on longitudinal motion.

Consider the equation (2.1.33) and (2.1.34) which can be expressed in the form

$$\sin\beta = \frac{a+bt}{c+dt} \approx \frac{1}{c}(a+bt)\left(1-\frac{dt}{c}\right) \qquad \left(\frac{dt}{c}\right)^2 \ll 1$$

$$\approx \frac{1}{c}\left[a+\left(b-\frac{ad}{c}\right)t\right] \qquad \frac{bd}{c^2}t^2 \ll 1$$

$$\equiv K_3 - K_4 t$$

Similarly

$$\cos\beta = \frac{e+ft}{c+dt} \approx \frac{1}{c}(e+ft)\left(1-\frac{dt}{c}\right)$$

$$= \frac{1}{c}\left[e+\left(f-\frac{ed}{c}\right)t\right]$$

$$\equiv K_5 - K_6 t$$

These linear approximations are required since the solution to be performed on the guidance computer must be linear in the steering constants to avoid iterative computations. Note that while terms involving t^2 could be easily added to the representations of both the sine and the cosine, there would be an insufficient number of boundary conditions to provide the constants without employing the dependence of these functions (at prescribed epochs) in the form

$$\sin^2 \beta + \cos^2 \beta = 1$$

However, since this identity has not been employed, it is highly probable that the resultant equations will be valid only for relatively small ranges of initial and terminal states. No attempt has been made to determine if this limitation exist or the severity of its effect on the steering program. One saving grace exists though since the results of the sample problems indicate that the steering angle β is quadratic (roughly) in time. Thus, $\dot{\beta}$ is roughly linear and

$$\dot{\beta} \cos \beta = \frac{d}{dt} (\sin \beta)$$

$$= - K_4$$

or

$$\cos \beta = - \frac{K_4}{p + gt} \approx - \frac{K_4}{p} \left(1 - \frac{gt}{p}\right) \qquad \left(\frac{gt}{p}\right)^2 \ll 1$$

$$= K_5 - K_6 t$$

Since this result is the same as that obtained in the previous series of approximations, it is assumed that the result is sufficiently accurate to allow for an approximate solution to the equations of motion.

Now, under the assumptions that these approximations are valid for some trajectories, the steering constants K_3, K_4, K_5 and K_6 can be evaluated by employing the solution presented in equations (2.1.37) and (2.1.38) by simply changing notation.

The result is

$$\begin{Bmatrix} \Delta \dot{r} \\ \Delta r \\ \Delta \dot{S} \\ \Delta S \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 \\ 0 & 0 & A_{11} & A_{12} \\ 0 & 0 & A_{21} & A_{22} \end{bmatrix} \begin{Bmatrix} K_4 \\ K_3 \\ K_6 \\ K_5 \end{Bmatrix}$$

where

$$\Delta \dot{r} = \dot{r}(T) - \dot{r}(O) - \vec{g}_{avg} \cdot \hat{R}\, T$$

$$\Delta r = r(T) - r(O) - \vec{g}_{avg} \cdot \hat{R}\, \frac{T^2}{2} - \dot{r}(O)\, T$$

$$\Delta \dot{s} = \dot{s}(T) - \dot{s}(O) - \vec{g}_{avg} \cdot \hat{S}\, T$$

$$\Delta s = s(T) - s(O) - \vec{g}_{avg} \cdot \hat{s}\, \frac{T^2}{2} - \dot{s}(O)\, T$$

A_{11}, A_{12}, A_{21}, A_{22} defined in equation (2.1.39)

Finally, since the coefficient matrix relating the steering constants is partitional as it is, the solutions for the constants is

$$\begin{Bmatrix} K_4 \\ K_3 \end{Bmatrix} = A^{-1} \begin{Bmatrix} \Delta \dot{r} \\ A_r \end{Bmatrix}$$

$$\begin{Bmatrix} K^6 \\ K^5 \end{Bmatrix} = A^{-1} \begin{Bmatrix} \Delta \dot{s} \\ \Delta s \end{Bmatrix}$$

where

$$A^{-1} = \frac{1}{\Delta} \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}$$

$$\Delta = A_{11} A_{12} - A_{12} A_{21}$$

and the "optimum" steering angle is given by

$$Tan\, \beta = \frac{K_3 - K_4 t}{K_5 - K_6 t}$$

Note that because of the previous approximations, the solutions for the sine and cosine of β are independent and that the constants K_3 and K_4 are identically K_1 and K_2, respectively. This fact allows the approximate constraint on the longitudinal motion to be added or deleted from the guidance solution as desired without modification to the formulation by simply bypassing a portion of the logic. Care must, however, be exercised to assure that the assumptions implicit in the development are not violated.

2.2 EXPLICIT GUIDANCE EMPLOYING GUIDANCE POLYNOMIALS

2.2.1 Introduction

The development of rocket vehicles that are capable of injecting multi-ton payloads into orbit has established the need for a guidance system that differs from those that were developed for ballistic missiles. Further, the high cost of the vehicles, along with their utility in the Apollo program to carry astronauts makes it necessary to have a guidance system with a high degree of reliability. These objectives can be attained by increasing the flexibility of the guidance system to make it compatible with the characteristics of these vehicles. For example, the enormous thrusts that are generated by the early stages are normally achieved by clustering engines. Therefore, if one of the engines fails a discrete variation in the thrust would result at an unpredictable point on the trajectory. This failure would not, however, effect the total energy available to complete the flight but would result in a lower rate at which energy is available; this fact would in turn alter the shape of the trajectory which can be flown. However, if the guidance system has been designed to accommodate this type of failure, it is still possible to complete the specified mission.

Further, the large vehicles are expected to be used for a variety of different missions. Thus, a guidance system with sufficient flexibility to be used for each of these missions without major redesign or modification is highly desirable due both to considerations of cost and reliability. And finally, the complexity of these vehicles are such that the probability of their achieving lift-off at a specific instant is small. Thus, the guidance system must be designed to compensate for these variations.

The objective of this section is to explore one approach to the problem of providing a guidance logic adequate for all of these requirements. To this end, the following paragraphs have been prepared.

2.2.2 Preliminary Considerations

In the discussion to follow, it will be assumed that a navigation system is available that indicates the vehicle position, velocity, acceleration, and attitude in a continuous mode. For example, such a system could be either inertial or radio navigation. If an inertial navigation system is used, then accelerometers are mounted on a gyro stabilized platform and the position and velocity of the vehicle are computed from the accelerations that are measured using these instruments. If the radio guidance system is used, one or more radar units will observe the distance of and direction to the vehicle, and this information will then be used to compute the position velocity and acceleration.

The guidance process is defined as the plan by which the navigation information is used to control the flight of the vehicle. For a rocket vehicle this process will take the form of two sets of equations. The first set is defined as the steering equations and is used to compute the direction the thrust vector should have for the vehicle to achieve the desired flight path. The second set of equations is used to compute the engine throttle setting. In general, this second set includes (1) the time of thrust initiation, (2) instantaneous thrust magnitude, (3) time of thrust termination. However, the large liquid vehicles presently being built have fixed thrust engines so that the instantaneous thrust magnitude can be controlled only in a small region about the nominal. As a consequence, only the times of thrust initiation and termination is generally computed.

A number of different guidance processes have been developed. Among this set, are those for which the steering equations have been intuitively selected. For example, one of these schemes orients the engines so that the thrust acceleration is in the same direction as the velocity-to-be-gained. (The difference in the velocity vector required to achieve the specified terminal position at some future time in free flight and the instantaneous velocity vector). In this system the navigation system is used to monitor the vehicle flight, and the thrust is terminated when the velocity-to-be-gained has been driven sufficiently close to zero. The time of thrust initiation is selected to give some desired end condition, with the aid of a simulation of the process. It is probably possible to freeze a process of this nature and engineer extensions that will handle particular missions. However, the previously mentioned objections still persist, the (1) the knowledge of the applicability of a guidance process is limited to one mission; (2) the degree to which the process reaches its theoretical optimum is not known.

For reasons such as those mentioned, the decision was made to start afresh with new concepts that were independent of vehicle and mission configuration. The objective of this new approach was to develop a guidance process which would adapt to the particular vehicle and mission being flown (thus, the designation path-adaptive). In this approach, the same basic guidance plan will be applied to all missions and configurations, with the steering and throttle equation being selected to minimize a particular loss function.

2.2.3 Formulation of the Guidance Equations

In order to gain some insight into the areas over which an optimization of the guidance mode may be made, consider Figure 2.2.1

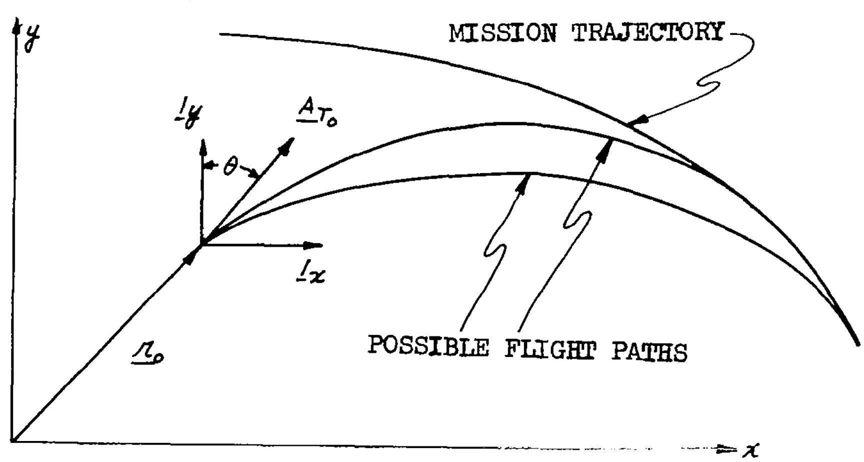


Figure 2.2.1 Choice of Flight Path

In this figure, the flight space is assumed to be two-dimensional, and the objective is to achieve the mission trajectory. The mission will be represented mathematically by the set of equations

$$f_i(x, y, \dot{x}, \dot{y}, t) = 0 \qquad i = 1, 2, 3 \tag{2.2.1}$$

where the solution to the system of equations in x, y, $\dot{x}$, $\dot{y}$, and t is, of course, not unique, since any point on the curve represents a solution. Therefore, rather than choose one particular solution as a standard injection point, the mission is left specified in a functional form. This degree of generality assures that guidance may be generated at any given point in the flight, $\underline{r}_0$, which is which is "best" in the sense of some scalar comparison function and which will satisfy the mission.

If desired, an arbitrary form of thrust direction history could be assumed, and the corresponding optimization for the best solution of the system of mission equations (2.2.1) made. It seems wisest, however, to seek that form which is optimum over all possible functional forms and to use the one function of that family which also provides the optimum solution to the mission equations. That is, among all steering functions which result in flight paths that have end points that satisfy the mission conditions, the one that minimizes some specified quantity is sought. This type of problem is treated in the Calculus of Variations, where theory exists that is useful for singling out the one particular desired function.

2.2.4 Sample Steering Function Solution

In order to obtain a guide to the approach and methods necessary to implement the adaptive guidance mode, using the concepts of the Calculus of Variations, it is helpful to consider a simplified problem; thus drawing attention to those steps that are characteristic of the methods applied to actual flight problems.

The particular problem considered was selected for two reasons. First, a closed form solution can be given for it. Second, the simplification of the problem allows a closer exposition of the salient features of the approach taken.

Consider the flight of a vehicle in an inverse square gravitational field that is propelled by a rocket having a constant thrust and mass flow rate. This physical situation can be represented by the following system of equations.

$$A_T = \frac{A_{T_0}}{1 + \left(\frac{\dot{m}}{m}\right)_0 (t - t_0)} \tag{2.2.2}$$

$$\ddot{\underline{r}} = \underline{A}_T - k^2 r^{-3} \underline{r} \tag{2.2.3}$$

where

$$\ddot{x} = \underline{l}_x \cdot \ddot{\underline{r}} = A_T \sin\theta - \frac{K^2}{r^3} x \tag{2.2.4}$$

$$\ddot{y} = \underline{l}_y \cdot \ddot{\underline{r}} = A_T \cos\theta - \frac{K^2}{r^3} y \tag{2.2.5}$$

Assume the initial conditions

$$t_o, \underline{r}_o, \underline{v}_o, \underline{A}_{T_o}, \left(\frac{\dot{m}}{m}\right)_o \tag{2.2.6}$$

to be known and that the mission criteria expressed by equations (2.2.1) are to be achieved with a minimum thrust time ($t_f - t_o$). The steering function which does this can be determined via the calculus of variations to satisfy the following equations

$$\ddot{\lambda}_1 = \frac{k^2}{r^5}\left[\lambda_1(2x^2 - y^2) + 3\lambda_2 xy\right] \tag{2.2.7}$$

$$\ddot{\lambda}_2 = \frac{k^2}{r^5}\left[3\lambda_1 xy + \lambda_2(2y^2 - x^2)\right] \tag{2.2.8}$$

$$0 = \lambda_1 \cos\theta - \lambda_2 \sin\theta \tag{2.2.9}$$

(where the initial conditions for the Lagrange multipliers for each component of the terminal state which is unconstrained may be selected arbitrarily and where the remaining multipliers must be determined by trial and error) together with equations (2.2.2), (2.2.4), (2.2.5), and the end conditions expressed by equations (2.2.1) and (2.2.6).

This set of equations constitutes a two-point boundary value problem. One form of the solution would be the set of functions $\theta(t)$, $\underline{r}(t)$, $\dot{r}(t)$ that meet the stated requirements, based on the absence of any disturbances in the interval from t_f to t_o. If a solution is obtained based on a state which is taken later on along the mission trajectory, the same set of functions $\underline{r}(t)$, $\dot{r}(t)$ and $\theta(t)$ would be obtained.

However, as soon as a disturbance occurs, a new set of solutions will result. Thus, the solutions $\dot{r}(t)$, $r(t)$, and $\theta(t)$ are in turn functions of the initial conditions and could be written explicitly as functions of these parameters. In particular, the function $\theta(t)$ evaluated at $t = t_o$ is

$$\theta(t_o) = \theta\left[t_o, r_o, v_o, A_{T_o}, \left(\frac{\dot{m}}{m}\right)_o\right] \tag{2.2.10}$$

This equation expresses the solution of the desired thrust direction in terms of the measured state at the initial epoch. The remaining end conditions, while they are not given explicitly, can be expressed similarly.

The resulting set of equations for the unknown end values in terms of those that are known may be thought of as a second form of the solution to the two-point boundary value problem. The most important of these, however, are equation (2.2.9) and a similar expression for t_f. Thus equation (2.2.9) is the steering function and the expression for t_f would be the thrust termination function.

The success of path-adaptive guidance is determined largely by the quality of the equations that are mechanized in the guidance computer. This, in turn, depends on the nature of the function necessary to accomplish the desired results and upon the manner in which it is prepared for the computer. However, guidance functions that represent the optimum exactly (to measurable accuracy) can be expected to require more computer complexity and weight than simplified approximations to these functions that produce approximate optimums. Thus, trade-off studies are required to determine the allowable degree of approximation for each of the applications.

2.2.5 Analytical Approach to Guidance Function Representation

The form of the solution of the two-point boundary value problem discussed previously (equations (2.2.1) to (2.2.9)), may be attempted analytically. Conceptually, this solution could proceed as in the following simplified example. (Analytic solutions to more complicated problems are discussed in Reference 2.1 and 2.2)

Consider the problem of steering to a point in the (μ, V) space from arbitrary initial conditions

$$\mu_o , V_o \tag{2.2.11}$$

to the fulfillment of the mission criteria.

$$\mu_f - \mu = 0 \tag{2.2.12}$$

$$V_f - V = 0 \tag{2.2.13}$$

If the equations of motion are

$$\dot{\mu} = \sin \alpha \tag{2.2.14}$$

$$\dot{V} = \cos \alpha \tag{2.2.15}$$

and if the objective of the guidance system is to minimize $t_f - t_o$, application of the calculus of variation provides the equations that define α as

$$\dot{\ell}_1 = 0 \tag{2.2.16}$$

$$\dot{\ell}_2 = 0 \tag{2.2.17}$$

$$\ell_1 \cos\alpha - \ell_2 \sin\alpha = 0 \tag{2.2.18}$$

These equations have the solution

$$\alpha = \alpha_0 \tag{2.2.19}$$

Thus, upon substitution of this solution into the equations of motion and integrating from t_0 to t_f .

$$u_f - u_0 = (t_f - t_0) \sin\alpha_0 \tag{2.2.20}$$

$$V_f - V_0 = (t_f - t_0) \cos\alpha_0 \tag{2.2.21}$$

This system is then solved for μ_f and V_f substitution made into equations (2.2.12) and (2.2.13).

$$u_f - u_0 = (t_f - t_0) \sin\alpha_0 \tag{2.2.22}$$

$$V_f - V_0 = (t_f - t_0) \cos\alpha_0 \tag{2.2.23}$$

The simultaneous solution provides the steering function

$$\alpha_0 = \arctan \frac{u_f - u_0}{V_f - V_0} \tag{2.2.24}$$

and the cutoff function

$$t_f = t_0 + \sqrt{(u_f - u_0)^2 + (V_f - V_0)^2} \tag{2.2.25}$$

Now consider the more realistic problem discussed in section (2.1) in its most complete or in its simplified form. For this problem, the equations of motion are more involved and the variables are coupled with the result that no simple solution for the Lagrange multipliers or the corresponding steering angle can be obtained. Thus, the analytic solution (or simple iterative solution) required for an onboard guidance computer cannot be realized. This fact has lead to an empirical approach to the problem; this approach will be presented in the following section.

2.2.6 Guidance Function Generation Using An Empirical Approach

For this approach, the two-point boundary value problem represented by equations (2.2.1) through (2.2.8) is solved numerically for a particular assumed set of initial conditions, equation (2.2.6). This process is repeated for a large variety of values that lie within a region that contains all of the disturbances that the vehicle is designed to withstand. Thus, a tabular representation of the guidance functions is constructed which may then be approximated by a polynomial. This task has thus become a curve fitting problem for a function of several variables. No general theory presently exists to accomplish this objective; however, some relatively good results have been obtained using intuitive procedures.

For evaluation of this approach, a test problem was postulated. The task was to derive a steering function for the vacuum flight of the second stage of a vehicle. Thus, the initial point is the cutoff point of the first stage. No cutoff function was employed, rather it was assumed that the fuel would be burned to depletion. (The application of this technique to other tasks is discussed in References 2.4 and 2.5.)

The procedure used to build the table for the steering function was to isolate a family of optimum trajectories originating from the area of the first stage cutoff conditions and satisfying the terminal end conditions. The family consisted of 126 trajectories for which variation had been made in the upper stage thrust level. It should be noted that each point on any one of the trajectories was in fact another initial conditions, thus providing another value for the table. For each trajectory of the family, points were read every five seconds to provide a time history of the steering function.

$$\theta_o \left[\underline{r}_o , \underline{\dot{r}}_o , \underline{A}_{T_o} , (\dot{m}/m)_o \right] \tag{2.2.26}$$

Since this mission is independent of time, t_o does not occur in the steering function.

The curve fitting procedures chosen to represent (2.2.28) was the method of least squares, since it was felt that this process would lead to a polynomial form that is especially convenient for an on-board computer. Further, the polynomial that was used to approximate equation (2.2.26) was of the form

$$\theta_j = a_1 W_{1j} + a_2 W_{2j} + a_3 W_{3j} \cdots a_i W_{ij} \tag{2.2.27}$$

where W_{ij} are generalized product functions of the type

$$W_{ij} = x_j^h \, y_j^k \, \dot{x}_j^p \, \dot{y}_j^q \, A_{T_j}^r \left(\frac{\dot{m}}{m} \right)^s \tag{2.2.28}$$

and where the indices range over the set that includes all postulated powers for the various factors in the series (h, k, p, q, r and s; generally these powers must be assumed, the guidance function determined, flight simulated, and the exponents empirically optimized by trial and error).

The choice of the particular powers for the variables to be contained in the set of indices will depend on the function being simulated. A method for selecting these indices is not yet available, however, for any particular choice it is desired to find the best set of coefficients, a_i . Thus, at this point it is assumed that the polynomial has been empirically defined and that the problem is now to define the best set of coefficients in the sense that the sum of the squares of the differences between the values provided by the polynomial and optimum values of Θ are a minimum; i.e.,

$$L = \sum_{i=1}^{h} \left[\theta_i' - \theta_i\right]^2 \tag{2.2.29}$$

is a minimum (where, Θ' , is the optimum value of the thrust attitude). But the computational algorithm for defining the a's is

$$\underline{a} = \left[M M^T\right]^{-1} M \underline{\theta}' \tag{2.2.30}$$

where

$$\underline{a} \equiv \begin{matrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_L \end{matrix} \qquad \overset{j \times 1}{\underline{\theta}'} \equiv \begin{matrix} \theta_1' \\ \theta_2' \\ \theta_3' \\ \vdots \\ \theta_j' \end{matrix}$$

$$\overset{\ell \times j}{M} \equiv \begin{bmatrix} W_{11} & W_{12} & \cdots & W_{1j} \\ W_{21} & W_{22} & \cdots & W_{2j} \\ W_{31} & W_{32} & \cdots & W_{3j} \\ \vdots & & & \vdots \\ W_{\ell 1} & & & W_{\ell j} \end{bmatrix}$$

Note that the index j ranges over all of the tabulated points and that the number of tabulated points exceeds the number of coefficients, i.e., . (This latter requirement assures that a solution can be generated)

By varying both the selection of the tabulated points and the specific polynomial form used in the fitting, the residuals can be modified so that the loss function of equation (2.2.29) is as small as possible. The final criteria of goodness of fit, however, must be a weighing of the number and complexity of terms in the polynomial against the degree of optimization and accuracy of meeting mission conditions achieved by the polynomial. For fitting the 12,600 point tabulation, various approximating polynomials up through third order were investigated, (Reference 2.3) and it was found that a third order polynomial of eighty-four terms proved the best compromise representation. This conclusion is predicated on the observation that the errors in the control deflections relative to the true optimum policy were of the order of .15 degrees while assuring that the storage limitations of the on-board computer were not exceeded.

In conclusion it is noted that while the entire discussion of this section of the monograph has been slanted toward the two dimensional problem, the results are valid for the three dimensional problem as well. The solution will in all cases be more complex due to the fact that additional terms in the equations will be required (unless the in-plane and out-of-plane motions are separated as discussed in section 2.1). However, the concepts are unaltered.

2.3 PERTURBATION GUIDANCE

This section of the monograph will deal with a class of guidance schemes which will be referred to as perturbation guidance. This name has been selected because each of the schemes in some way relies on the closeness of the true flight to precomputed information that has been computed for a specific reference flight. In other words, a significant amount of computation is done on the ground prior to the flight for a particular mission in order to minimize the amount of on-board computation. Since the preflight computation is performed with a particular mission in mind and with a nearly exact model for all forces acting on the system, it is assumed that the preflight results will be in the "neighborhood" of the actual flight results. It is this "closeness" of the preflight computation that is the basis of any perturbation guidance scheme and which affords accuracies comparable to those obtained if a large and very fast computer was used for real time computation during the actual flight. As long as the actual flight does not differ significantly from this "nominal" or reference trajectory, the precomputed information is sufficiently accurate to allow the guidance scheme, which is designed by approximating some quantity by a Taylor series, to correct for measured deviations in position and velocity. The original assumption of closeness to a nominal trajectory permits the Taylor series to be truncated after first order terms in most cases without introducing intolerable errors. This approximation then permits the use of the more powerful techniques of linear analysis in many cases.

Although the techniques presented herein have many similarities, certain peculiar features have been singled out in order to distinguish the schemes. It should be noted, however, that any attempted organization based on these peculiarities is artificial since similar mathematical techniques are used in conjunction with all of the guidance schemes. In this light, the organization of the section will now be presented. These discussions are divided into two major parts. The first part, 2.3.1, deals with guidance schemes that use the required velocity concept (defined in text). The second part, 2.3.2, presents a class of guidance schemes that reduce the nonlinear equations to linear perturbation equations. Following the derivation of the equations for each of these guidance schemes, a control or steering section is presented. Since it is usually desired to optimize the control in some sense, the determination of the optimum control policy involves the use of calculus of variations, Pontryagin's Maximum Principle, and/or dynamic programming. A sample application of each of these approaches is included. Section 2.3.1.2.3.1, Optimum Steering for C* Guidance, is an application of Pontryagin's Principle. Sections 2.3.2.2.1 and 2.3.2.2.2 are respectively applications of calculus of variations and dynamic programming. It should be noted that a detailed discussion of the variational techniques used in these sections is beyond the scope of the monograph. However, all of the required information may be found in other monographs of the series (references 3.22, 3.23 and 3.24).

Before continuing with a detailed analysis of the various Perturbation Guidance schemes, it is worthy to note some of the features of each so that the advantages and disadvantages of each can be kept in mind as they are presented. Figure 2.3.1 is a table that compares and contrasts the three main types of Perturbation Guidance schemes, Delta Guidance, C* (or Q) Guidance, and Linearized Perturbation Guidance. These three schemes are compared on the basis of performance, optimization, errors, application and mechanization.

2.3.1 Required Velocity Approach

The guidance schemes in this section share the use of the required velocity concept. (The required velocity is defined as the velocity that is needed by a vehicle in order for it to reach a specified position at some specified future time under the assumption of free flight from its present position.) The two schemes presented in this section are two different mechanizations of the equations for the required velocity. The idea behind both of these methods is to provide a simple means of updating the required velocity as the vehicle progresses along the powered flight. By measurement of acceleration of the vehicle the first concept, Delta Guidance, expands the required velocity in a Taylor series about some nominal burn-out point. A good approximation of the required velocity for a point other than the nominal burn-out point can then be found by substituting the position coordinates of the vehicle into the Taylor expansion. The position of the vehicle at subsequent times is then determined by integrations of the total vehicle acceleration. This scheme provides a continuous knowledge of the required velocity for purposes of steering and is capable for compensating for errors at earlier epochs.

The second guidance scheme that used the required velocity concept is C* (sometimes called Q) Guidance. This scheme is a means of continuously updating the velocity-to-be-gained (the difference between the required velocity and the current velocity.) Measurement of the thrust acceleration provides the information that is necessary to continuously update the velocity-to-be-gained such that the vehicle can be steered properly.

Acknowledgement is given to C. W. Sarture (reference 3.3) whose material on Delta Guidance was of significant assistance in the preparation of this section of the monograph.

2.3.1.1 Delta Guidance

2.3.1.1.1 General Discussion. The object of Delta Guidance is to provide a reasonably accurate value of the required velocity throughout all phases of a powered flight. However, since the expressions for the required velocity are quite complex and non-linear, an "exact" knowledge of the required velocity as the vehicle progresses in its powered flight would require a tremendously large and fast computer performing real time computation. To remove this problem, it has been found that a Taylor series expansion as a function of position and time about the nominal burnout point provides a sufficiently accurate value of the required velocity for any position or time along the nominal trajectory and at the same time produces a significant reduction in on-board computation.

Figure 2.3.1 A Comparison of Perturbation Guidance Schemes

	Mechanization	Optimization	Application	Errors
Delta Guidance	Compute required velocity by substituting present position and time into a polynomial. Steering involves taking cross product of two vectors or simple integration.	Nominal trajectory is non-optimal; steering can be optimal.	Ballistic Vehicles (Minuteman)	Highly accurate as long as vehicle remains close to nominal. Moderately inaccurate if widely off nominal.
C* (or Q)	Continuously solve a set of first-order differential equations with thrust acceleration as a forcing function. Steering involves taking cross product of two vectors.	Reference trajectory can be near optimal. Steering will be near optimal if burning time is short compared to the constant for the system.	Boost-Coast Injection	Accuracy deteriorates if trajectory differs from nominal for which matrices were computed.
Linearized Perturbation Guidance	Precomputed matrices and vectors are fed from a computer. The state is compared to nominal and a control deviation is generated by a matrix multiplication.	Nominal trajectory is near optimal. Perturbation control can also be near optimal.	Boost-Coast Injection	Highly accurate in all respects as long as deviations from nominal are not too large.

It should be noted that while such an expansion has advantages from the point of view of real time computation, it has corresponding disadvantages in that there is no guarantee that the vehicle will adhere to a nominal trajectory, especially at the beginning of the guided flight where the vehicle is its farthest from the burnout point (Taylor expansion point). It should also be noted that since most computation is done prior to the actual flight, (the results of these computations are the Taylor series coefficients), any mission change requires complete pre-flight reprogramming including, at a minimum, a new set of Taylor coefficients.

In general, the expressions for the required velocity are quite complex and cannot be written explicitly so that differentiation and expansion in the Taylor Series is possible. A grossly simplified problem employing a flat earth can be analyzed in order to demonstrate the theory involved. Following the flat earth analysis, the more practical numerical techniques that are utilized in the preflight computations will be discussed.

2.3.1.1.2 Derivation of Equations

2.3.1.1.2.1 General. The required velocity expression in the most general case will be a function of position and time, i.e., $V_R = F(x,y,z,t)$. Each component of V_R can be expressed as a Taylor Series of a function of four variables as follows:

$$f(x,y,z,t) = f(x_o, y_o, z_o, t) + \left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y} + \Delta z \frac{\partial}{\partial z} + \Delta t \frac{\partial}{\partial t}\right) f(x,y,z,t)$$
$$+ \frac{1}{2!}\left[\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y} + \Delta z \frac{\partial}{\partial z} + \Delta t \frac{\partial}{\partial t}\right] f(x,y,z,t) + \ldots \quad (2.3.1)$$

The x component of V_R then becomes

$$V_{R_x} = V_{R_{x_o}} + \frac{\partial F}{\partial x}(x-x_o) + \frac{\partial F}{\partial y}(y-y_o) + \frac{\partial F}{\partial z}(z-z_o) + \frac{\partial F}{\partial t}(t-t_o)$$
$$+ \frac{1}{2!}\left[\frac{\partial^2}{\partial x^2}(x-x_o)^2 + \frac{\partial^2}{\partial y^2}(y-y_o)^2 + \frac{\partial^2}{\partial z^2}(z-z_o)^2 + \frac{\partial^2}{\partial t^2}(t-t_o)^2\right. \quad (2.3.2)$$
$$+ \frac{\partial^2 F}{\partial x \partial y}(x-x_o)(y-y_o) + \frac{\partial^2 F}{\partial x \partial z}(x-x_o)(z-z_o) + \frac{\partial^2 F}{\partial x \partial t}(x-x_o)(t-t_o)$$

$$+\frac{\partial^2 F}{\partial y \partial x}(y-y_o)(x-x_o)+\frac{\partial^2 F}{\partial y \partial z}(y-y_o)(z-z_o)+\frac{\partial^2 F}{\partial y \partial t}(y-y_o)(t-t_o)$$

$$+\frac{\partial^2 F}{\partial z \partial x}(z-z_o)(x-x_o)+\frac{\partial^2 F}{\partial z \partial y}(z-z_o)(y-y_o)+\frac{\partial^2 F}{\partial z \partial t}(z-z_o)(t-t_o) \quad (2.3.2)$$

$$\left.+\frac{\partial^2 F}{\partial t \partial x}(t-t_o)(x-x_o)+\frac{\partial^2 F}{\partial t \partial y}(t-t_o)(y-y_o)+\frac{\partial^2 F}{\partial t \partial z}(t-t_o)(z-z_o)\right]+\ldots$$

where $V_{R_{x_o}}$ is the nominal burnout velocity in the x direction. Similar expressions result for V_{R_y} and V_{R_z}. As mentioned above, the partial derivatives are usually evaluated at the nominal burnout point (x_o, y_o, z_o, t_o) so that the accuracy in the representation improves as the desired terminal state is approached. These partials are retained as constants throughout the thrust phase. The required velocity then is a function of present position and present time.

2.3.1.1.2.2 Simple Flat Earth Example. An over-simplified flat earth problem employing a uniform gravitational field will now be analyzed so that the previous theory can be interpreted clearly. Consider a short range ballistic vehicle whose target coordinates are designated by (x_T, y_T, t_T), where t_T is the desired time of impact. The free flight rectilinear equations of motion for this problem are:

$$x_T = x_o + V_x\,(t-t_o) \quad (2.3.3)$$

$$y_T = y_o + V_y\,(t-t_o) - \frac{1}{2} g\,(t-t_o)^2 \quad (2.3.4)$$

More specifically, the equations for the required velocity for free flight target impact at the designated time from any burnout point (x_b, y_b, t_b) are

$$V_{R_x} = \frac{x_T - x_b}{t_T - t_b} \quad (2.3.5)$$

$$V_{R_y} = \frac{y_T - y_b + \frac{1}{2} g\,(t_T - t_b)^2}{(t_T - t_b)} \quad (2.3.6)$$

The partial derivatives for these expressions can now be formed in a straightforward manner by treating the burnout coordinates as variables.

$$\frac{\partial V_{Rx}}{\partial x_b} = \frac{-1}{t_T - t_b} \qquad \frac{\partial V_{Rx}}{\partial y_b} = 0 \qquad \frac{\partial V_{Rx}}{\partial t_b} = \frac{x_T - x_b}{(t_T - t_b)^2}$$

$$\frac{\partial V_{Ry}}{\partial x_b} = 0 \qquad \frac{\partial V_{Ry}}{\partial y_b} = \frac{-1}{(t_T - t_b)} \qquad \frac{\partial V_{Ry}}{\partial t_b} = \frac{y_T - y_b + \frac{1}{2} g (t_T - t_b)^2}{(t_T - t_b)^2} - g$$

If the nominal burnout and target parameters are assumed to be

x_b = 85,000 feet $\quad y_b$ = 123,000 feet $\quad t_b$ = 70 sec.

x_T = 1,212,000 feet $\quad y_T$ = 0 feet $\quad t_T$ = 392 sec.

then the partial derivatives become

$$\frac{\partial V_{Rx}}{\partial x_b} = -3.10 \times 10^{-3} \qquad \frac{\partial V_{Rx}}{\partial y_b} = 0 \qquad \frac{\partial V_{Rx}}{\partial t_b} = 10.9$$

$$\frac{\partial V_{Ry}}{\partial x_b} = 0 \qquad \frac{\partial V_{Ry}}{\partial y_b} = -3.1 \times 10^{-3} \qquad \frac{\partial V_{Ry}}{\partial t_b} = -17.3$$

The only constants remaining to be determined are thus the nominal required velocities at the nominal burnout point. These can easily be determined from equations 2.3.5 and 2.3.6 by employing the nominal burnout and target parameters. The result is

$$V_{Rx_0} = 3500 \text{ ft./sec.}$$

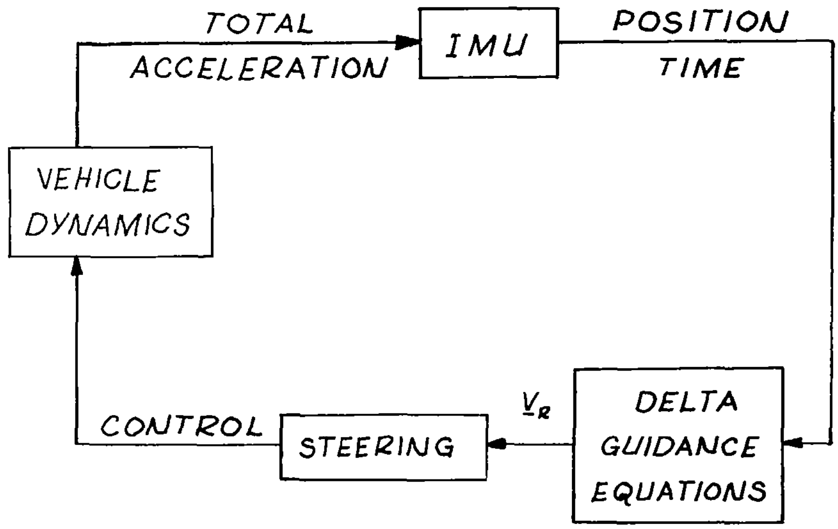
$$V_{Ry_0} = 4800 \text{ ft./sec.}$$

Hence, the Delta Guidance equations for the required velocity in this particular case become

$$V_{R_x} = 3500 - (3.1 \times 10^{-3})(x - 85{,}000) + (10.9)(t - 70) \qquad (2.3.7)$$

$$V_{R_y} = 4800 - (3.1 \times 10^{-3})(y - 123{,}700) - (17.3)(t - 70) \qquad (2.3.8)$$

The required velocity is used by the steering law in order to insure that the thrust is in the correct direction in order to satisfy the desired terminal condition. The following sketch illustrates the guidance loop for Delta Guidance.



The discussion of steering policies for this and other schemes is presented in section 2.3.1.3.

2.3.1.1.2.3 Numerical Techniques. In a more realistic problem, the equations for the required velocity cannot be found in a form which will allow the partial derivatives to be determined analytically. However, the numerical values for the partial derivatives at the burnout point can be determined by simulation techniques. The name given to the most frequently used technique is "targeting", a method of determining the best values for the partial derivatives in a least squares sense by equating the Taylor Series for the required velocity at some perturbed burnout point to the corresponding value of the required velocity for the same perturbed point as calculated from the best available equations. If the number of perturbed points investigated is equal to or greater than the number of unknown constants in the Taylor Series, then a "best" estimate can be found for the partial derivatives evaluated at the point of interest (burnout point).

The required velocity for a point slightly perturbed from the nominal burnout point can be mathematically stated as

$$V_{R_x} = V_{R_x}(x, y, z, t) \tag{2.3.9}$$

$$V_{R_y} = V_{R_y}(x, y, z, t) \tag{2.3.10}$$

$$V_{R_z} = V_{R_z}(x, y, z, t) \tag{2.3.11}$$

Thus, the Taylor series (retaining only linear terms) for the required velocity can be expressed as

$$V_{R_x} = V_{R_{x_0}} + K_{xx}(x - x_0) + K_{xy}(y - y_0) + K_{xz}(z - z_0) + K_{xt}(t - t_0) + \ldots \tag{2.3.12}$$

where $V_{R_{x_o}}$ = the required velocity of the nominal burnout point,

$$K_{xx} = \left.\frac{\partial V_{Rx}}{\partial x}\right|_{nom.}$$

$$K_{xy} = \left.\frac{\partial V_{Rx}}{\partial y}\right|_{nom.}$$

$$K_{xz} = \left.\frac{\partial V_{Rx}}{\partial z}\right|_{nom.}$$

$$K_{xt} = \left.\frac{\partial V_{Rx}}{\partial t}\right|_{nom.}$$

and x_o, y_o, z_o, t_o = nominal burnout conditions. Similar Taylor series can be written for V_{R_y} and V_{R_z}. It should be noted that there is no limitation to the process which would preclude the inclusion of higher order terms in the series. Indeed, these terms can be added simply once the coefficients are determined on the ground.

Now, since the perturbed point is assumed to be specified (neglecting errors in the estimate resulting from IMU errors, etc.), the only unknowns in equation 2.3.12 are the partial derivatives. The problem now becomes one of finding the best numbers to use for the partial derivatives such that the values chosen form a "best" fit in the least squares sense.

The classical method that is used on this type of problem involves the minimization of the mean square error between the values predicted by the Taylor Series and those calculated. Thus, if the error is defined as

$$\begin{aligned}\epsilon_{x_i} = V_{Rx_i}(x, y, z, t) - V_{Rx_{o_i}} - K_{xx}(x_i - x_o)\\ - K_{xy}(y_i - y_o) - K_{xz}(z_i - z_o) - K_{xt}(t_i - t_o)\end{aligned} \qquad (2.3.13)$$

where i represents the i^{th} perturbed point and if N perturbed points are investigated (N = the number of unknowns), then the mean squared value of the error of all N points is

$$\epsilon_x^2 = \epsilon_{x1}^2 + \epsilon_{x2}^2 + \ldots \epsilon_{xN}^2 \qquad (2.3.14)$$

The best least squares choice for the unknown constants can be found by setting the partial derivatives with respect to each of the unknown parameters equal to zero. The result is a set of simultaneous equations in which the number of equations is exactly equal to the number of unknown constants. Expressed mathematically,

$$2\epsilon_x \frac{\partial \epsilon_x}{\partial K_{xx}} = 2\epsilon_{x1} \frac{\partial \epsilon_{x1}}{\partial K_{xx}} + 2\epsilon_{x2} \frac{\partial \epsilon_{x2}}{\partial K_{xx}} + \ldots + 2\epsilon_{xN} \frac{\partial \epsilon_{xN}}{\partial K_{xx}} = 0 \quad (2.3.15a)$$

$$2\epsilon_x \frac{\partial \epsilon_x}{\partial K_{xy}} = 2\epsilon_{x1} \frac{\partial \epsilon_{x1}}{\partial K_{xy}} + 2\epsilon_{x2} \frac{\partial \epsilon_{x2}}{\partial K_{xy}} + \ldots + 2\epsilon_{xN} \frac{\partial \epsilon_{xN}}{\partial K_{xy}} = 0 \quad (2.3.15b)$$

$$2\epsilon_x \frac{\partial \epsilon_x}{\partial K_{xz}} = 2\epsilon_{x1} \frac{\partial \epsilon_{x1}}{\partial K_{xz}} + 2\epsilon_{x2} \frac{\partial \epsilon_{x2}}{\partial K_{xz}} + \ldots + 2\epsilon_{xN} \frac{\partial \epsilon_{xN}}{\partial K_{xz}} = 0 \quad (2.3.15c)$$

$$2\epsilon_x \frac{\partial \epsilon_x}{\partial K_{xz}} = 2\epsilon_{x1} \frac{\partial \epsilon_{x1}}{\partial K_{xt}} + 2\epsilon_{x2} \frac{\partial \epsilon_{x2}}{\partial K_{xt}} + \ldots + 2\epsilon_{xN} \frac{\partial \epsilon_{xN}}{\partial K_{xt}} = 0 \quad (2.3.15d)$$

or

$$\epsilon_{x1} \frac{\partial \epsilon_{x1}}{\partial K_{xx}} + \epsilon_{x2} \frac{\partial \epsilon_{x2}}{\partial K_{xx}} + \ldots + \epsilon_{xn} \frac{\partial \epsilon_{xN}}{\partial K_{xx}} = 0 \quad (2.3.16a)$$

$$\epsilon_{x1} \frac{\partial \epsilon_{x1}}{\partial K_{xy}} + \epsilon_{x2} \frac{\partial \epsilon_{x2}}{\partial K_{xy}} + \ldots + \epsilon_{xn} \frac{\partial \epsilon_{xN}}{\partial K_{xy}} = 0 \quad (2.3.16b)$$

$$\epsilon_{x1} \frac{\partial \epsilon_{x1}}{\partial K_{xz}} + \epsilon_{x2} \frac{\partial \epsilon_{x2}}{\partial K_{xz}} + \ldots + \epsilon_{xN} \frac{\partial \epsilon_{xN}}{\partial K_{xz}} = 0 \quad (2.3.16c)$$

$$\epsilon_{x1} \frac{\partial \epsilon_{x1}}{\partial K_{xt}} + \epsilon_{x2} \frac{\partial \epsilon_{x2}}{\partial K_{xt}} + \ldots + \epsilon_{xN} \frac{\partial \epsilon_{xN}}{\partial K_{xt}} = 0 \quad (2.3.16d)$$

Now, since the partial derivatives in these expressions are

$$\frac{\partial \epsilon_{xi}}{\partial K_{xx}} = -(x_i - x_o) \quad (2.3.17a)$$

$$\frac{\partial \epsilon_{xi}}{\partial K_{xy}} = -(y_i - y_o) \quad (2.3.17b)$$

$$\frac{\partial \epsilon_{xi}}{\partial K_{xz}} = -(z_i - z_o). \quad (2.3.17c)$$

$$\frac{\partial \epsilon_{xi}}{\partial K_{xt}} = -(t_i - t_o) \quad (2.3.17d)$$

equations (7.3.16) are seen to be a set of simultaneous equations in the unknowns K_{xx}, K_{xy}, K_{xz}, K_{xt}.

The least squares solution to equations (2.3.16) is

$$\bar{K}_x = (B^T W B)^{-1} B^T W (\Delta \bar{V}_{R_x})$$

where

$$\bar{K}_x = \begin{bmatrix} K_{xx} \\ K_{xy} \\ K_{xz} \\ K_{xT} \end{bmatrix}$$

$$B = \begin{bmatrix} x_1 - x_0 & y_1 - y_0 & z_1 - z_0 & t_1 - t_0 \\ & \vdots & & \\ x_n - x_0 & y_n - y_0 & z_n - z_0 & t_n - t_0 \end{bmatrix}$$

W = weighting matrix which can be used if desired

ΔV_{R_x} = vector of errors in series representation $V_{R_x}(x,y,z,t) - V_{R_x}$.

with an identical form of solution for the vector constants $\bar{K}_y$ and $\bar{K}_z$.

2.3.1.2 C* Guidance

2.3.1.2.1 General Description

The C* (sometimes called Q) Guidance scheme is another method that uses the required velocity concept discussed in the previous section on Delta Guidance. More specifically, this scheme uses a differential equation in terms of the velocity-to-be-gained as a means of updating target information. The velocity-to-be-gained is defined as the difference between the present true velocity and the present required velocity, i.e.,

$$\underline{V}_g = \underline{V}_R(r,t) - \underline{V} \tag{2.3.18}$$

The continuous knowledge of the present velocity-to-be-gained then provides the information required for steering. The differential equation developed in the following analysis is a second order differential equation in $\underline{V}_g$ with the measured thrust as the forcing function. Hence, the velocity-to-be-gained is known as a function of time providing the C* guidance equation is forced properly. The appropriate steering for C* guidance will be presented in section 2.3.1.3.

2.3.1.2.2 Derivation of C* Guidance Equation

The velocity-to-be-gained is defined as the difference between the required velocity and the current velocity, i.e.,

$$\underline{V}_g = \underline{V}_R(\underline{r},t) - \underline{V} \tag{2.3.19}$$

Thus, the derivative of equation 2.3.19 is

$$\dot{\underline{V}}_g = \dot{\underline{V}}_R(r,t) - \dot{\underline{V}} \tag{2.3.20}$$

Now, since $\underline{V}_R$ is a function of both position and time, its derivative is

$$\dot{\underline{V}}_R(\underline{r},t) = \frac{\partial \underline{V}_R}{\partial \underline{r}} \frac{d\underline{r}}{dt} + \frac{\partial \underline{V}_R}{\partial t} \tag{2.3.21}$$

so that equation 2.3.20 becomes

$$\dot{\underline{V}}_g = \frac{\partial \underline{V}_R}{\partial \underline{r}} \underline{V} + \frac{\partial \underline{V}_R}{\partial t} - \dot{\underline{V}} \tag{2.3.22}$$

where $\underline{V}$ has been used for $\frac{d\underline{r}}{dt}$.

The total acceleration of the vehicle is given by $\dot{\underline{V}}$. This function is the sum of gravitational and thrust acceleration, i.e.,

$$\dot{\underline{V}} = \underline{a}_T + \underline{g} \tag{2.3.23}$$

Substituting equations 2.3.19 and 2.3.23 in equation 2.3.22 yields

$$\dot{\underline{V}}_g = \frac{\partial \underline{V}_R}{\partial \underline{r}} (\underline{V}_R - \underline{V}_g) + \frac{\partial \underline{V}_R}{\partial t} - \underline{a}_T - \underline{g} \tag{2.3.24}$$

or

$$\dot{\underline{V}}_g = \frac{\partial \underline{V}_R}{\partial \underline{r}} \underline{V}_R - \frac{\partial \underline{V}_R}{\partial \underline{r}} \underline{V}_g + \frac{\partial \underline{V}_R}{\partial t} - \underline{a}_T - \underline{g} \tag{2.3.25}$$

This equation can be simplified if the definition of required velocity is employed. Consider two vehicles, A and B, at the same point in space. Both vehicles are to satisfy the same terminal conditions at the same time; however, vehicle A has already acquired the necessary velocity in order to terminate correctly and is presently in free flight, having terminated thrust. Vehicle B, on the other hand, has not acquired its required velocity and is still thrusting. Since vehicle A is in free flight and experiences no thrust, the following is true:

$$\underline{V}_{g_A} = 0$$

$$\underline{\dot{V}}_{g_A} = 0$$

$$\underline{V}_{R_A} = \underline{V}_A$$

$$\underline{a}_{T_A} = 0$$

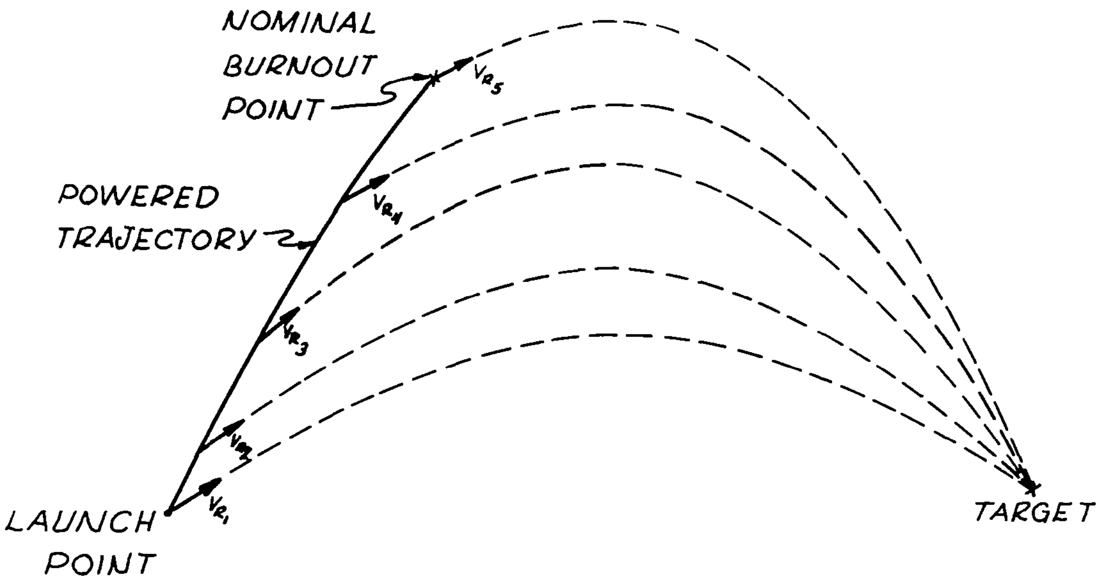
and equation 2.3.25 for vehicle A becomes

$$0 = \frac{\partial \underline{V}_{R_A}}{\partial \underline{r}_A} \underline{V}_{R_A} + \frac{\partial \underline{V}_{R_A}}{\partial t} - \underline{g}_A \qquad (2.3.26)$$

or

$$\underline{g}_A = \frac{\partial \underline{V}_{R_A}}{\partial \underline{r}_A} \underline{V}_{R_A} + \frac{\partial \underline{V}_{R_A}}{\partial t} = \frac{d}{dt} \underline{V}_{R_A} = \frac{d}{dt} \underline{V}_A \qquad (2.3.27)$$

This result is expected since it states that the only acceleration that vehicle A experiences is that of gravity. However, since vehicle A is at the same position as vehicle B, and since both have the same terminal constraints, they must have the same required velocity vectors. Furthermore, both vehicles are experiencing the same gravitational field so that each point on the thrust trajectory can be compared to a free flight vehicle that has exactly the same required velocity and gravity vectors. Of course, each point must be compared to a different free flight trajectory since the required velocity is continuously changing.



Finally, if all points on the powered trajectory instantaneously satisfy the required velocity and gravity vectors of some free flight vehicle at the same point, then equation 2.3.27 is also true for the powered flight, i.e.,

$$\underline{g} = \frac{\partial \underline{V}_R}{\partial \underline{r}} \underline{V}_R + \frac{\partial \underline{V}_R}{\partial t} \tag{2.3.28}$$

along both a powered and free flight trajectory. In this light, equation 2.3.25 becomes

$$\dot{\underline{V}}_g = -\frac{\partial \underline{V}_R}{\partial \underline{r}} \underline{V}_g - \underline{a}_T \tag{2.3.29}$$

or

$$\dot{\underline{V}}_g + \frac{\partial \underline{V}_R}{\partial \underline{r}} \underline{V}_g = -\underline{a}_T \tag{2.3.30}$$

The matrix $\frac{\partial \underline{V}_R}{\partial \underline{r}}$ evaluated along the nominal trajectory is called the C* or Q matrix. Thus, in conventional notation

$$\boxed{\dot{\underline{V}}_g + C^* \underline{V}_g = -\underline{a}_T} \tag{2.3.31}$$

This equation provides a scheme for computing the velocity-to-be-gained from the measurement of the thrust acceleration. The object of the steering policy will now be to use the velocity-to-be-gained and its time rate at various epochs to drive $\underline{V}_g$ to zero. The choice of the steering policy depends on the quantity that is desired to be optimized; thus, families of logics can be proposed. Steering will be discussed in section 2.3.1.3. In fact, an optimal program including a discussion of the closed loop will be presented in section 2.3.1.3.2.1.

2.3.1.3 Steering

Guidance schemes, in general, and those that use the required velocity concept in particular are usually divided into two phases, the atmospheric phase and the vacuum phase. The reason for this division is that the vehicle cannot tolerate excessive structural loads due to aerodynamic effects. Thus, since the major strengths of the vehicle are axial, the loads must be near axial during atmospheric flight (i.e., if there is steering, it must be "gentle" in nature.) However, after the vehicle leaves the atmosphere (or more correctly, after the aerodynamic loads are reduced below a specific level), it may be subjected to the more violent maneuvering that may be commanded by the guidance system. For this reason, the steering is usually run in an open loop manner during the atmospheric phase in order to prevent any violent commands that might occur during this phase. The open loop steering is, in general, designed to keep the vehicle as close to the near optimum nominal path as possible. The nature of perturbation guidance schemes requires this closeness to the nominal in order to assure the accuracy during the steering phase to be guaranteed.

Acknowledgement is given to D. F. McAllister, D. R. Grier, and J. T. Wagner whose work on the optimum steering for the powered phases of the Apollo Mission (reference 3.5) was used in the preparation of this section.

2.3.1.3.1 Atmospheric Phase

The atmospheric portion of the flight usually consists of a vertical rise for a prescribed period followed by a transition turn. The object of the transition turn is to rotate the vehicles attitude and velocity vector by an amount known as the "kick" angle. This kick angle defines a zero lift (gravity turn) trajectory through the atmosphere such that near nominal (optimal) conditions are attained upon entry into the vacuum phase. One of the simplest ways of implementing the atmospheric phase of steering is to mechanize a program of vehicle attitude or attitude rate based on sensed data. During the kick, yaw attitude is usually held to zero, while pitch attitude is commanded.

A more complicated loop steering scheme that is used during the atmospheric phase is called velocity steering. In this scheme, the desired vertical velocity is written in terms of position and time for the atmospheric phase of the flight. The true vertical velocity is then compared to the desired value and a pitch perturbation command that is proportional to the difference is generated. The pitch perturbation command is then added to the preflight nominal pitch command (which is also a function of position and time) to generate the total command. The primary advantage to the velocity steering method is that trajectory perturbations, as reflected in the variations with respect to the nominal trajectory, are greatly reduced during the atmospheric phase. On the other hand, this method introduces smaller stability margins at vehicle vibration frequencies because of the more active attitude control system.

2.3.1.3.2 Vacuum Phase

Once the vehicle is out of the atmosphere, structural constraints can be relaxed and the steering system can begin to perform its primary function, that of reducing the velocity-to-be-gained vector to zero. One obvious method of driving $\underline{V}_g$ to zero is by thrusting in the direction of $\underline{V}_g$. This method could be used for either Delta Guidance or C* Guidance since both methods calculate $\underline{V}_g$ either directly or indirectly. A simple mechanization of this steering law is obtained if the vehicle is given an attitude rate command that is proportional to $\underline{a}_T \times \underline{V}_g$, where $\underline{a}_T$ is the thrust acceleration vector. The verification of $\underline{a}_T \times \underline{V}_g$ steering can be made by considering the rate commands that would be generated for various orientations of $\underline{a}_T$ and $\underline{V}_g$. The vector $\underline{a}_T \times \underline{V}_g$ is zero if $\underline{a}_T$ and $\underline{V}_g$ are aligned and no attitude change is commanded. If they are not aligned, however, an attitude rate command in the $\underline{a}_T \times \underline{V}_g$ direction is given and the vectors begin to realign. Since the C* Guidance Scheme generates $\dot{\underline{V}}_g$ as well as $\underline{V}_g$, another type of steering that nulls the cross product of these two vectors is suggested. In other words, $\underline{V}_g$ can also be driven to zero by keeping the $\dot{\underline{V}}_g$ vector antiparallel to $\underline{V}_g$ instead of $\underline{a}_T$. As a matter of fact, it is shown in section 2.3.1.3.2.1 that the $\underline{V}_g \times \dot{\underline{V}}_g$ method of steering is optimum for C* Guidance in the sense that a minimum amount of fuel is consumed.

A comparison of the steering schemes indicates that these two methods in general are not equivalent. This can easily be seen by considering the expressions for $\underline{a}_T$ and $\underline{\dot{V}}_g$ in each case. For the $\underline{a}_T$ x $\underline{V}_g$ case

$$\frac{\underline{a}_T}{a_T} = \frac{\underline{V}_g}{V_g}$$

or

$$\underline{a}_T = a_T \frac{\underline{V}_g}{V_g}$$

Employing the C* Guidance equation (2.3.31) it is seen that

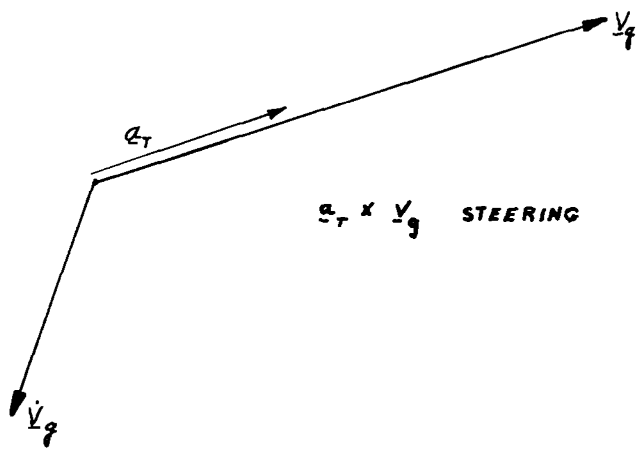
$$\underline{\dot{V}}_g = -\underline{a}_T - C^* \underline{V}_g = -\left(\frac{a_T}{V_g} I + C^* \right) \underline{V}_g$$

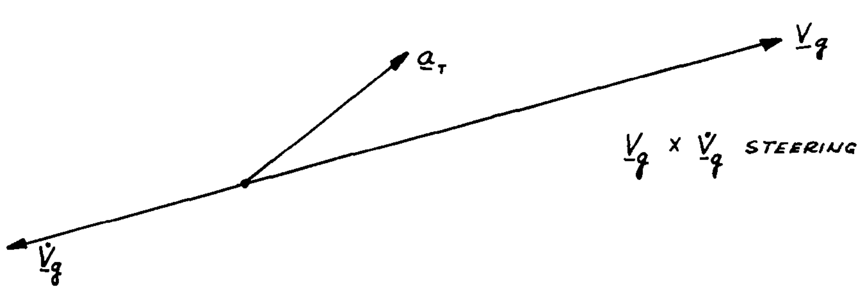
Similarly, for the $\underline{V}_g$ x $\underline{\dot{V}}_g$ case

$$\underline{\dot{V}}_g = - \frac{\dot{V}_g}{V_g} \underline{V}_g$$

$$\underline{a}_T = \left(\frac{\dot{V}_g}{V_g} I - C^* \right) \underline{V}_g$$

So in general, the steering schemes are not equivalent. The following sketches give a pictorial interpretation of the two steering schemes.

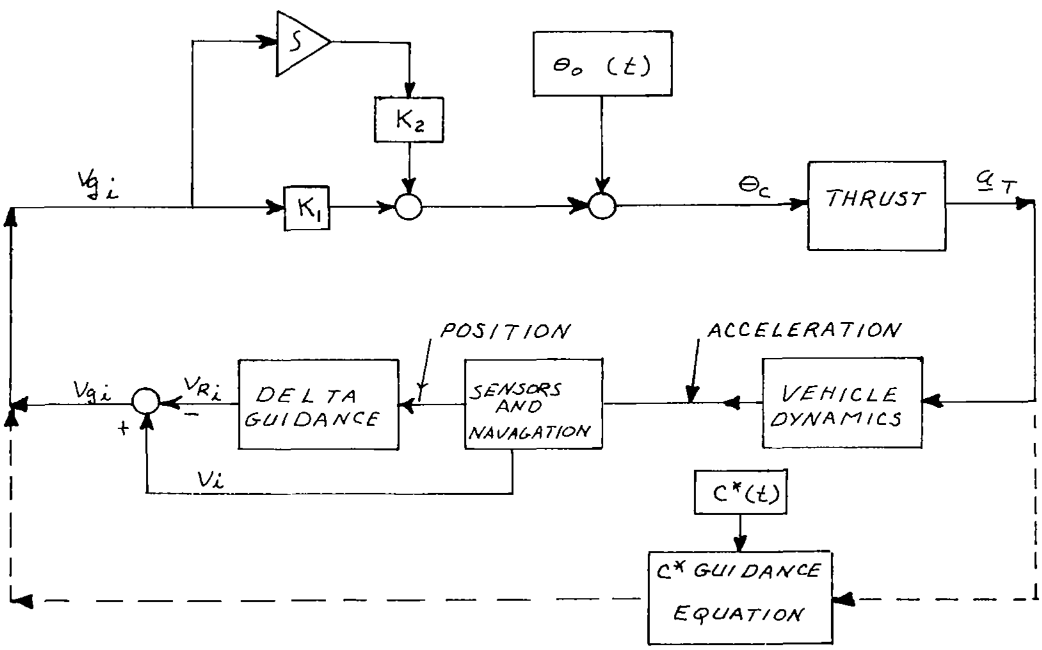




In each of these sketches the nulled condition is assumed. The appropriate steering can be verified if misalignments are considered.

An interesting adaptation of $\underline{V}_g \times \dot{\underline{V}}_g$ steering to Delta Guidance by a numerical differentiation of the velocity-to-be-gained for $\dot{\underline{V}}_g$ could be conjectured. (It is recalled that $\underline{a}_T \times \underline{V}_g$ Steering could be applied readily since $\dot{\underline{V}}_g$ was not needed). Such a differentiation process would introduce a noisy $\dot{\underline{V}}_g$, however, and may deteriorate the end performance in spite of the more optimum steering law. Simulation studies would indicate the best method to be used for a particular mission.

Other steering schemes using the velocity-to-be-gained require a knowledge of nominal steering commands for the entire flight. It should be noted that cross product steering made no such requirements. $\underline{V}_g$ can be driven to zero by interpreting its components as error signals. A suitable attitude command would consist of the sum of the nominal command and some combination of the appropriate component of $\underline{V}_g$ (typically proportional plus integral). The following sketch shows how such a scheme could be used to null one component of $\underline{V}_g$ for either C* or Delta Guidance.

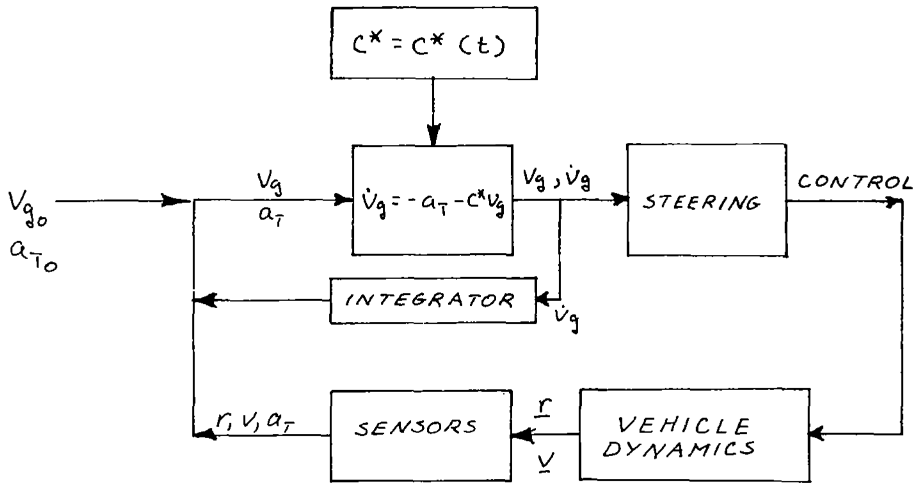


If a vehicle wanders from the nominal trajectory by a substantial amount, the steering commands could be larger than tolerable. This results in nonoptimum use of propellant. In order to alleviate this problem, a nominal velocity-to-be-gained can be precalculated and a less stringent error signal such as the deviation from nominal of $\underline{V}_g$ can be used. Further, by introducing a weighting technique, the error signal can be weighted lighter at the beginning of the flight when $\underline{V}_r$ (required velocity) is not known so accurately (as in Delta Guidance).

2.3.1.3.2.1 Optimum Steering for C* Guidance. In section 2.3.1.2.2 the C* guidance equation was shown to be

$$\dot{\underline{V}}_g + C^* \underline{V}_g = -\underline{a}_T \qquad (2.3.3.3)$$

The mechanization of this equation with a steering law constitutes a closed loop guidance scheme as illustrated in the following sketch:



The object of the following analysis will be to determine a steering policy for this loop. The simplest steering policy would be to drive the thrust acceleration in the direction of the velocity-to-be-gained. Usually, however, it is desired to maximize the performance in some sense. The following derivation presents the formulation of conditions necessary for a steering scheme to be optimum and shows that cross product steering ensures the reduction of the velocity-to-be-gained vector to zero while maximizing the burnout mass of the spacecraft (minimize propellant consumed). It should be noted that such an optimization problem requires the use of Pontryagin's Maximum Principle. Since it is beyond the scope of this monograph to present this principle in rigorous detail, the reader is referred to reference 3.23 for an introductory explanation.

There are many ways of formulating the state equations for the variable mass vehicle. However, the ideal approach involves the use of variables which uncouple the equations being processed. Appendix A presents several approaches designed to accomplish this objective and substantiates the choice of variables used for the following analysis. Let

$$X_1 = V_{g_1} \qquad (2.3.34a)$$

$$X_2 = V_{g_2} \qquad (2.3.34b)$$

$$X_3 = V_{g_3} \qquad (2.3.34c)$$

$$X_4 = g\, I_{sp} \ln\left[\frac{m(t)}{m(o)}\right] = V_e \ln\left[\frac{m(t)}{m(o)}\right] \qquad (2.3.34d)$$

$$u_1 = \frac{a_{T_1}}{a_T} \tag{2.3.34e}$$

$$u_2 = \frac{a_{T_2}}{a_T} \tag{2.3.34f}$$

$$u_3 = \frac{a_{T_3}}{a_T} \tag{2.3.34g}$$

$$u_4 = -\frac{|\dot{m}(t)|}{m(t)} = \frac{\dot{m}(t)}{m(t)} \quad ; (\dot{m} < 0) \tag{2.3.34h}$$

where

V_{g_i} = components of $\underline{V}_g$

a_{T_i} = components of $\underline{a}_T$

$a_T = |\underline{a}_T|$ = thrust magnitude

V_e = escape velocity of exhaust gas

It is desirable to express equation 2.3.34 in principle coordinates, i.e., eigenvector directions of C*, in order to simplify the algebra involved in the analysis. If such a coordinate system is employed, equation 2.3.3.3 becomes

$$\begin{bmatrix} \dot{V}_{g_1} \\ \dot{V}_{g_2} \\ \dot{V}_{g_3} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} V_{g_1} \\ V_{g_2} \\ V_{g_3} \end{bmatrix} - \begin{bmatrix} a_{T_1} \\ a_{T_2} \\ a_{T_3} \end{bmatrix} \tag{2.3.35}$$

or

$$C^* = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{2.3.36}$$

where all components are taken to be along principle axes.

Now, Note that

$$\dot{x}_4 = V_e \left[\frac{m(0)}{m(t)} \; \frac{\dot{m}(t)}{m(0)} \right] = V_e \frac{\dot{m}(t)}{m(t)} \tag{2.3.37}$$

But, since $V_e\ \dot{m}(t)$ is the thrust of the rocket, i.e.,

$$V_e \frac{\dot{m}(t)}{m(t)} = -a_T \qquad (2.3.38)$$

and since

$$\frac{\dot{m}(t)}{m(t)} = u_4 , \qquad (2.3.39)$$

$$\dot{x}_4 = V_e u_4$$

Thus, equation 2.3.34 can be written in the following state variable form:

$$\dot{x}_1 = \lambda_1 x_1 + V_e u_4 u_1 \qquad (2.3.40a)$$

$$\dot{x}_2 = \lambda_2 x_2 + V_e u_4 \cdot u_2 \qquad (2.3.40b)$$

$$\dot{x}_3 = \lambda_3 x_3 + V_e u_4 u_3 \qquad (2.3.40c)$$

$$\dot{x}_4 = V_e u_4 \qquad (2.3.40d)$$

or in matrix notation as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} V_e u_4 & 0 & 0 & 0 \\ 0 & V_e u_4 & 0 & 0 \\ 0 & 0 & V_e u_4 & 0 \\ 0 & 0 & 0 & V_e \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \qquad (2.3.41)$$

or

$$\underline{\dot{X}} = A \underline{X} + B \underline{U} \qquad (2.3.42)$$

Finally, the adjoint equations to this set are

$$\begin{bmatrix} \dot{P}_1 \\ \dot{P}_2 \\ \dot{P}_3 \\ \dot{P}_4 \end{bmatrix} = - \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} \qquad (2.3.43)$$

or

$$\underline{\dot{P}} = - A^T \underline{P} \qquad (2.3.44)$$

See section 2.3.2.1.3 for a discussion of adjoint equations.

At this point the generalized Hamiltonian defined below is introduced

$$\begin{aligned} H &= \underline{P}^T \cdot \dot{\underline{X}} = P_1 \dot{X}_1 + P_2 \dot{X}_2 + P_3 \dot{X}_3 + P_4 \dot{X}_4 \\ &= P_1(\lambda_1 X_1 + V_e u_4 u_1) + P_2(\lambda_2 X_2 + V_e u_4 u_2) \\ &\quad + P_3(\lambda_3 X_3 + V_e u_4 u_3) + P_4(V_e u_4) \end{aligned} \tag{2.3.45}$$

and the performance function is defined to be the burnout mass, i.e.,

$$S_T = X_4(T) = V_e \ln \left[\frac{m(T)}{m(0)}\right] \tag{2.3.46}$$

In standard functional notation, Equation 2.3.46 is

$$S_T = \underline{C} \cdot \underline{X}_T = [0\ 0\ 0\ 1] \begin{bmatrix} X_{1T} \\ X_{2T} \\ X_{3T} \\ X_{4T} \end{bmatrix} \tag{2.3.47}$$

Now, a necessary and sufficient condition for maximizing the functional, S_T, is that the Hamiltonian, H, be a minimum with respect to the control vector at every point of the path subject to the terminal condition that

$$\underline{P}_T^T = \underline{P}^T(t=T) = -\underline{C}^T \tag{2.3.48}$$

Thus

$$\underline{P}^T(t=T) = -\underline{C}^T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \tag{2.3.49}$$

or

$$P_4(t=T) = -1 \tag{2.3.50}$$

But, since $\dot{P}_4 = 0$, then $P_4 = -1$ for all t so that equation 2.3.45 can be rewritten as

$$\begin{aligned} H &= \lambda_1 P_1 X_1 + \lambda_2 P_2 X_2 + \lambda_3 P_3 X_3 + V_e u_4 \left[P_1 u_1 + P_2 u_2 + P_3 u_3 - 1\right] \\ &= [P_1\ P_2\ P_3] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + V_e u_4 \left\{ [P_1\ P_2\ P_3] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - 1 \right\} \end{aligned} \tag{2.3.51}$$

or

$$H = \underline{P}^{*T}(-C^{*})\,\underline{X}^{*} + V_e u_4 \left[\underline{P}^{*T}\underline{u}^{*} - 1\right] \tag{2.3.52}$$

where

$$\underline{u}^{*} = \frac{a_T}{|a_T|}$$

This is the H function that is to be minimized with respect to the control vector at every point of the path. Before proceeding, it must be remembered that u_4, which is defined to be $\dot{m}/m$, is limited to the allowable range of

$$-D \le u_4 \le 0 \tag{2.3.53}$$

where

$$D = max\left[\frac{|\dot{m}|}{m}\right]$$

This constraint states that there is a finite amount of thrust available and that the direction of thrust cannot be reversed. In order to determine the control that minimizes H, it is necessary to investigate all possible values of the second term in the above equation: i.e., $V_e u_4 \; \underline{P}^{*T}\underline{u}^{*} - 1$. Consider the two cases, (1) $|\underline{P}^{*}| > 1$ and (2) $0 < |\underline{P}^{*}| < 1$:

(1) $|\underline{P}^{*}| > 1$. For this case, two choices of control are possible. One choice of control would be $_4 = 0$. Under this choice the control term of equation (2.3.52) would be zero. A smaller value of H can be realized, however, if u_4 is chosen to be -D and $[\underline{P}^{*T}\underline{u}^{*} - 1]$ is chosen to be some positive quantity. Under this choice of control

$$-D\left[\underline{P}^{*T}\underline{u}^{*} - 1\right] \tag{2.3.54}$$

is most negative if $\underline{P}^{*T}u^{*} - 1$ is made as large as possible; i.e., in the limit

$$\underline{P}^{*T}\underline{u}^{*} = |\underline{P}^{*}| \qquad \text{or} \qquad \underline{u}^{*} = \frac{\underline{P}^{*}}{|\underline{P}^{*}|}$$

Hence, if $\underline{P}^{*} = 1$ the control that minimizes H is

$$u_4 = -D \qquad\qquad \underline{u}^{*} = \frac{\underline{P}^{*}}{|\underline{P}^{*}|} \tag{2.3.55}$$

(2) $0 < |\underline{P}^{*}| < 1$. For this case, since $0 < |\underline{P}^{*}| < 1$, then $[\underline{P}^{*T}\underline{u}^{*} - 1]$ must be a negative quantity ($\underline{P}^{*T}\underline{u}^{*} < 1$). Thus. u_4 cannot be positive and the minimum value of H is attained for the choice $u_4 = 0$, i.e., if $0 < |\underline{P}^{*}| < 1$, the control that minimizes H is $u_4 = 0$.

In summary, the optimal control law is to burn at maximum thrust in the $\underline{P}^{*}$ direction as long as $|\underline{P}^{*}| > 1$. When $|\underline{P}^{*}| < 1$, terminate thrust. Thus, the thrust vector during powered flight may be written as

$$\underline{a}_T = a_T \frac{\underline{P}^*}{|\underline{P}^*|} \qquad (2.3.56)$$

where

$$a_T = |\underline{a}_T|$$

It is desired to determine the optimum steering in terms of the velocity-to-be-gained. Thus far, it has been determined in terms of the adjoint variables. In order to accomplish this transformation, it is first necessary to write the solution to the adjoint equations.

$$P_1 = \alpha e^{-\lambda_1 t}. \qquad (2.3.57a)$$

$$P_2 = \beta e^{-\lambda_2 t} \qquad (2.3.57b)$$

$$P_3 = \gamma e^{-\lambda_3 t} \qquad (2.3.57c)$$

$$Z = |\underline{P}^*| = \left[\alpha^2 e^{-2\lambda_1 t} + \beta^2 e^{-2\lambda_2 t} + \gamma^2 e^{-2\lambda_3 t}\right]^{1/2} \qquad (2.3.57d)$$

Now, the state equations (2.3.35) can be written as

$$\dot{V}g_1 = \lambda_1 Vg_1 - a_T \frac{P_1}{Z} \qquad (2.3.58a)$$

$$\dot{V}g_2 = \lambda_2 Vg_2 - a_T \frac{P_2}{Z} \qquad (2.3.58b)$$

$$\dot{V}g_3 = \lambda_3 Vg_3 - a_T \frac{P_3}{Z} \qquad (2.3.58c)$$

Using results of Section 2.3.2.1.3, these equations can be integrated from t = 0 to t to yield

$$Vg_1(t) = Vg_1(0)\, e^{\lambda_1 t} - \int_0^t e^{\lambda_1(t-\tau)} \frac{a_T(\tau)}{Z(\tau)} P_1(\tau)\, d\tau. \qquad (2.3.59a)$$

$$Vg_2(t) = Vg_2(0)\, e^{\lambda_2 t} - \int_0^t e^{\lambda_2(t-\tau)} \frac{a_T(\tau)}{Z(\tau)} P_2(\tau)\, d\tau \qquad (2.3.59b)$$

$$V_{g_3}(t) = V_{g_3}(0)\, e^{\lambda_3 t} - \int_0^t e^{\lambda_3(t-\tau)} \frac{a_T(\tau)}{Z(\tau)} P_3(\tau)\, d\tau \qquad (2.3.59c)$$

This set of equations can be evaluated from time t = 0 to the cutoff time, T, by substituting the terminal value of the velocity-to-be-gained (zero), i.e.,

$$V_{g_1}(T) = 0 = V_{g_1}(0)\, e^{\lambda_1 T} - \int_0^T e^{\lambda_1(T-\tau)} \frac{a_T(\tau)}{Z(\tau)} P_1(\tau)\, d\tau \qquad (2.3.60a)$$

$$V_{g_2}(T) = 0 = V_{g_2}(0)\, e^{\lambda_2 T} - \int_t^T e^{\lambda_2(T-\tau)} \frac{a_T(\tau)}{Z(\tau)} P_2(\tau)\, d\tau \qquad (2.3.60b)$$

$$V_{g_3}(T) = 0 = V_{g_3}(0)\, e^{\lambda_3 T} - \int_t^T e^{\lambda_3(T-\tau)} \frac{a_T(\tau)}{Z(\tau)} P_3(\tau)\, d\tau \qquad (2.3.60c)$$

or

$$V_{g_1}(0)\, e^{\lambda_1 T} = \alpha \int_0^T e^{\lambda_1(T-\tau)} \frac{a_T(\tau)}{Z(\tau)} e^{-\lambda_1 \tau}\, d\tau \qquad (2.3.62a)$$

$$V_{g_2}(0)\, e^{\lambda_2 T} = \beta \int_0^T e^{\lambda_2(T-\tau)} \frac{a_T(\tau)}{Z(\tau)} e^{-\lambda_2 \tau}\, d\tau \qquad (2.3.62b)$$

$$V_{g_3}(0)\, e^{\lambda_3 T} = \gamma \int_0^T e^{\lambda_3(T-\tau)} \frac{a_T(\tau)}{Z(\tau)} e^{-\lambda_3 \tau}\, d\tau \qquad (2.3.62c)$$

Now the constants α, β, and γ can be found as soon as the integral in equations (2.3.62) is evaluated; and therefore, equation 2.3.56 can be used to find the thrust policy as a function of the velocity-to-be-gained. In general, the integrals of equation (2.3.62) are not easily evaluated. However, the exponential can be expressed as a Taylor series

$$e^{-2\lambda_1 \tau} = 1 - 2\lambda_1 \tau + \frac{4\lambda_1^2 \tau^2}{2!} \quad \cdots \qquad (2.3.63)$$

and higher-order terms can be neglected if $\lambda_1 \tau << 1$. This approximation is not very unrealistic when the burn time is compared to the time constants of the adjoint equations. Under this assumption, the first two terms of the Taylor series suffices in equation (2.3.62).

$$\alpha \int_0^T (1 - 2\lambda_1 \tau) \frac{a_T(\tau)}{z(\tau)} d\tau = V_{g_1}(0) \tag{2.3.64a}$$

$$\beta \int_0^T (1 - 2\lambda_2 \tau) \frac{a_T(\tau)}{z(\tau)} d\tau = V_{g_2}(0) \tag{2.3.64b}$$

$$\gamma \int_0^T (1 - 2\lambda_3 \tau) \frac{a_T(\tau)}{z(\tau)} d\tau = V_{g_3}(0) \tag{2.3.64c}$$

Since the method of integration of equations (2.3.64) is the same, only one integration will be performed. The solution to the others will be written by analogy. This integration will be performed by parts after first writing the equation in the following form:

$$\alpha \left[\int_0^T \frac{a_T(\tau)}{z(\tau)} d\tau - \int_0^T 2\lambda_1 \tau \frac{a_T(\tau)}{z(\tau)} d\tau \right] = V_{g_1}(0) \tag{2.3.65}$$

$$\alpha \left\{ \int_0^T \frac{a_T(\tau)}{z(\tau)} d\tau - 2\lambda_1 \left[T \int_0^T \frac{a_T(\tau)}{z(\tau)} d\tau \right] \right\} = V_{g_1}(0) \tag{2.3.66}$$

and substituting
$$I(T) = \int_0^T \frac{a_T(\tau)}{z(\tau)} d\tau$$

to obtain

$$\alpha \left\{ I(T) - 2\lambda_1 \left[T\, I(T) - \int_0^T I(\tau)\, d\tau \right] \right\} = V_{g_1}(0) \tag{2.3.67}$$

Now, assuming that $I(\tau)$ can be approximated linearly over the interval $0 < t < T$ as

$$\int_0^T I(\tau)\, d\tau \cong \frac{1}{2} I(T) \cdot T \tag{2.3.68}$$

(This again is not an unrealistic approximation since the burn time has been assumed small compared to $1/\lambda$.) allows equation (2.3.67) to be written as

$$\alpha\left\{I(T) - 2\lambda_1 T \cdot I(T) + \cancel{2}\lambda_1 \frac{1}{\cancel{2}} T \cdot I(T)\right\} = Vq_1(0) \qquad (2.3.69)$$

or

$$\alpha\left[I(T) - \lambda_1 T \cdot I(T)\right] = \alpha\left[1 - \lambda_1 T\right] I(T) = Vq_1(0) \qquad (2.3.70)$$

A similar technique can be used for β, and γ. The results are stated below.

$$\alpha\left[1 - \lambda_1 T\right] I(T) = Vq_1(0) \qquad (2.3.71a)$$

$$\beta\left[1 - \lambda_2 T\right] I(T) = Vq_2(0) \qquad (2.3.71b)$$

$$\gamma\left[1 - \lambda_3 T\right] I(T) = Vq_3(0) \qquad (2.3.71c)$$

Thus, the solutions for α, β, and γ are

$$\alpha = \frac{Vq_1(0)}{(1 - \lambda_1 T) I(T)} \qquad (2.3.72a)$$

$$\beta = \frac{Vq_2(0)}{(1 - \lambda_2 T) I(T)} \qquad (2.3.72b)$$

$$\gamma = \frac{Vq_3(0)}{(1 - \lambda_3 T) I(T)} \qquad (2.3.72c)$$

But, since the approximations $\lambda T \ll 1$ have been made

$$\frac{1}{1 - \lambda_i T} \cong 1 + \lambda_i T$$

is valid, and equations (2.3.72) become

$$\alpha = (1+\lambda_1 T)\ \frac{Vg_1(0)}{I(T)} \tag{2.3.73a}$$

$$\beta = (1+\lambda_2 T)\ \frac{Vg_2(0)}{I(T)} \tag{2.3.73b}$$

$$\gamma = (1+\lambda_3 T)\ \frac{Vg_3(0)}{I(T)} \tag{2.3.73c}$$

Now that the coefficients to be adjoint equations have been determined in terms of the initial velocity-to-be-gained and the eigenvalues of the C* matrix, the initial unit control vector $u^*(0)$ can be determined by the definition in equation (2.3.55b) as a unit vector in the $\underline{P}^*$ direction.

$$u^*(0) = \frac{1}{I(T)\ Z(0)} \begin{bmatrix} (1+\lambda_1 T)\ Vg_1(0) \\ (1+\lambda_2 T)\ Vg_2(0) \\ (1+\lambda_3 T)\ Vg_3(0) \end{bmatrix} \tag{2.3.74}$$

However, since u^* is a unit vector, it is not necessary to include the coefficient $\frac{1}{I(T)Z(0)}$. Rather, it is sufficient to write u^* as

$$u^*(0) = \text{unit} \left\{ \begin{vmatrix} (1+\lambda_1 T) & 0 & 0 \\ 0 & (1+\lambda_2 T) & 0 \\ 0 & 0 & (1+\lambda_3 T) \end{vmatrix} \underline{V}g(0) \right\} \tag{2.3.75}$$

with the understanding that once the vector

$$\begin{bmatrix} (1+\lambda_1 T) & Vg_1(0) \\ (1+\lambda_2 T) & Vg_2(0) \\ (1+\lambda_3 T) & Vg_3(0) \end{bmatrix}$$

is determined, it will be scaled to a unit vector by dividing by its magnitude. For subsequent times, the optimum control is defined by the condition

$$u^*(t) = \text{unit} \left\{ \begin{vmatrix} [1-\lambda_1(T-t)] & & \\ & [1-\lambda_2(T-t)] & \\ & & [1-\lambda_3(T-t)] \end{vmatrix} \right\}$$

which, under the substitution of the time-to-go (t_{go}) defined by $t_{go} = T-t$ becomes

$$u^*(t) = \text{unit} \left\{ \begin{vmatrix} (1+\lambda_1 t_{go}) & 0 & 0 \\ 0 & (1+\lambda_2 t_{go}) & 0 \\ 0 & 0 & (1+\lambda_3 t_{go}) \end{vmatrix} \underline{V}g(t) \right\} \tag{2.3.76}$$

Equation (2.3.76) gives the optimum thrust vector orientation to drive $\underline{V}_g$ to zero at time T while minimizing propellant consumption for a C^* guidance scheme.

It is now desired to find a steering scheme (no attempt will be made here to show that the resultant scheme is unique) that satisfies the condition of equation (2.3.76). This objective will be accomplished by assuming a candidate steering law for C^* guidance based on nulling the vector product

$$\underline{V}_g \times \dot{\underline{V}}_g$$

and proving that this steering satisfies the conditions of equation (2.3.76) (thus, is optimum in the sense that it minimizes the propellant consumption). Nulling the quantity $\underline{V}_g \times \dot{\underline{V}}_g$ in effect forces $\underline{V}_g$ to be anti-parallel to $\dot{\underline{V}}_g$ for all t, $0 \leq t \leq t_{go}$. This behavior is due to the fact that the cross product is zero if the two vectors are parallel, anti-parallel, or if one or both vectors are zero. While thrusting, the only possibility for nulling this cross product is if $\dot{\underline{V}}_g$ is anti-parallel to V_g. That is, this steering policy produces a thrust vector in the $\dot{\underline{V}}_g$ direction which is anti-parallel to a unit vector in the $\underline{V}_g$ direction (for all t), i.e.,

$$\frac{\dot{\underline{V}}_g}{\dot{V}_g} = -\frac{\underline{V}_g}{V_g}$$

or

$$\dot{\underline{V}}_g = -\frac{\dot{V}_g}{V_g}\underline{V}_g \qquad (2.3.77)$$

The desired thrust can now be found by substituting equation (2.3.77) into equation (2.3.33) to obtain

$$-\frac{\dot{V}_g}{V_g}\underline{V}_g + C^*\underline{V}_g = -\underline{a}_T \qquad (2.3.78)$$

or

$$\underline{a}_T = -C^*\underline{V}_g + \frac{\dot{V}_g}{V_g}\underline{V}_g \qquad (2.3.79)$$

But, in principal coordinates, the C^* matrix is

$$-C^* = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \qquad (2.3.80)$$

Thus, equation (2.3.79) becomes

$$\underline{a}_T = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} V_{g_1} \\ V_{g_2} \\ V_{g_3} \end{bmatrix} + \frac{\dot{V}_g}{V_g} \begin{bmatrix} V_{g_1} \\ V_{g_2} \\ V_{g_3} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 + \frac{\dot{V}_g}{V_g} & 0 & 0 \\ 0 & \lambda_2 + \frac{\dot{V}_g}{V_g} & 0 \\ 0 & 0 & \lambda_3 + \frac{\dot{V}_g}{V_g} \end{bmatrix} \begin{bmatrix} V_{g_1} \\ V_{g_2} \\ V_{g_3} \end{bmatrix}$$

$$= \frac{\dot{V}_g}{V_g} \begin{bmatrix} 1 + \frac{\lambda_1 V_g}{\dot{V}_g} & 0 & 0 \\ 0 & 1 + \frac{\lambda_2 V_g}{\dot{V}_g} & 0 \\ 0 & 0 & 1 + \frac{\lambda_3 V_g}{\dot{V}_g} \end{bmatrix} \begin{bmatrix} V_{g_1} \\ V_{g_2} \\ V_{g_3} \end{bmatrix} \tag{2.3.81}$$

Finally, since the time-to-go can be approximated as $(V_g / \dot{V}_g) \approx t_{go}$, equation (2.3.81) can be written as

$$\underline{a}_T = \frac{\dot{V}_g}{V_g} \begin{bmatrix} 1 + \lambda_1 t_{go} & 0 & 0 \\ 0 & 1 + \lambda_2 t_{go} & 0 \\ 0 & 0 & 1 + \lambda_3 t_{go} \end{bmatrix} \underline{V}_g \tag{2.3.82}$$

and the unit vector in the thrust direction is

$$\underline{u}^* = \text{unit} \left\{ \begin{bmatrix} 1 + \lambda_1 t_{go} & 0 & 0 \\ 0 & 1 + \lambda_2 t_{go} & 0 \\ 0 & 0 & 1 + \lambda_3 t_{go} \end{bmatrix} \underline{V}_g \right\} \tag{2.3.83}$$

But, this result is exactly the equation that must be satisfied for "optimal" performance as specified by equation (2.3.76). Thus, the cross-product steering scheme is optimal in the sense of minimum fuel consumption for C^* guidance.

The use of $\underline{V}_g \times \dot{\underline{V}}_g$ to generate a proportional steering rate command suffers from two related sources of error. First, as $\underline{V}_g$ approaches zero, the cross product also approaches zero, thus reducing the gain of the control loop. Second, since $\underline{V}_g$ is not exactly coincident with the acceleration vector, a small turning rate must be given to the acceleration vector in order to drive $\dot{\underline{V}}_g$ to be anti-parallel to $\underline{V}_g$. This turning rate can be calculated by considering the time derivative of a unit vector in the $\underline{V}_g$ direction, $\underline{1}_{vg}$.

$$\frac{d}{dt}\left(\underline{1}_{vg}\right) = \frac{d}{dt}\left(\frac{\underline{V}_g}{V_g}\right) = \frac{V_g \dot{\underline{V}}_g - \underline{V}_g \dot{V}_g}{V_g^2} \tag{2.3.84}$$

Since $\underline{1}_{vg}$ is a unit vector, its magnitude does not change in time and its total derivative is

$$\frac{d}{dt}\left(\underline{1}_{vg}\right) = \underline{\omega} \times \underline{1}_{vg} \tag{2.3.85}$$

Further, if $\underline{\omega}$ is taken as the pitch and yaw rotation rates of the unit vector, then $\underline{\omega}$ must be perpendicular to $\underline{1}_{vg}$. In this light

$$\underline{\omega} = \underline{1}_{vg} \times \frac{d}{dt}\left(\underline{1}_{vg}\right) = \underline{1}_{vg} \times \left[\underline{\omega} \times \underline{1}_{vg}\right] \tag{2.3.86}$$

So

$$\underline{\omega} = \underline{1}_{vg} \times \frac{V_g \dot{\underline{V}}_g - \underline{V}_g \dot{V}_g}{V_g^2} = \frac{\underline{V}_g}{V_g} \times \frac{V_g \dot{\underline{V}}_g}{V_g^2} - \cancelto{0}{\frac{\underline{V}_g}{V_g} \times \frac{\dot{V}_g \underline{V}_g}{V_g^2}}$$

$$\underline{\omega} = \frac{\underline{V}_g \times \dot{\underline{V}}_g}{V_g^2} \tag{2.3.87}$$

Hence, the turning rate of the vehicle is proportional to the misalignment of $\underline{V}_g$ and $\dot{\underline{V}}_g$ which is the policy of cross-product steering.

Since equation (2.3.87) prescribes infinite turning rates as $V_g \rightarrow 0$, it is customary to command the turning rates as

$$\underline{\omega} = K\left(\underline{V}_g \times \dot{\underline{V}}_g\right) \tag{2.3.88}$$

where K may or may not be a function of $\underline{V}_g$. Writing $\underline{\omega}$ in pitch and yaw components, the command rates become

$$\omega_p = -K\left(V_{gx} \dot{V}_{gz} - V_{gz} \dot{V}_{gx}\right) \tag{2.3.89a}$$

$$\omega_y = K(V_{gx}\dot{V}_{gy} - V_{gy}\dot{V}_{gx}) \qquad (2.3.89b)$$

In summary, the optimum steering for the velocity-to-be-gained C* guidance has been found to be $\underline{V}_g \times \dot{\underline{V}}_g$ (cross product) steering. First, the steering conditions for minimum fuel consumption were determined. Then, $\underline{V}_g \times \dot{\underline{V}}_g$ steering was shown to satisfy these requirements. Finally, it was shown that vehicle-command rates must be proportional to $\underline{V}_g \times \dot{\underline{V}}_g$ in order to maintain the anti-parallelism of $\underline{V}_g$ and $\dot{\underline{V}}_g$ as required for cross-product steering.

2.3.2 Linearized Perturbation Guidance

Linearized Perturbation Guidance is a scheme that takes advantage of the assumed fact that if a reference trajectory and control is specified for a flight, the actual trajectory will be "close" to the nominal trajectory. This "closeness" enables the nominal-nonlinear equations to be reduced to a linear set of first-order differential equations with time-varying coefficients. The techniques of linear analysis can then be used in order to relate control deviations to position deviations from nominal as the vehicle travels along the trajectory. The section begins with the basic linearization technqiues and presents the solutions to the linear time-varying second-order differential equation. The solutions are then related to the guidance problem.

2.3.2.1 Formulation of Linearized Perturbation Equations

2.3.2.1.1 Linearization Technique. The nonlinear equations of motion for a vehicle can be written as

$$\dot{\underline{X}} = \underline{F}(t, \underline{X}(t), \underline{U}(t)) \qquad (2.3.90)$$

where $\underline{X}(t)$ is the state vector

$\underline{U}(t)$ is the control vector

t is time

$\underline{F}$ is a function vector $\begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$

It is assumed that a numerical solution has been generated with some prespecified control law that is optimum in some sense, and that this nominal trajectory and the corresponding control as functions of time are

$$\underline{X}^*(t)$$

$$\underline{U}^*(t)$$

During the actual flight, the vehicle will follow a trajectory $\underline{X}(t)$ which differs from $X^*(t)$ due to errors in the system and in the model used to generate the nominal solution. Thus, the associated control, $\underline{U}(t)$, required to correct for these errors must be computed. The perturbations in the state and control vectors are

$$\delta \underline{X}(t) = \underline{X}(t) - \underline{X}^*(t) \tag{2.3.91a}$$

$$\delta \underline{U}(t) = \underline{U}(t) - \underline{U}^*(t) \tag{2.3.92b}$$

So

$$\frac{d}{dt}[\delta \underline{X}(t)] = \frac{d}{dt}\underline{X}(t) - \frac{d}{dt}\underline{X}^*(t) \tag{2.3.92a}$$

and

$$\frac{d}{dt}[\delta \underline{U}(t)] = \frac{d}{dt}\underline{U}(t) - \frac{d}{dt}\underline{U}^*(t) \tag{2.3.92b}$$

Now, the nominal trajectory and control relationship is

$$\frac{d}{dt}\underline{X}^*(t) = \underline{F}(t,\underline{X}^*(t), \underline{U}^*(t)) \tag{2.3.93}$$

Thus, from equation (2.3.92a)

$$\frac{d}{dt}[\delta\underline{X}(t)] = \underline{F}(t,\underline{X}(t), \underline{U}(t)) - \underline{F}(t,X^*(t), U^*(t)) \tag{2.3.94}$$

However, since the actual trajectory is assumed to be close to the nominal trajectory, the term $\underline{F}(t,\underline{X}(t), \underline{U}(t))$ can be expanded in a first-order truncated Taylor series about the nominal; i.e.,

$$\underline{F}(t,\underline{X}(t), \underline{U}(t)) = \underline{F}(t,\underline{X}^*(t), \underline{U}^*(t)) + \left.\frac{\partial \underline{F}}{\partial \underline{X}}\right|_* \delta\underline{X}(t) + \left.\frac{\partial \underline{F}}{\partial \underline{U}}\right|_* \delta\underline{U}(t) \tag{2.3.95}$$

where

$$\delta\underline{X}(t) = \underline{X}(t) - \underline{X}^*(t)$$

$$\delta\underline{U}(t) = \underline{U}(t) - \underline{U}^*(t)$$

$\left.\frac{\partial \underline{F}}{\partial \underline{X}}\right|_{*} =$ the Jacobian matrix of $\underline{F}$ with respect to $\underline{X}$ evaluated along the nominal trajectory

$\left.\frac{\partial \underline{F}}{\partial \underline{U}}\right|_{*} =$ the Jacobian matrix of $\underline{F}$ with respect to $\underline{U}$ evaluated at the nominal control law

Note that terms for deviation in time are not included. This deletion is the result of the fact that a fixed final time is assumed.

Equation (2.3.94) now becomes

$$\frac{d}{dt}(\delta \underline{X}(t)) = \underline{F}(t, \underline{X}^*(t), \underline{U}^*(t)) + \left.\frac{\partial \underline{F}}{\partial \underline{X}}\right|_{*} \delta \underline{X}(t) + \left.\frac{\partial \underline{F}}{\partial \underline{U}}\right|_{*} \delta \underline{U}(t) - \underline{F}(t, \underline{X}^*(t) \underline{U}^*(t)) \tag{2.3.96}$$

or

$$\frac{d}{dt}(\delta \underline{X}(t)) = \left.\frac{\partial \underline{F}}{\partial \underline{X}}\right|_{*} \delta \underline{X}(t) + \left.\frac{\partial \underline{F}}{\partial \underline{U}}\right|_{*} \delta \underline{U}(t) \tag{2.3.97}$$

At this point, a rotational change will be accomplished by substituting

$$A(t) = \left.\frac{\partial \underline{F}}{\partial \underline{X}}\right|_{*} \qquad \underline{x}(t) = \delta \underline{X}(t)$$

$$B(t) = \left.\frac{\partial \underline{F}}{\partial \underline{X}}\right|_{*} \qquad \underline{u}(t) = \delta \underline{U}(t)$$

so that equation (2.3.97) assumes the familiar time-varying linear differential perturbation equation

$$\dot{\underline{x}}(t) = A(t)\,\underline{x}(t) + B(t)\,\underline{u}(t) \tag{2.3.98}$$

2.3.2.1.2 <u>Homogeneous Solution</u>. The homogeneous solution to equation (2.3.98) is

$$\underline{x}(t) = \Phi(t, t_o)\,\underline{x}_o \tag{2.3.99}$$

where $\underline{x}_0$ is the state vector at time $t = t_0$ and $\phi(t, t_0)$ is the state-transition matrix which is defined as the solution to the following matrix differential equation

$$\frac{d}{dt} \Phi(t, t_o) = A(t) \Phi(t, t_o) , \qquad (2.3.100)$$

Equation (2.3.99) can easily be seen as the homogeneous solution by taking the derivative of equation (2.3.99) and employing the definition of $\phi(t, t_0)$

$$\begin{aligned} \dot{\underline{x}}(t) &= \dot{\Phi}(t, t_o) \underline{x}_o = A(t) \Phi(t, t_o) \underline{x}_o \\ &= A(t) \underline{x}(t) \end{aligned} \qquad (2.3.101)$$

So

$$\dot{\underline{x}}(t) = A(t) \underline{x}(t), \qquad (2.3.102)$$

which is the homogeneous equation.

Two of the properties of the state-transition matrix will be useful later in the development. They are

$$\Phi(t_3, t_1) = \Phi(t_3, t_2) \cdot \Phi(t_2, t_1) \qquad (2.3.103)$$

$$\Phi(t_1, t_1) = I \text{ (the identity matrix)} \qquad (2.3.104)$$

2.3.2.1.3 General Solution. The state deviations can be controlled by proper choice of the control deviation, $\underline{u}(t)$. However, in order to determine the effect of the control on the state at a later time, it is necessary to know the general solution to equation (2.3.98). As will be apparent, the general solution to equation (2.3.98) alone is not sufficient knowledge for determining a control law in itself.

The general solution to equation (2.3.98) is obtained by the introduction of another set of equations called the adjoint equations. These equations reduce the problem to a straightforward integration problem in which there is no cross coupling in the state variables. The proper choice of the adjoint equations can be determined by considering the derivative of the product $\Lambda^T(t, t_o) \underline{x}(t)$, where $\Lambda(t, t_0)$ is a matrix of variables for the adjoint system. (This matrix will be defined during the steps which follow. At present, the elements of this matrix are unknown.), and where "T" represents the (conjugate) transpose of a matrix.

That is,

$$\frac{d}{dt}\left[\Lambda^T(t,t_o)\,\underline{x}(t)\right] = \dot{\Lambda}^T(t,t_o)\,\underline{x}(t) + \Lambda^T(t,t_o)\,\dot{\underline{x}}(t)$$

$$= \dot{\Lambda}^T(t,t_o)\,\underline{x}(t) + \Lambda^T(t,t_o)\,A(t)\,\underline{x}(t) + \Lambda^T(t,t_o)\,B(t)\,\underline{u}(t) \quad (2.3.105)$$

Now, since the elements of $\Lambda(t, t_o)$ have not been specified, they can be selected to satisfy the equation

$$\dot{\Lambda}^T(t,t_o) = -\Lambda^T(t,t_o)\,A(t)\,, \quad (2.3.106)$$

with the boundary conditions $\Lambda(t_o, t_o) = I$

Thus, equation (2.3.105) reduces to

$$\frac{d}{dt}\left[\Lambda^T(t,t_o)\,\underline{x}(t)\right] = \Lambda^T(t,t_o)\,B(t)\,\underline{u}(t) \quad (2.3.107)$$

The integration of this equation is now seen to be straightforward. The result is

$$\Lambda^T(t,t_o)\,\underline{x}(t) = \Lambda^T(t_o,t_o)\,\underline{x}(t_o) + \int_{t_o}^{t}\Lambda^T(\xi,t_o)\,B(\xi)\,\underline{u}(\xi)\,d\xi \quad (2.3.108)$$

Now, employing the fact that $\Lambda(t_o, t_o) = I$, the solution for $\underline{x}(t)$ is obtained as

$$\underline{x}(t) = \Lambda^{-T}(t,t_o)\left[\underline{x}(t_o) + \int_{t_o}^{t}\Lambda^T(\xi,t_o)\,B(\xi)\,\underline{u}(\xi)\,d\xi\right] \quad (2.3.109)$$

Hence, the general solution for $\underline{x}(t)$ for all $t > t_o$ has been determined in terms of the adjoint matrix.

Now, consider the derivative of the product

$$\Lambda^T(t,t_o)\,\Phi(t,t_o)\,.$$

That is,

$$\frac{d}{dt}\left[\Lambda^T(t,t_o)\,\Phi(t,t_o)\right] = \dot{\Lambda}^T(t,t_o)\,\Phi(t,t_o) + \Lambda^T(t,t_o)\,\dot{\Phi}(t,t_o) \quad (2.3.110)$$

But, substituting equations (2.3.100) and (2.3.106) reduces this equation to

$$\begin{aligned}\frac{d}{dt}\left[\Lambda^T(t,t_o)\,\Phi(t,t_o)\right] &= -\Lambda^T(t,t_o)\,A(t)\,\Phi(t,t_o) \\ &\quad + \Lambda^T(t,t_o)\,A(t)\,\Phi(t,t_o) = 0 \quad (\textit{Null matrix})\end{aligned} \quad (2.3.111)$$

so that

$$\Lambda^T(t,t_o)\,\Phi(t,t_o) = \Lambda^T(t_o,t_o)\,\Phi(t_o,t_o) + \int_{t_o}^{t} 0\,dt = I\cdot I = I \quad (2.3.112)$$

But, if

$$\Lambda^T(t,t_o)\,\Phi(t,t_o) = I \quad (2.3.113)$$

then

$$\Phi(t,t_o) = \Lambda^{T^{-1}}(t,t_o) \quad (2.3.114)$$

or

$$\Phi^{-1}(t,t_o) = \Lambda^T(t,t_o) \quad (2.3.115)$$

Now using the relationship between $\phi(t, t_o)$ and $\Lambda(t, t_o)$, equation (2.3.109) can be written in terms of $\phi(t, t_o)$ as

$$\underline{x}(t) = \Phi(t,t_o)\,\underline{x}(t_o) + \Phi(t,t_o)\int_{t_o}^{t} \Lambda^T(\xi,t_o)\,B(\xi)\,\underline{u}(\xi)\,d\xi \quad (2.3.116)$$

$$= \Phi(t,t_o)\,\underline{x}(t_o) + \int_{t_o}^{t} \Phi(t,t_o)\,\Lambda^T(\xi,t_o)\,B(\xi)\,\underline{u}(\xi)\,d\xi \quad (2.3.117)$$

But

$$\Lambda^T(\xi, t_o) = \Phi^{-1}(\xi, t_o) = \Phi(t_o, \xi) \tag{2.3.118}$$

So, finally

$$\boxed{x(t) = \Phi(t, t_o)\, \underline{x}(t_o) + \int_{t_o}^{t} \Phi(t, \xi) B(\xi) \underline{u}(\xi) d\xi} \tag{2.3.119}$$

Equation (2.3.119) is the general solution to the differential equation

$$\dot{\underline{x}}(t) = A(t)\, \underline{x}(t) + B(t)\, \underline{u}(t) \tag{2.3.120}$$

Note that this solution is independent of the adjoint parameters and requires no inversions.

2.3.2.1.4 Fundamental Guidance Equation. The derivation in Section 2.3.2.1.3 gave the general solution to the first-order perturbation equation. A slightly modified form of this equation, known as the fundamental guidance equation, is used extensively for performing error analyses, generating requirements, commanding corrective maneuvers, making linear prediction, etc. This section will derive this equation and give some of the interpretations of it.

The general first-order perturbation equations will be rewritten for convenience.

$$\frac{d}{dt}\, \underline{x}(t) = A(t)\, \underline{x}(t) + B(t)\, \underline{u}(t) \tag{2.3.121}$$

Now, since

$$\dot{\Lambda}(t, t_o) = -A^T(t)\, \Lambda(t, t_o)$$

an adjoint system can be defined as

$$\dot{\underline{\lambda}}(t) = -A^T(t)\, \underline{\lambda}(t) \tag{2.3.122}$$

where $\lambda(t)$ is the state vector for the adjoint system.

Thus, an analysis similar to that used in Section 2.3.2.1.3 can be used to derive the useful combination of the two systems of equations.

Consider the derivative of the scalar product of $\underline{\lambda}$ and $\underline{x}$.

$$\begin{aligned}\frac{d}{dt}\langle \underline{\lambda}, \underline{x}\rangle &= \langle \dot{\underline{\lambda}}, \underline{x}\rangle + \langle \underline{\lambda}, \dot{\underline{x}}\rangle \\ &= \langle -A^T\underline{\lambda}, \underline{x}\rangle + \langle \underline{\lambda}, A\underline{x} + B\underline{u}\rangle \\ &= \langle -A^T\underline{\lambda}, \underline{x}\rangle + \langle \underline{\lambda}, A\underline{x}\rangle + \langle \underline{\lambda}, B\underline{u}\rangle \\ &= \langle \underline{\lambda}, B\underline{u}\rangle = \langle B^T\underline{\lambda}, \underline{u}\rangle \end{aligned} \tag{2.3.123}$$

where $\langle \underline{\lambda}, \underline{x}\rangle = \underline{\lambda}\cdot\underline{x}$ = inner product of two vectors

Integration of equation (2.3.123) from some initial time, $t = t_o$, to a final time, $t = T$, yields

$$\underline{\lambda}^T(T)\,\underline{x}(T) - \underline{\lambda}^T(t_o)\,\underline{x}(t_o) = \int_{t_o}^{T} \underline{\lambda}^T(\tau)\,B(\tau)\,\underline{u}(\tau)\,d\tau$$

or

$$\underline{\lambda}^T(T)\,\underline{x}(T) = \underline{\lambda}^T(t_o)\,\underline{x}(t_o) + \int_{t_o}^{T} \underline{\lambda}^T(\tau)\,B(\tau)\,\underline{u}(\tau)\,d\tau \tag{2.3.124}$$

This equation is generally denoted by the fundamental guidance equation.

So far, no constraints on the systems of equations have been specified. However, the proper specification of these constraints reduces equation (2.3.124) to a more useful form. One constraint that can be specified is the terminal state deviation, $x(T)$. This constraint can be expressed as some maximum allowable deviation at the end of the powered trajectory. It is noted that though $x(T)$ does not explicitly appear in equation (2.3.124), it does appear in the form of the inner product of $\langle \lambda(T), x(T)\rangle$. Thus, if the appropriate choices of the components of $\underline{\lambda}(t)$ are made, a particular component of $\underline{x}(t)$ can result from this inner product. For instance, if the inner product is desired to be $x_1(t)$, the choice of $\lambda(T)$ is

$$\underline{\lambda}(T) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

It should be apparent that in order for the left side of equation (2.3.124) to be written explicitly in terms of $\underline{x}(T)$, a different adjoint vector is needed for each component of the state vector.

$$\underline{\lambda}_1^T(T)\ \underline{x}(T) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1(T) \\ x_2(T) \\ \vdots \\ x_n(T) \end{bmatrix} = x_1(T)$$

$$\underline{\lambda}_2^T(T)\ \underline{x}(T) = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1(T) \\ x_2(T) \\ \vdots \\ x_n(T) \end{bmatrix} = x_2(T)$$

$$\underline{\lambda}_n^T(T)\ \underline{x}(T) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1(T) \\ x_2(T) \\ \vdots \\ x_n(T) \end{bmatrix} = x_n(T)$$

Equation (2.3.124) can be written for each adjoint vector that was selected in these constraint equations. Written in matrix form

$$\Lambda^T(T)\ \underline{x}(T) = \Lambda^T(t_0)\ \underline{x}(t_0) + \int_{t_0}^{T} \Lambda^T(\tau)\, B(\tau)\, \underline{u}(\tau)\, d\tau \qquad (2.3.125)$$

$$\Lambda^T(T) = \begin{bmatrix} \underline{\lambda}_1^T(T) \\ \underline{\lambda}_2^T(T) \\ \vdots \\ \underline{\lambda}_2^T(T) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ & \vdots & & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix} = I$$

$$\Lambda^T(t_o) = \begin{bmatrix} \underline{\lambda}_1^T(t_o) \\ \underline{\lambda}_2^T(t_o) \\ \vdots \\ \underline{\lambda}_n^T(t_o) \end{bmatrix} = \begin{bmatrix} \lambda_{11}(t_o) & \lambda_{12}(t_o) \ldots \lambda_{1n}(t_o) \\ \lambda_{21}(t_o) & \lambda_{22}(t_o) \ldots \lambda_{2n}(t_o) \\ \vdots & \vdots \qquad \vdots \\ \lambda_{n1}(t_o) & \lambda_{n2}(t_o) \ldots \lambda_{nn}(t_o) \end{bmatrix}$$

$$\Lambda^T(t) = \begin{bmatrix} \underline{\lambda}_1^T(t) \\ \underline{\lambda}_2^T(t) \\ \vdots \\ \underline{\lambda}_n^T(t) \end{bmatrix} = \begin{bmatrix} \lambda_{11}(t) & \lambda_{12}(t) \ldots \lambda_{1n}(t) \\ \lambda_{21}(t) & \lambda_{22}(t) \ldots \lambda_{2n}(t) \\ \vdots & \\ \lambda_{n1}(t) & \ldots\ldots \lambda_{nn}(t) \end{bmatrix}$$

Now that the boundary conditions of the adjoint equations have been specified, these equations can be integrated in reverse time from t = T.

$$\dot{\Lambda}(t) = -A^T(t)\,\Lambda(t) \qquad (2.3.126)$$

$$\Lambda(T) = I \qquad (2.3.127)$$

Thus, for a given state deviation at an arbitrary time, t, the only unknown term in equation (2.3.125) is the integral. (All other terms are known since the terminal and present state deviations are specified and the adjoint equations have been integrated.) Hence, the control that is necessary for a prescribed terminal-state deviation is implicitly contained in the integral of equation (2.3.125). This information is useful in determining possible control for the remaining portion of the powered flight.

A simple example would be the determination of a constant control vector (for the remaining powered flight time) that would satisfy the terminal state deviation specification. Consider the following:

$$\int_{t_o}^{T} \Lambda(\tau) B(\tau) \underline{u}\, d\tau = \underline{D}$$

where $\underline{D}$ is known from the specification of present and terminal-state deviations and the adjoint equations.

Since $\underline{u}$ was arbitrarily selected to be a constant vector for this simple example, it can be factored out of the integral

$$\left[\int_{t_0}^{T} \Lambda^T(\tau)\, B(\tau)\, d\tau\right] \underline{u} = \underline{D}$$

The integral can now be found since $\Lambda(\tau)$ and $B(\tau)$ are known. The constant control vector is thus

$$\underline{u} = \left[\int_{t_0}^{T} \Lambda^T(\tau)\, B(\tau)\, d\tau\right]^{-1} \underline{D}$$

2.3.2.2 Optimum Control

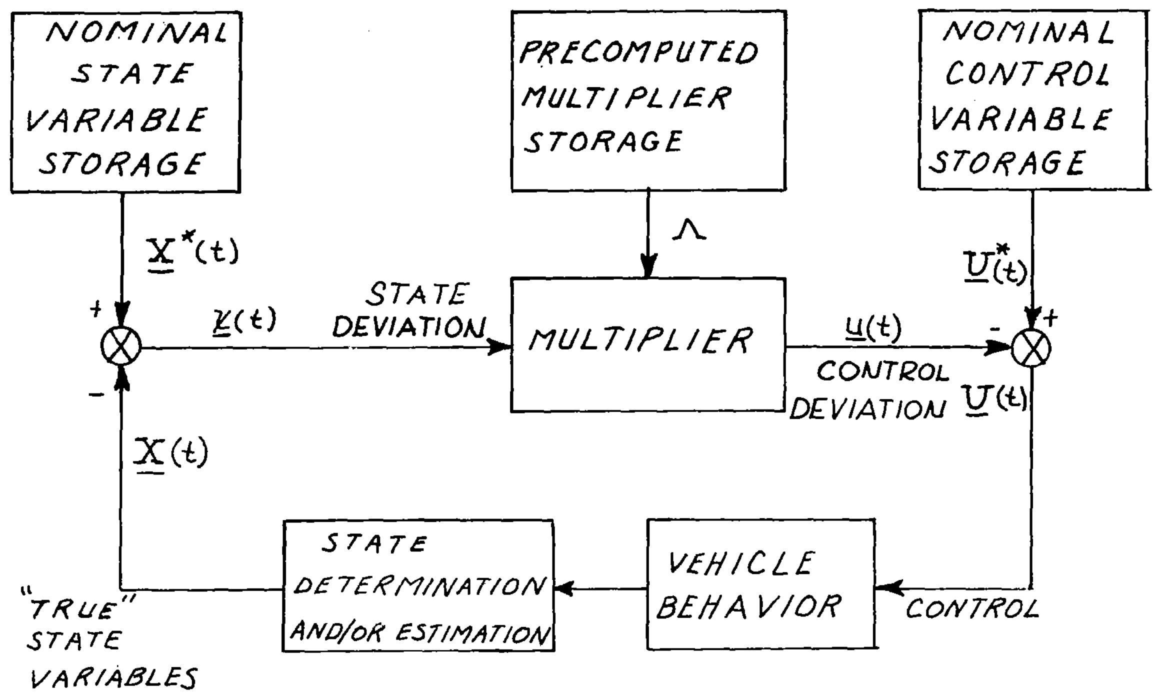
The basic formulation and solution to the linearized perturbation equation has been presented in Section 2.3.2.1. Although this information is necessary for a complete understanding of the problem, it does not explicitly give a solution to the control problem. In other words, the material presented so far is suitable for use with a control that is known beforehand but is not sufficient to determine a control law as such. The theory must be extended further in order for it to be useful in control law determination. This extension will be realized by optimizing the performance in some sense over the entire trajectory.

The optimum control problem has been studied extensively in recent years, and as a result, a large amount of literature is available on the subject. However, since the methods of analysis, while different, are analogous and since it is impossible to discuss all of the available material, several of the more important variations will be presented.

Acknowledgement is given to W. F. Denham and A. E. Bryson whose work on terminal control for a minimum-mean square deviation from a nominal path (Reference 3.19) was used in the preparation of this section. The material presented an optimal linear control generally seen in text books on optimal control of discrete-time systems. Reference 3.26 presents more advanced concepts in optimum control theory as well as the basic deviations.

2.3.2.2.1 <u>Terminal Control for a Minimum-Mean Square Deviation from Nominal</u>. In Section 2.3.2.1 the linearized perturbation equations for a nonlinear system were derived and the general solutions were written in terms of the state transition matrix and an integral that involved the control deviation. However, the determination of the control deviation that is necessary to correct a state deviation is not easily accomplished since the control must be known beforehand in order to evaluate the effect of the control on the state deviation at a later time. Further, it is usually desired that a control policy be optimum (in some sense) and that the on-board computation minimized. Thus, this discussion will be initiated with the presentation of a convenient control scheme for the linearized perturbation equations.

In this scheme, it is only necessary to multiply the state deviation by a precomputed matrix in order to find the desired optimum control deviation required to guarantee the correct terminal state. However, in order to achieve this simplicity, the burden of the computation has been placed on the pre-flight



simulation. Thus, in addition to the optimum nominal trajectory computation that must be performed, it is also necessary to determine the matrix multiplier, $\Lambda(t)$, as a function of flight time. This latter objective is not easily realized since the solution depends on many parameters such as terminal-condition constraints, optimization desired, weighting, etc. However, the following discussion is designed to present the derivation of the $\Lambda(t)$ matrix for the control such that the mean value of a positive definite quadratic form in the control variable deviations is minimum.

As in any terminal control problem, it is necessary to define the terminal conditions that must be satisfied at the end of the control time. If the nominal trajectory and nominal control are followed for the entire powered flight, the vehicle will arrive at the terminal point with the desired terminal conditions

$$\underline{\Psi}\left[\underline{X}(T), T\right]$$

where $\underline{\Psi}$ is a column matrix of "P" known functions of $\underline{X}(t)$ and t, ($P \leq N$) and T = terminal time.

Since nominal conditions are not followed exactly, it is necessary to define the terminal point by choosing some scalar function $\Omega\left[X(T), T\right]$, which satisfies the desired terminal constraint, i.e., control is terminated when Ω is satisfied. Such a formulation enables the terminal point to be defined in terms of a desired constraint equation rather than in terms of the independent variable, t. This feature is advantageous since the nominal-terminal time, T, may not be the time at which Ω is satisfied once deviations in state and control are experienced at any point in the trajectory.

Using previous results, the linearized perturbation equations are

$$\frac{d}{dt}\underline{x}(t) = A(t)\,\underline{x}(t) + B(t)\,\underline{u}(t) \tag{2.3.128}$$

In Section 2.3.2.1.4, the general solution to this equation was found (with the aid of the adjoint equations) to be

$$\Lambda^T(T)\,\underline{x}(T) = \Lambda^T(t_o)\,\underline{x}(t_o) + \int_{t_o}^{T} \Lambda^T(\tau)B(\tau)\,\underline{u}(\tau)\,d\tau \tag{2.3.129}$$

where $\Lambda^T(t)$ is a matrix of multiplier functions.

The boundary conditions must now be given to the matrix of functions, $\Lambda^T(t)$. This step is accomplished by introducting two different matrices, $\Lambda^T_\psi(t)$ and $\Lambda^T_\Omega(t)$ each of which satisfies the terminal conditions

$$\Lambda^T_\psi(T) = \left(\frac{\partial \underline{\psi}}{\partial \underline{x}}\right)_{t=T}$$

$$\Lambda^T_\Omega(T) = \left(\frac{\partial \Omega}{\partial \underline{x}}\right)_{t=T} \tag{2.3.130}$$

If these conditions are used in equation (2.3.129), the following is obtained

$$(\delta \underline{\psi})_{t=T} = \int_{t_o}^{T} \Lambda^T_\psi(t)B(t)\,\underline{u}(t)\,dt + (\Lambda^T_\psi\,x)_{t=t_o} \tag{2.3.131}$$

$$(\delta \Omega)_{t=T} = \int_{t_o}^{T} \Lambda^T_\Omega(t)\,B(t)\,\underline{u}(t)\,dt + (\Lambda^T_\Omega\,x)_{t=t_o} \tag{2.3.132}$$

where

$$(\delta \underline{\psi})_{t=T} = (\Lambda^T_\psi\,x)_{t=T}$$

$$(\delta \Omega)_{t=T} = (\Lambda^T_\Omega\,x)_{t=T}$$

and where

$\Lambda_{\psi}^{T}(t)$, $\Lambda_{\Omega}^{T}(t)$ denote the matrix of multipliers defined by matching the boundary conditions for both ψ and Ω , respectively.

The terminal point is defined to be that point when Ω is equal to some specified value, i.e., the control is terminated when $d\Omega = 0$. However, small deviations from the nominal trajectory produce changes in the value of T, dT and in the total derivatives of ψ and Ω .

$$d\underline{\psi} = (\delta\underline{\psi})\Big|_{t=T} + \dot{\underline{\psi}}\,dT \tag{2.3.133}$$

$$d\Omega = (\delta\Omega)\Big|_{t=T} + \dot{\Omega}\,dT \tag{2.3.134}$$

where

$$\dot{\underline{\psi}} = \frac{\partial\underline{\psi}}{\partial t} + \frac{\partial\underline{\psi}}{\partial\underline{X}}\frac{d\underline{X}}{dt}\Bigg|_{t=T} \tag{2.3.135}$$

$$\dot{\Omega} = \frac{\partial\Omega}{\partial t} + \frac{\partial\Omega}{\partial\underline{X}}\frac{d\underline{X}}{dt}\Bigg|_{t=T} \tag{2.3.136}$$

Substituting equation (2.3.131) into equation (2.3.132) yields

$$d\underline{\psi} = \int_{t_o}^{T}\Lambda_{\psi}^{T}B(t)\underline{U}(t)\,dt + \dot{\underline{\psi}}\,dT + \Lambda_{\psi}^{T}(t_o)\,\underline{X}(t_o) = 0 \tag{2.3.137}$$

$$d\Omega = \int_{t_o}^{T}\Lambda_{\Omega}^{T}B(t)\underline{U}(t)\,dt + \dot{\Omega}\,dT + \Lambda_{\Omega}^{T}(t_o)\,\underline{X}(t_o) = 0 \tag{2.3.138}$$

It is now desired to have these terminal constraint equations independent of dT. This objective is achieved by solving equation (2.3.138) for dT and substituting the result into equation (2.3.137).

$$d\psi = \int_{t_o}^{T}\Lambda_{\psi\Omega}^{T}B(t)\underline{U}(t)\,dt + \Lambda_{\psi\Omega}^{T}(t_o)\,\underline{X}(t_o) = 0 \tag{2.3.139}$$

where

$$\Lambda^T_{\psi\Omega} = \Lambda^T_{\psi} - \frac{\dot{\psi}}{\dot{\Omega}} \Lambda^T_{\Omega} . \qquad (2.3.140)$$

$\Lambda^T_{\psi\Omega}$ is again a matrix of Lagrange multipliers. Equation (2.3.140) determines the values of the elements of $\Lambda^T_{\psi\Omega}$, the boundary conditions being given by

$$\Lambda^T_{\psi\Omega}(T) = \left(\frac{\partial \psi}{\partial \underline{X}} - \frac{\dot{\psi}}{\dot{\Omega}} \frac{\partial \Omega}{\partial \underline{X}} \right)_{t=T} . \qquad (2.3.141)$$

Now that the constraint equations have been determined, a performance index must be specified in order to establish some measure of optimization. In general, though it is noted that the performance index need not be limited or to the control deviation over the nominal trajectory, these criteria serve as very effective measures of the performance of a system. (Any positive definite quadratic form can be considered to be an allowable term in the performance index as long as it is associated in some way with the performance.)

For purposes of illustration, a simple quadratic form of the control deviation will be used for the performance index. This form is the type which would be employed if fuel consumption was of major concern. Assume

$$V = \int_{t_0}^{T} \underline{u}(t)^T \gamma \, \underline{u}(t) \, dt \qquad (2.3.142)$$

where γ is an arbitrary symmetric, non-negative weighting matrix chosen by the control engineer and rewrite equation (2.3.139) as

$$d\psi - \int_{t_0}^{T} \Lambda^T_{\psi\Omega} B(t) \underline{U}(t) \, dt - \Lambda^T_{\psi\Omega}(t_0) \underline{X}(t_0) = 0 \qquad (2.3.143)$$

Now, variations in V with respect to $\underline{u}$ can be found once the equations are joined by another matrix of multipliers, ν^T.

$$V = \int_{t_0}^{T} (\underline{u}^T \gamma \underline{u} - \nu^T \Lambda^T_{\psi\Omega} B \underline{u}) \, dt + \nu^T \left[d\psi - \Lambda^T_{\psi\Omega}(t_0) \underline{X}(t_0) \right] \qquad (2.3.144)$$

$$\delta V = \int_{t_0}^{T} (2 \underline{u}^T \gamma - \nu^T \Lambda^T_{\psi\Omega} B) \, \delta \underline{u}(t) \, dt \qquad (2.3.145)$$

But, for an extremum in , = 0. This equality occurs when the integrand of equation (2.3.145) vanished, i.e.,

$$2\underline{u}^T \gamma - \nu^T \Lambda^T_{\psi\Omega} B = 0$$
$$\underline{u}^T = \frac{1}{2} \nu^T \Lambda^T_{\psi\Omega} B \gamma^{-1} \qquad (2.3.146)$$

Now equation (2.3.148) can be written

$$d\psi - \int_{t_o}^{T} \Lambda^T_{\psi\Omega} B \frac{1}{2} \gamma^{-1^T} B^T \Lambda_{\psi\Omega} \nu \, dt - \Lambda^T_{\psi\Omega}(t_o) \underline{X}(t_o) = 0 \qquad (2.3.147)$$

or

$$d\psi - \Lambda^T_{\psi\Omega}(t_o) \underline{X}(t_o) = \frac{1}{2} J(t_o) \nu \qquad (2.3.148)$$

where

$$J(t_o) = \int_{t_o}^{T} \Lambda^T_{\psi\Omega} B \gamma^{-1} B \Lambda_{\psi\Omega} \, dt \qquad (2.3.149)$$

Now the multiplier matrix which was introduced can be seen to be

$$\nu(t_o) = 2J^{-1}(t_o)\left[d\psi - \Lambda^T_{\psi\Omega}(t_o) X(t_o)\right] \qquad (2.3.150)$$

Finally, the control deviation can be found in terms of the state deviation from the transpose of equation (2.3.146).

$$\underline{u}^T = \frac{1}{2}\left\{2\left[d\psi - \Lambda^T_{\psi\Omega}\underline{X}(t_o)\right]^T J^{-1^T}(t_o)\right\} \Lambda^T_{\psi\Omega} B \gamma^{-1}$$
$$\underline{u} = \gamma^{-1} B^T(t) \Lambda_{\psi\Omega}(t) J^{-1}(t_o)\left[d\psi - \Lambda^T_{\psi\Omega}(t_o)\underline{X}(t_o)\right] \qquad (2.3.151)$$

Equation (2.3.151) leads to the final relation between the control and state deviation when $d\psi$ is set equal to zero.

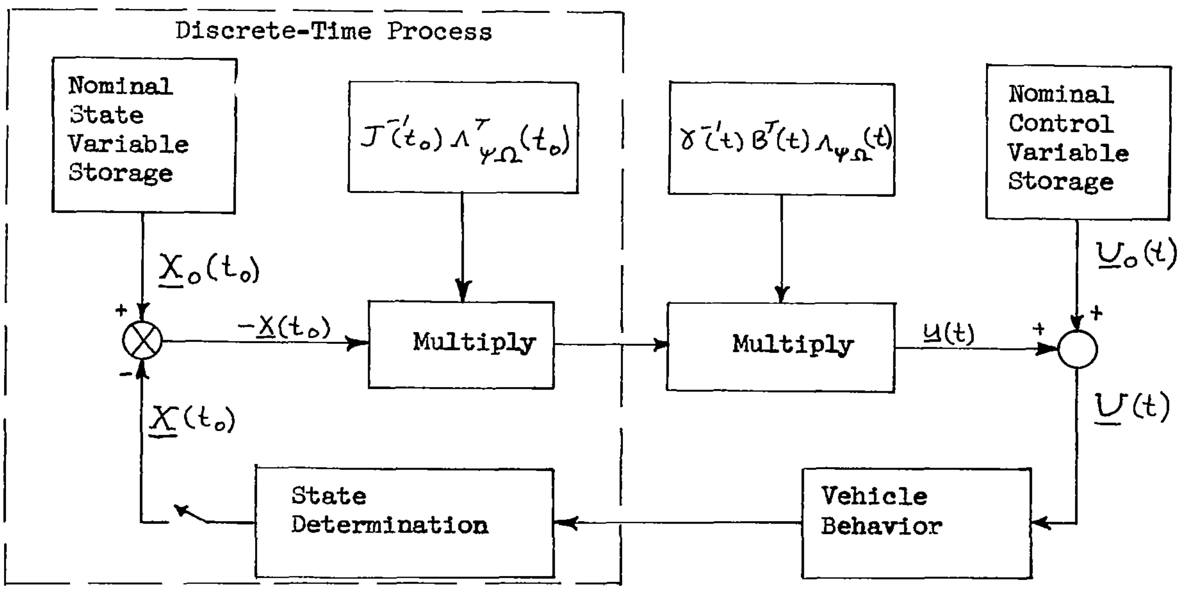
$$\boxed{\underline{u}(t) = -\gamma^{-1} B^T(t) \Lambda_{\psi\Omega}(t) J^{-1}(t_o) \Lambda^T_{\psi\Omega}(t_o) \underline{X}(t_o)} \qquad (2.3.152)$$

In the way of explanation, $d\underline{\psi}$ specifies the terminal constraint deviations that are desired based on currently available information. For simplicity, these desired deviations will not be considered here. The interested reader will find information regarding terminal constraint modifications in Reference 3.19.

Equation (2.3.152) can be mechanized in two methods. The first method can be thought of as a discrete-time system in which a major computation cycle occurs at intervals that are spaced by minor computation cycles. The second method utilizes the computer's speed to run the problem as though it were a continuous control problem. The following paragraphs will describe the calculations that are necessary to mechanize this guidance scheme in both the continuous and discrete modes.

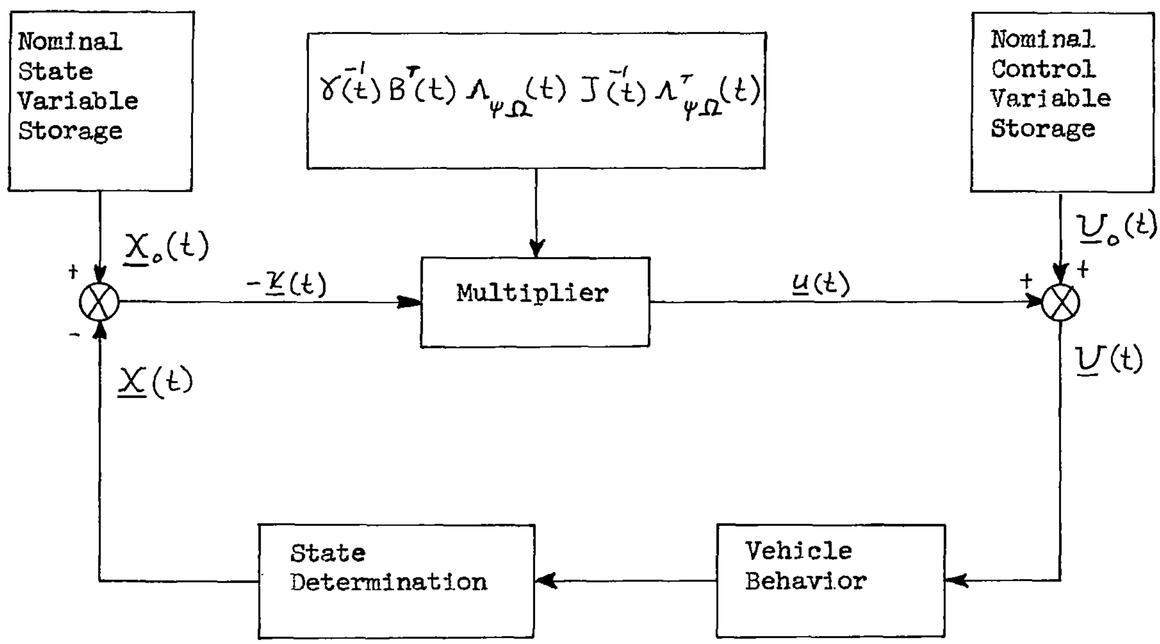
First, the choice of the nominal trajectory that satisfies terminal conditions must be made. Then, the elements of the A(t) and B(t) matrices must be determined along the entire nominal trajectory. Having determined A(t), the matrix of functions $\Lambda_{\psi\Omega}(t)$ can be determined by integrating the adjoint equations backward in time from the terminal point with the boundary conditions of equation (2.3.141). Now, $J(t_o)$ can be determined, since all components of its integrand have been determined. Further, the limits for the integral are from t_o to T, so J can be evaluated at the same time the adjoint equations are integrated, since t_o varies and T is fixed. It is also convenient to tabulate $\gamma^{-1}(t)\, B^T(t)\, \Lambda_{\psi\Omega}(t)$ as the integration progresses since this information is needed during the execution of the guidance scheme.

The remaining parameters in equation (2.3.152), $J^{-1}(t_o)\, \Lambda^T_{\psi\Omega}(t_o)$, can also be determined while the integration of the adjoint equation is being performed. However, if the discrete-time approach is taken, these parameters need only be known at certain epochs of the flight. If the time t_o is taken as a sample point, the following mechanization can be implemented for the discrete-time case during a time interval.



In the previous mechanization, the state is sampled at predetermined epochs. The sampled state is then compared to the nominal state for the same epoch, and a state deviation for that epoch is determined. The state deviation then undergoes two multiplications. The first matrix is precalculated for the particular sample epoch,but the second matrix is a sequence of stored matrices that are fed to the guidance system at such a rapid rate that it makes the matrix look as though it is a time-varying matrix. The remainder of the scheme is straightforward and will not be discussed further here.

The mechanization of the continuous-time approach is shown below.

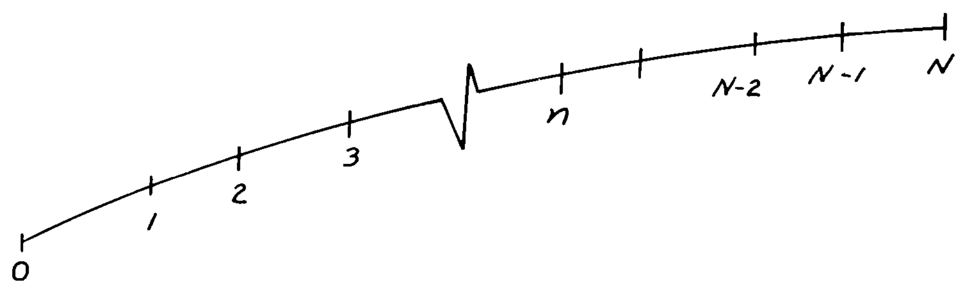


The main distinctions of this type of mechanization are: (1) the precomputed multiplier matrix is stored as a sequence of matrices for the entire flight, whereas in the discrete-time case, part of this multiplier was known only at sampling epochs; and (2) the state variables must be known as a continuous function of time (or as close as possible) for the continuous-time case; the state was only needed at sampling epochs in the discrete-time case.

2.3.2.2.2 Optimum Linear Control. Optimum linear control, as discussed here, is a method of determining control deviations by a simple multiplication of a precomputed matrix by the state deviation. The technique is analogous to a discrete-time sampled-data system in that it divides the control interval into a number of smaller increments and determines the control for each increment as a separate problem. The control that is selected must then extremize the performance index which represents the cumulated performance over the entire trajectory.

Since the previous control scheme and the one suggested in the preceding paragraph are simply different implementations of the same idea, it is expected that the schemes themselves are similar. First, it is noted that both schemes determine the control deviation by the multiplication of the state deviation and a precomputed matrix. Second, as it will be seen, both schemes also use the same type of performance index, i.e., a positive definite quadratic form. Thus, the primary differences exists due to variations in the methods used to compute the multiplier matrix in the two schemes; the results should agree, at least to the first-order, with those provided by the previous formulation. It is recalled that the multiplier matrix in the previous scheme was determined by the multiplication of the matrix of functions (which were integrated backwards in time from some terminal condition) and other parameters of the system. In contrast, the multiplier matrix in this section is determined by a dynamic programming approach. That is, the multiplier matrix is determined for the last step first. Then it is determined for the next to the last step based on extremizing the performance index for the last two steps. The process continues in a step by step decision process to the epoch of flight initiation. The result, as in the previous scheme, is a series of multiplier matrices for each increment. With these introductory remarks in mind, the optimum linear control derivation will now be presented.

Consider the following nominal trajectory which has been divided into N control increments



and consider a general performance index which accounts for state and control deviation for the increments from n to N

$$J_{N-n+1} = \sum_{i=n}^{N} \left[\underline{x}^T(i)\, Q_i\, \underline{x}(i) + \underline{u}^T(i-1)\, \gamma_i\, \underline{u}(i-1) \right] \quad (2.3.153)$$

where Q_i and γ_i are arbitrarily weighting matrices determined by the control engineer. Finally, define the minimum loss associated with J_{N-n+1} as

$$I_{N-n+1} = \min_{\text{all } u} J_{N-n+1} \tag{2.3.154}$$

That is, I_{N-n+1} is the value that J_{N-n+1} takes for the sequence of controls from t_{n-1} to t_{N-1} that minimizes J_{N-n+1} . Any other sequence of controls during the same time interval will produce a loss function that is greater than or equal to I_{N-n+1} Now, if the minimum loss from t_{n+1} to t_N is I_{N-n} , then the minimal loss from t_n to t_N is

$$I_{N-n+1} = \min_{u(n-1)} \left[\underline{X}^T(n)\, Q_n\, \underline{X}(n) + \underline{u}^T(n-1)\, \gamma_n\, \underline{u}(n-1) + I_{N-n} \right] \tag{2.3.155}$$

This statement is obvious in that it states that if an optimum control policy from time t_{n+1} to t_N, i.e., $\{\underline{u}(n) \dots \underline{u}(N-1)\}$, has been determined, then the minimum loss for the interval $[t_n, t_N]$ is determined by the choice of $\underline{u}(n-1)$. Of all the possible choices for $\underline{u}(n-1)$, there is one that will minimize equation (2.3.153) and the value of I_{N-n+1} is thus determined.

Now consider the loss for the last step

$$I_1 = \min_{u(N-1)} \left[\underline{X}^T(N)\, Q_N\, \underline{X}(N) + \underline{u}^T(N-1)\, \gamma_N\, \underline{u}(N-1) \right] \tag{2.3.156}$$

But the state vector and control vectors of one time can be related to the next by the discrete-time form of equation (2.3.119), i.e.,

$$\underline{X}(N) = \Phi_{N-1}\, \underline{X}(N-1) + \Delta_{N-1}\, \underline{u}(N-1) \tag{2.3.157}$$

where $\phi_{N-1} = \exp(At_s) \equiv$ the state transition matrix

$$\Delta_{N-1} = \int_0^{t_s} \left[\exp A(t_s - \tau) \right] B\, d\tau$$

Thus, substituting equation(2.3.157) into equation (2.3.156), the loss for the last increment is found to be

$$I_1 = \min_{u(N-1)} \Big\{ \left[\Phi_{N-1}\, \underline{X}(N-1) + \Delta_{N-1}\, \underline{u}(N-1) \right]^T Q_N \left[\phi_{N-1}\, \underline{X}(N-1) + \Delta_{N-1}\, \underline{u}(N-1) \right] \tag{2.3.158}$$
$$+ \underline{u}^T(N-1)\, \gamma_N\, \underline{u}(N-1) \Big\}$$

Now, expanding and collecting terms yields

$$I_1 = \min_{u(N-1)} \left\{ \underline{x}^T(N-1)\, \Phi_{N-1}^T q_N \Phi_{N-1} \underline{x}(N-1) + 2\,\underline{u}^T(N-1)\, \Delta_{N-1}^T q_N \phi_{N-1}\, \underline{x}(N-1) + \underline{u}^T(N-1) \left[\Delta_{N-1}^T q_N \Delta_{N-1} + \gamma_N \right] \underline{u}(N-1) \right\} \tag{2.3.159}$$

so that the minimum of J_1, which is I_1, can be found by determining the solution to

$$\frac{\partial J_1}{\partial \underline{u}^T(N-1)} = 0$$

or

$$2\,\Delta_{N-1}^T q_N \Phi_{N-1}\, \underline{x}(N-1) + 2\left[\Delta_{N-1}^T q_N \Delta_{N-1} + \gamma_N \right] \underline{u}(N-1) = 0 \tag{2.3.160}$$

Solving for the optimum control for the last step yields

$$\underline{u}(N-1) = -\left[\Delta_{N-1}^T q_N \Delta_{N-1} + \gamma \right]^{-1} \Delta_{N-1}^T q_N \Phi_{N-1}\, \underline{x}(N-1) \tag{2.3.161}$$

or

$$\underline{u}(N-1) = a_1^T\, \underline{x}(N-1) \tag{2.3.162}$$

where a_1^T is defined from equation (2.3.166).

The optimum control for the last step is thus seen to be a linear combination of the state deviation of the $(N-1)^{th}$ step. /If the previous state deviation must be estimated, it can be shown that the optimum estimate of $\underline{x}(N-1)$ suffices for $\underline{x}(N-1)$./

Equations (2.3.161) and (2.3.167) can now be used to determine the minimum loss for the last step.

$$I_1 = \left\{ \underline{x}^T(N-1)\, \phi_{N-1}^T q_N \phi_{N-1} \underline{x}(N-1) + 2\,\underline{x}^T(N-1)\, a_1 \Delta_{N-1}^T q_N \phi_{N-1} \underline{x}(N-1) + \underline{x}^T(N-1)\, a_1 \left[\Delta_{N-1}^T q_N \Delta_{N-1} + \gamma_N \right] a_1^T\, \underline{x}(N-1) \right\}$$

$$= \underline{x}^T(N-1) \left\{ \phi_{N-1}^T q_N \phi_{N-1} + 2\, a_1 \Delta_{N-1}^T q_N \phi_{N-1} + a_1 \left[\Delta_{N-1}^T q_N \Delta_{N-1} + \gamma_N \right] a_1^T \right\} \underline{x}(N-1) \tag{2.3.163}$$

$$= \underline{x}^T(N-1)\, P_1\, \underline{x}(N-1)$$

It is thus seen that the minimum loss for the last step is a quadratic function of the state deviation at time N-1 (the beginning of the last step).

The optimum control for the last two steps can be found by employing equation (2.3.150).

$$I_2 = \min_{\underline{u}(N-2)} \left\{ \underline{X}^T(N-1)\, Q_{N-1}\, \underline{X}(N-1) + \underline{u}^T(N-2)\, \gamma_{N-1}\, \underline{u}(N-2) + I_1 \right\} \quad (2.3.164)$$

but

$$I_1 = \underline{X}^T(N-1)\, P_1(N-1)$$

so

$$I_2 = \min_{u(N-2)} \left\{ \underline{X}^T(N-1)\, Q_{N-1}\, \underline{X}(N-1) + \underline{u}^T(N-2)\, \gamma_{N-1}\, \underline{u}(N-2) + \underline{X}^T(N-1)\, P_1\, \underline{X}(N-1) \right\} \quad (2.3.165)$$

or

$$I_2 = \min_{u(N-2)} \left\{ \underline{X}^T(N-1) \left[Q_{N-1} + P_1 \right] \underline{X}(N-1) + \underline{u}^T(N-2)\, \gamma_{N-1}\, \underline{u}(N-2) \right\} \quad (2.3.166)$$

But

$$\underline{X}(N-1) = \Phi_{N-2}\, \underline{X}(N-2) + \Delta_{N-2}\, \underline{u}(N-2) \quad (2.3.167)$$

Therefore, as before, the optimal $\underline{u}$(N-2) can be from the solution of

$$\frac{\partial J_2}{\partial \underline{u}^T(N-2)} = 0$$

$$\Delta^T_{N-2} \left[Q_{N-1} + P_1 \right] \Phi_{N-2}\, \underline{X}(N-2) + \left[\Delta^T_{N-2} (Q_{N-1} + P_1)\, \Delta_{N-2} + \gamma_{N-1} \right] \underline{u}(N-2) = 0 \quad (2.3.168)$$

Finally,

$$\underline{u}(N-2) = -\left[\Delta^T_{N-2} (Q_{N-1} + P_1)\, \Delta_{N-2} + \gamma_{N-1} \right]^{-1} \Delta^T_{N-2} (Q_{N-1} + P_1)\, \Phi_{N-2}\, \underline{X}(N-2)$$

or

$$\underline{u}(N-2) = a_2^T\, \underline{X}(N-2) \quad (2.3.169)$$

and the minimum loss can be found from equation (2.3.166) as

$$I_2 = \underline{X}^T(N-1)\left[Q_{N-1}+P_1\right]\underline{X}(N-1) + \underline{X}^T(N-2)\,a_2\,\gamma_{N-1}\,a_2^T\,\underline{X}(N-2) \quad (2.3.170)$$

Now, recalling that

$$\begin{aligned}\underline{X}(N-1) &= \phi_{N-2}\underline{X}(N-2) + \Delta_{N-2}\underline{u}(N-2)\\ &= \phi_{N-2}\underline{X}(N-2) + \Delta_{N-2}\,a_2^T\,\underline{X}(N-2) \end{aligned} \quad (2.3.171)$$

reduces equation (2.3.170) to

$$\begin{aligned} I_2 &= \left[\underline{X}^T(N-2)\,\Phi_{N-2}^T + \underline{X}^T(N-2)a_2\,\Delta_{N-2}^T\right]\left[Q_{N-1}+P_1\right]\left[\phi_{N-2}\,\underline{X}(N-2) + \Delta_{N-2}a_2^T\underline{X}(N-2)\right]\\ &\quad + \underline{X}^T(N-2)\,a_2\,\gamma_{N-1}\,a_2^T\,\underline{X}(N-2)\\ &= \underline{X}^T(N-2)\left\{\Phi_{N-2}^T\left[Q_{N-1}+P_1\right]\Phi_{N-2} - a_2F_1a_2^T\right\}\underline{X}(N-2) \end{aligned} \quad (2.3.172)$$

where

$$F_1 = \Delta_{N-2}^T\,(Q_{N-1}+P_1)\,\Delta_{N-2} + \gamma_{N-1} \quad (2.3.173)$$

Thus, writing equation (2.3.172) in the form of equation (2.3.163)

$$I_2 = \underline{X}^T(N-2)\,P_2\,\underline{X}(N-2) \quad (2.3.174)$$

where

$$P_2 = \phi_{N-2}^T\,(Q_{N-1}+P_1)\,\Phi_{N-2} - a_2F_1a_2^T \quad (2.3.175)$$

This procedure can be extended to the n^{th} step as follows:

$$\underline{u}(n) = a_{N-n}^T\,\underline{X}(n) \quad (2.3.176a)$$

$$a^T_{N-n+1} = -F^{-1}_{N-n}\,\Delta^T_{n-1}\left[Q_n + P_{N-n}\right]\phi_{n-1} \tag{2.3.176b}$$

$$F_{N-n} = \left[\Delta^T_{n-1}(Q_n + P_{N-n})\Delta_{n-1} + \gamma_n\right] \tag{2.3.176c}$$

$$P_{N-n+1} = \phi^T_{n-1}\left[Q_n + P_{N-n}\right]\phi_{n-1} - a_{N-n+1}F_{N-n}a^T_{N-n+1} \tag{2.3.176d}$$

$$P_0 = 0 \tag{2.3.176e}$$

The use of these equations should be apparent from the previous discussion. The feedback matrix, a^T_{N-n}, can be found by starting with n=N and iterating back to n=0.

The sequence of steps is shown below:

1. Compute F_o

$$F_o = \Delta^T_{N-1}(Q_N + \cancel{P_o}^{\,0})\Delta_{N-1} + \gamma_N$$

2. Compute a^T_1

$$a^T_1 = -F_o^{-1}\Delta^T_{N-1}(Q_N + \cancel{P_o}^{\,0})\phi_{N-1}$$

$$= -\left[\Delta^T_{N-1}Q_N\Delta_{N-1} + \gamma_N\right]^{-1}\Delta^T_{N-1}Q_N\phi_{N-1}$$

3. Control for time t_{N-1} to t_N is

$$\underline{u}(N-1) = a^T_1\,\underline{X}(N-1)$$

4. Compute P_1

$$P_1 = \phi^T_{N-1}\left[Q + \cancel{P_o}^{\,0}\right]\phi_{N-1} - a_1 F_o a^T_1$$

5. Compute F_1

$$F_1 = \Delta^T_{N-2}(Q_{N-1} + P_1)\Delta_{N-2} + \gamma_{N-1}$$

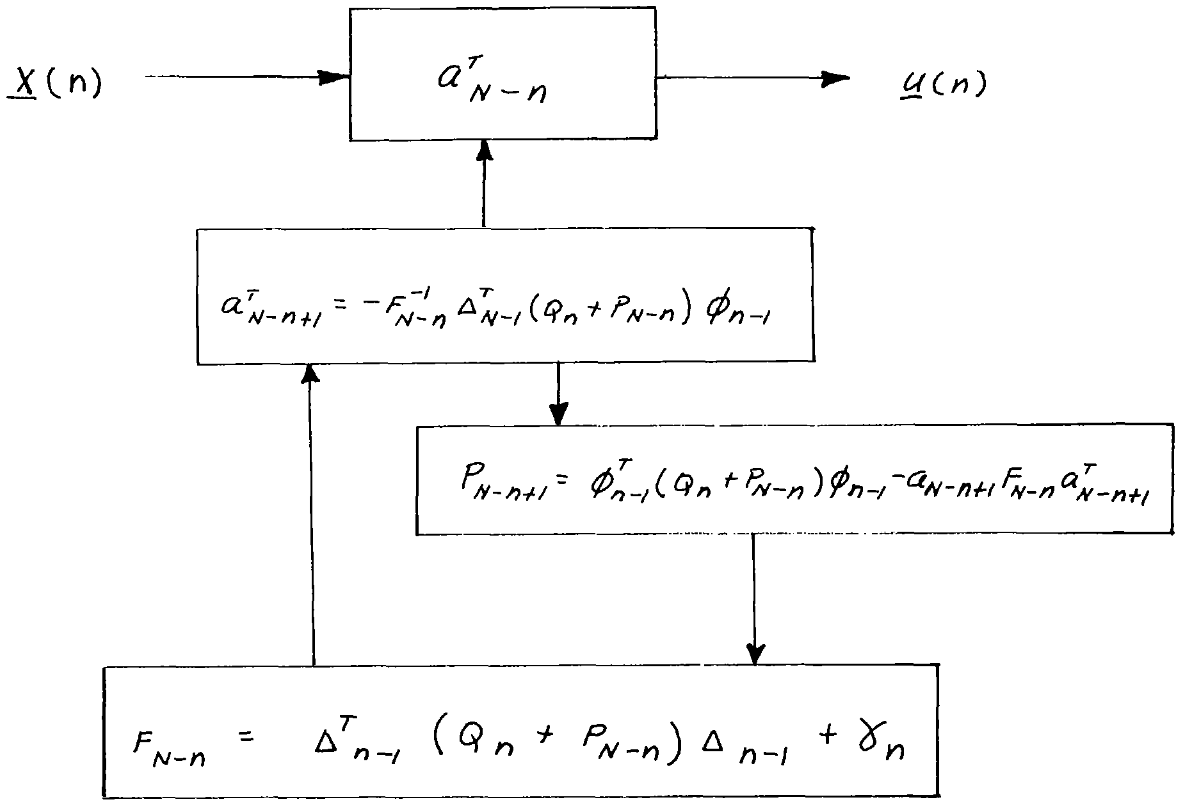
6. Compute a_2^T

$$a_2^T = -F_1^{-1}\Delta_{N-2}^T(Q_{N-1}+P_1)\phi_{N-2}$$

7. Control is

$$\underline{u}(N-2) = a_2^T\,\underline{X}(N-2)$$

The general flow chart for this process is



The actual mechanization of this optimal guidance control scheme would consist of the storage of all of the elements for the "a" matrices from time t_1 to t_N. The control deviation can then be established by a multiplication of the state deviation for the particular time of interest by the current feedback matrix, a_{N-n}.

An example of the application of optimum linear control follows: Let $\delta\underline{r}$ and $\delta\dot{\underline{r}}$ be the position and velocity perturbations from some nominal trajectory. The deviations at time t, can be related to those at time t_o as follows:

$$\begin{bmatrix} \delta \underline{r}_1 \\ \delta \dot{\underline{r}}_1 \end{bmatrix} = \left[\begin{array}{c|c} \phi_1 & \phi_2 \\ \hline \phi_3 & \phi_4 \end{array}\right] \begin{bmatrix} \delta \underline{r}_o \\ \delta \dot{\underline{r}}_o \end{bmatrix} + \left[\begin{array}{c} \phi_2 \\ \hline \phi_4 \end{array}\right] \delta \underline{V}_o \tag{2.3.177}$$

where $\delta \underline{V}_o$ is some velocity correction that can be made at time t_o. This equation can be thought of as the discrete-time form of the general solution to the linear system

$$\underline{X}(1) = \phi_o \underline{X}(0) + \Delta_o \underline{U}(0) \tag{2.3.178}$$

A loss function for this process can be defined as the square of the position deviation at time t_1, i.e.,

$$J_1 = \delta \underline{r}_1 \cdot \delta \underline{r}_1 = \begin{bmatrix} \delta \underline{r}_1 & \delta \dot{\underline{r}}_1 \end{bmatrix} \left[\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right] \begin{bmatrix} \delta \underline{r}_1 \\ \delta \dot{\underline{r}}_1 \end{bmatrix} \tag{2.3.179}$$

Under this particular choice of a loss function, no penalty is placed on the fuel needed in order to perform the control correction. In "real life" problems, this choice of a loss function would be less than ideal since the loss function in effect forces the position deviation to be minimum at any fuel cost. Ideally, it is desirable to use a loss function that not only weighs the position deviation, but also the control deviation, since the nominal trajectory is optimal.

Using the standard notation, the following definitions can be made from equations (2.3.177) and (2.3.178).

$$\underline{X} = \left[\begin{array}{c} \delta \underline{r}_1 \\ \hline \delta \dot{\underline{r}}_1 \end{array}\right] \qquad \underline{U} = \delta \underline{V}_o \qquad \Delta_o = \left[\begin{array}{c} \phi_2 \\ \hline \phi_4 \end{array}\right]$$

$$\phi_o = \left[\begin{array}{c|c} \phi_1 & \phi_2 \\ \hline \phi_3 & \phi_4 \end{array}\right]$$

The loss function, equation (2.3.179), may also be written as

$$J_2 = \underline{X}_1^T Q_1 \underline{X}_1 \tag{2.3.180}$$

so

$$Q = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \qquad \gamma_1 = 0 \qquad P_0 = 0$$

Thus, the previous results of this section, in conjunction with consideration of the problem of one interval, define the control as follows:

$$\underline{u}(0) = a_1^T \underline{X}(0) \tag{2.3.181a}$$

$$a_1^T = -F_0^{-1} \Delta_0^T Q_1 \phi_0 \tag{2.3.181b}$$

$$F_0 = \Delta_0^T Q_1 \Delta_0 = \left[\phi_2^T \cdot \phi_4^T\right] \left[\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right] \begin{bmatrix} \phi_2 \\ \phi_4 \end{bmatrix} = \phi_2^T \phi_2 \tag{2.3.181c}$$

$$\begin{aligned} a_1^T &= -\left[\phi_2^T \phi_2\right]^{-1} \left[\phi_2^T \,\middle|\, \phi_4^T\right] \left[\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c|c} \phi_1 & \phi_2 \\ \hline \phi_3 & \phi_4 \end{array}\right] \\ &= -\left[\phi_2^T \phi_2\right]^{-1} \left[\phi_2^T \,\middle|\, \phi_4\right] \left[\begin{array}{c|c} \phi_1 & \phi_2 \\ \hline 0 & 0 \end{array}\right] \\ &= -\phi_2^{-1} \left[\phi_1 \,\middle|\, \phi_2\right] \end{aligned} \tag{2.3.182b}$$

The optimum linear control is therefore

$$\underline{u}(0) = a_1^T \begin{pmatrix} \delta \underline{r}_0 \\ \delta \underline{\dot{r}}_0 \end{pmatrix} \tag{2.3.183}$$

or

$$\delta \underline{V}_0 = -\phi_2^{-1} \left[\phi_1 \,\middle|\, \phi_2\right] \begin{bmatrix} \delta \underline{r}_0 \\ \delta \underline{\dot{r}}_0 \end{bmatrix} \tag{2.3.184}$$

$$\delta \underline{V}_0 = -\phi_2^{-1} \phi_1 \, \delta \underline{r}_0 - \delta \underline{\dot{r}}_0 \tag{2.3.185}$$

A simple verification of this answer can be made by determining the position deviation, $\delta \underline{r}_1$. Substitution into equation (2.3.175) yields

$$
\begin{aligned}
\delta \underline{r}_1 &= \phi_1 \, \delta \underline{r}_0 + \phi_2 \, \delta \underline{\dot{r}}_0 + \phi_2 \, \delta V_0 \\
&= \phi_1 \, \delta \underline{r}_0 + \phi_2 \, \delta \dot{r}_0 + \phi_2 \left[-\phi_2^{-1} \phi_1 \, \delta \underline{r}_0 - \delta \underline{\dot{r}}_0 \right] \qquad (2.3.186) \\
&= 0
\end{aligned}
$$

It should be noted that the results to this over-simplified problem are identical to the "Fixed Time of Arrival" scheme that is presented in Reference 3.25. The reason for the similarity is that only one control increment was analyzed and that the only performance criteria for the mission was the position deviation at the end of the increment. The answer to the previous problem is not so obvious when a more complicated loss function is considered or when a "loss" is associated with the control deviation.

3.0 RECOMMENDED PROCEDURES

The material presented in the previous discussions is applicable to a series of boost vehicle guidance problems. However, since the guidance equations are but a small part of the total guidance loop and since the selection of the guidance equations should be based upon consideration of the required performance from the total loop, the reliability and cost of the system, the mission success criteria, etc., any attempt at this point to select a specific set of guidance equations for general application will fail. This observation is strengthened when it is realized that the primary differences in the results of these applications as observed in terms of the mission is in the flexibility derived. Thus, since no set of guidance equations appears to have a clearly defined advantage for all applications prior to consideration of the total guidance loop, a definitive decision will be deferred until the monograph on guidance system synthesis is prepared. However, general preferences for the guidance of boost vehicles of the present and near future can be presented.

The simplest of the schemes (perturbation guidance) is applicable in varying degrees of sophistication to the boost of ballistic missiles, small scientific payloads, and manned satellites (where the trajectories are well defined before launch and where retargeting has been ruled out as a possible event). In contrast, while the more complex mechanizations of the explicit and adaptive guidance approaches do not preclude application to the same missions, the cost of the system, resulting from more elaborate instrumentation and a larger guidance computer, will generally preclude application to those missions which do not require flexibility. Thus, the adaptive mode of guidance will probably be restricted, for the time being, to the more demanding missions such as manned lunar and planetary escape trajectories, and extraterrestrial soft landing. This application is justified for these mission phases due to the complex manner that the mission phases are tied together and due to the variety of abort and mission redefinition possibilities which exist (to derive usefulness from a partial success).

Finally, in the case of the adaptive type of guidance, the two forms would appear to have the same type of computational requirements and appear to impose the same order of mechanizational complexity. However, since the iterative path adaptive guidance mode of section 2.1 requires less targeting before the vehicle is launched, this approach is preferred for the more sophisticated applications over the guidance polynomial approach of section 2.2.

Mechanizations of the material presented in both sections 2.1 and 2.3 are illustrated in the respective sections. Thus, reference is made to these sections for such information. These discussions show the unique capabilities of the guidance equations and illustrate the degree of optimization attained in the process.

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APPENDIX A

Formulation of the Variable Mass Vehicle Problem

This appendix will present a straightforward formulation of the variable mass vehicle and then show why a different selection of the state variables is desirable in order to avoid cross coupling the state variables in the solution. A conventional formulation would assign the state variables as follows:

$$\dot{x}_1 = x_4 \qquad V_e = \text{EXHAUST VELOCITY}$$

$$\dot{x}_2 = x_5 \qquad u_1^2 + u_2^2 + u_3^2 = 1$$

$$\dot{x}_3 = x_6$$

$$\dot{x}_4 = -V_e \frac{\dot{m}(t)}{m(t)} u_1$$

$$\dot{x}_5 = -V_e \frac{\dot{m}(t)}{m(t)} u_2 \tag{A.1}$$

$$\dot{x}_6 = -V_e \cdot \frac{\dot{m}(t)}{m(t)} u_3$$

where $m(t)$ = vehicle mass

x_1, x_2, x_3 = position of vehicle with respect to target

x_4, x_5, x_6 = velocity of vehicle with respect to target

In addition, the following state and control definitions are made

$$m(t) = x_7$$
$$\dot{m}(t) = u_4 \tag{A.2}$$

So the state equations become

$$\dot{x}_1 = x_4$$
$$\dot{x}_2 = x_5 \tag{A.3}$$
$$\dot{x}_3 = x_6$$
$$\dot{x}_4 = V_e \frac{u_4}{x_7} u_1$$

$$\dot{x}_5 = V_e \frac{u_4}{x_7} u_2$$

$$\dot{x}_6 = V_e \frac{u_4}{u_7} u_3$$

$$\dot{x}_7 = - u_4$$

where the admissible control is

$$0 \leq u_4(t) \leq D$$

This formulation seems reasonable since the controls u_1, u_2, u_3 determine the direction of thrust and the control u_4 determines the magnitude. However, it will be shown to have an over complicated solution once the Maximum Principle is applied. This fact is seen in the following analysis.

It is required to determine the control vector components u_1, u_2, u_3 and u_4 such that the relative position and velocity between the vehicle and some target are reduced to zero while $x_7\,(t = T)$ is a maximum. Applying Pontryagin's Maximum Principle, it is desired to maximize $S_T = \underline{c} \cdot \underline{X}_7$ where $c_7 = 1$ and the remaining components of $\underline{c}$ are to be determined. For the system of equations, (A.3), the function $H(\underline{x}, \underline{p}, \underline{u}, t)$ becomes

$$H(\underline{x}, \underline{p}, \underline{u}, t) = p_1 x_4 + p_2 x_5 + p_3 x_6 + V_e \frac{u_4}{x_7}\left[u_1 p_4 + u_2 p_5 + u_3 p_6 - \frac{x_7}{V_e} p_1\right] \qquad \text{(A.4)}$$

The system $\underline{P}$, thus becomes

$$\begin{aligned}
\dot{p}_1 &= \dot{p}_2 = \dot{p}_3 = 0 \\
\dot{p}_4 &= -p_1 \\
\dot{p}_5 &= -p_2 \\
\dot{p}_6 &= -p_3 \\
\dot{p}_7 &= V_e \frac{u_4}{x_1^2}\left[u_1 p_4 + u_2 p_5 + u_3 p_6\right]
\end{aligned} \qquad \text{(A.5)}$$

Now, according to Pontryagin's Maximum Principle the optimum control vector is determined by minimizing $H(\underline{x}, \underline{p}, \underline{u}, t)$. However, equation (A.4) and (A.5) are somewhat "over" complicated since the determination of the optimum control vector components is dependent on x_7 and p_7, and p_7, in turn, is dependent on the control vector components and x_7. This "over" complication of equations (A.4 and A.5) resulted from the definition of x_7 as the mass of the vehicle. It is seen that the process of state variables for the system which involve products of system states and control vector components should be avoided wherever possible. Fortunately, it is possible to do so in the present problem if an alternate control vector component u_5 is defined such that

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$$\dot{x}_5 = V_e \frac{u_4}{x_7} u_2$$

$$\dot{x}_6 = V_e \frac{u_4}{u_7} u_3$$

$$\dot{x}_7 = - u_4$$

where the admissible control is

$$0 \leq u_4(t) \leq D$$

This formulation seems reasonable since the controls u_1, u_2, u_3 determine the direction of thrust and the control u_4 determines the magnitude. However, it will be shown to have an over complicated solution once the Maximum Principle is applied. This fact is seen in the following analysis.

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$$H(\underline{x}, \underline{p}, \underline{u}, t) = p_1 x_4 + p_2 x_5 + p_3 x_6 + V_e \frac{u_4}{x_7}\left[u_1 p_4 + u_2 p_5 + u_3 p_6 - \frac{x_7}{V_e} p_1\right] \qquad \text{(A.4)}$$

The system $\underline{P}$, thus becomes

$$\begin{aligned}
\dot{p}_1 &= \dot{p}_2 = \dot{p}_3 = 0 \\
\dot{p}_4 &= -p_1 \\
\dot{p}_5 &= -p_2 \\
\dot{p}_6 &= -p_3 \\
\dot{p}_7 &= V_e \frac{u_4}{x_7^2}\left[u_1 p_4 + u_2 p_5 + u_3 p_6\right]
\end{aligned} \qquad \text{(A.5)}$$

Now, according to Pontryagin's Maximum Principle the optimum control vector is determined by minimizing $H(\underline{x}, \underline{p}, \underline{u}, t)$. However, equation (A.4) and (A.5) are somewhat "over" complicated since the determination of the optimum control vector components is dependent on x_7 and p_7, and p_7, in turn, is dependent on the control vector components and x_7. This "over" complication of equations (A.4 and A.5) resulted from the definition of x_7 as the mass of the vehicle. It is seen that the process of state variables for the system which involve products of system states and control vector components should be avoided wherever possible. Fortunately, it is possible to do so in the present problem if an alternate control vector component u_5 is defined such that

$$
\begin{aligned}
\dot{x}_1 &= x_4 \\
\dot{x}_2 &= x_5 \\
\dot{x}_3 &= x_6 \\
\dot{x}_4 &= V_e u_5 u_1 \\
\dot{x}_5 &= V_e u_5 u_2 \\
\dot{x}_6 &= V_e u_5 u_3 \\
\dot{x}_7 &= V_e u_5
\end{aligned}
\qquad (A.6)
$$

The control vector component u_5 is introduced such that x_7 in equation (A.3) is essentially absorbed. It is important to note that x_7 in equation (A.6) is no longer the mass of the vehicle since u_5 is no longer the mass flow rate. Actually, u_5 is defined as

$$u_5(t) = -\frac{\dot{m}(t)}{m(t)} \qquad (A.7)$$

Thus,

$$x_7(\tau) = x_7(0) + \int_0^{\tau} \dot{x}_7\,dt \quad = x_7(0) - V_e \int_0^{\tau} \frac{dm}{m}$$

$$x_7(\tau) = x_7(0) - V_e \log_e m(t)\Big|_0^{\tau} = V_e \log_e \left[\frac{m(0)}{m(\tau)}\right] \qquad (A.8)$$

where $x(0)$ is arbitrarily selected as zero without loss of generality. Thus, for minimum mass expenditure it is desired to minimize $x_7(T)$ or, equivalently, it is desired to minimize $S_T = 1 \cdot x_7(T)$.

In terms of u_5 , the function $H(\underline{x},\underline{p},\underline{u},t)$ and the $\underline{P}$ system become

$$H(\underline{x},\underline{P},\underline{U},t) = p_1 x_4 + p_2 x_5 + p_3 x_6 + V_e u_5 \left[u_1 p_4 + u_2 p_5 + u_3 p_6 + p_7\right] \qquad (A.9)$$

$$
\begin{aligned}
\dot{p}_1 &= \dot{p}_2 = \dot{p}_3 = 0 \\
\dot{p}_4 &= -p_1 \\
\dot{p}_5 &= -p_2 \\
\dot{p}_6 &= -p_3 \\
\dot{p}_7 &= 0
\end{aligned}
\qquad (A.10)
$$

Now, since $\dot{p}_7 = 0$, $p_7(t) = p_7(T) = -1$, therefore,

$$H(\underline{x}, \underline{p}, \underline{u}, t) = p_1 x_4 + p_2 x_5 + p_3 x_6 + V_e u_5 \left[u_1 p_4 + u_2 p_5 + u_3 p_6 - 1 \right] \quad \text{(A.11)}$$

A comparison of these equations with equations (A.4) and (A.5) shows the advantage gained in the use of u_5 as a control vector component and the alternate definition of x_7.

The previous formulation was part of a more general study on the application of Pontryagin's Maximum Principle to optimum control of a variable mass space vehicle performed at NAA by D. R. Grier.

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