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GUIDANCE, FLIGHT MECHANICS AND TRAJECTORY OPTIMIZATION

Volume VII - The Pontryagin Maximum Principle

by J. E. McIntyre

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By J. E. McIntyre

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FOREWORD

This report was prepared under contract NAS 8-11495 and is one of a series intended to illustrate analytical methods used in the fields of Guidance, Flight Mechanics, and Trajectory Optimization. Derivations, mechanizations and recommended procedures are given. Below is a complete list of the reports in the series.

Volume I	Coordinate Systems and Time Measure
Volume II	Observation Theory and Sensors
Volume III	The Two Body Problem
Volume IV	The Calculus of Variations and Modern Applications
Volume V	State Determination and/or Estimation
Volume VI	The N-Body Problem and Special Perturbation Techniques
Volume VII	The Pontryagin Maximum Principle
Volume VIII	Boost Guidance Equations
Volume IX	General Perturbations Theory
Volume X	Dynamic Programming
Volume XI	Guidance Equations for Orbital Operations
Volume XII	Relative Motion, Guidance Equations for Terminal Rendezvous
Volume XIII	Numerical Optimization Methods
Volume XIV	Entry Guidance Equations
Volume XV	Application of Optimization Techniques
Volume XVI	Mission Constraints and Trajectory Interfaces
Volume XVII	Guidance System Performance Analysis

The work was conducted under the direction of C. D. Baker, J. W. Winch, and D. P. Chandler, Aero-Astro Dynamics Laboratory, George C. Marshall Space Flight Center. The North American program was conducted under the direction of H. A. McCarty and G. E. Townsend.

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1.0 STATEMENT OF THE PROBLEM

The purpose of an engineering analysis usually centers on the development of a system which performs in some best or optimum fashion. However, just what constitutes an optimum system is difficult to define, even for the simplest systems, since factors such as realiability, system complexity, and time and cost of development cannot generally be incorporated in a mathematical model of the system. Notwithstanding, at some point in the analysis, it is often beneficial to specify a single mathematical criterion, or series of criteria, with respect to which the system is to be optimized.

Frequently the system which optimizes these performance criteria is adjudged to be impractical or unsatisfactory from an engineering point of view since the criteria themselves, by virtue of their mathematical representation, may not contain all the engineering considerations and constraints which enter into the problem. Even in these cases, however, the device of specifying a system performance criterion serves two useful purposes. First, it provides a basis for systematically selecting values for the parameters which govern the system; and second, it gives one a yardstick for measuring how well the system could perform if all other engineering constraints were absent. The cost or penalty in system performance which these additional constraints impose is simply the difference in the value of the criterion function between the optimal and actual systems.

The simplest and most direct approach to determine the optimal system is one of trial and error. In this approach, several sets of values for the parameters of the problem are selected and each set is evaluated as to system performance on a digital computer. However, such an approach has several drawbacks since any set of parameters selected as optimal is only optimal with respect to all the other sets that have been tested (i.e., all the trials that have been made). Thus, since, in most problems, the parameters may assume an infinite number of different values, the determination of a true or absolute optimum would entail an infinite number of runs on the computer.

A second approach involves the use of standard mathematical techniques to determine the optimum system. Foremost among these techniques is the maxima-minima theory of the Differential Calculus by which a point or set of points can be determined at which a function takes on an extremum value. This technique has been used, and to good advantage, in engineering studies since time immemorial. However, in the early 1950's there arose a series of problems, particularly in the areas of trajectory analysis and control theory, which could not be handled within the maxima-minima framework. One such problem involves the maneuvering of a chemical rocket between two points in space. In this problem there may be an infinite number of paths or trajectories along which the vehicle could fly to accomplish its mission, due to the fact that the thrusting engine can be steered and throttled. Thus, the problem is to determine some sort of best or optimum path, and the corresponding optimum time history of the thrust vector. The mathematical technique for handling such problems is the Calculus of Variations.

The Variational Calculus has interested and motivated mathematicians for well over two hundred years, with the theory itself reaching a near level of completion during the 1930's at the hands of G. A. Bliss and others at the University of Chicago. This material was reviewed in one of the preceding monographs of this series (The Calculus of Variations and Modern Applications - SID 65-1200-4). After a relatively dormant period of approximately twenty years, the trajectory and control problems encountered in the performance of high speed aircraft and missiles provided a resurgence of interest in this area - a resurgence which brought about several new developments and theories. One of these theories, the Maximum Principle, is the subject of this monograph.

The Maximum Principle was developed by L. S. Pontryagin and his colleagues at the Steklov Mathematical Institute in Moscow. It is essentially an extension or generalization of the well known Weierstrass condition of the Calculus of Variations. In this regard, it leads to the same result as the Weierstrass condition for just about all engineering problems. Its primary advantages are that it simplifies the development of proofs, gives additional insight into the computational process of constructing a solution, and allows for the easy inclusion of certain types of inequality constraints frequently encountered in engineering problems.

The feature which distinguishes the modern from the classical variational problem is the occurrence of inequality constraints. The Maximum Principle was developed in order to handle a certain general type of inequality that occurs regularly in optimal trajectory and control problems. Unknown to Pontryagin and his associates were extensions of the classical theory which took place in this country and which allowed for the inclusion of inequality constraints using the Calculus of Variations. While these extensions were not as concise or general as the Maximum Principle in regard to the treatment of the most commonly encountered inequality, a control inequality, they were, however, directly applicable to other types of inequalities such as a state inequality. One of the shortcomings of the Maximum Principle is that it can not be applied without a major revision when these other inequalities are present.

This monograph is intended to provide the analytic framework in which the subtle distinction between this material and the more classic approaches can be appreciated. The Maximum Principle will then be used to formulate and solve a variety of problems encountered in optimal trajectory and control analysis. The treatment will include both linear and nonlinear systems, and problems with control inequalities and state inequalities. Several comparisons and parallels will be drawn with both the Calculus of Variations and the maxima-minima theory of the Differential Calculus.

2.0 STATE OF THE ART

2.1 Maxima-Minima Theory

The discussions will begin with a review of the pertinent concepts from maxima-minima theory since the methods used here have much in common with those used in optimal trajectory and control problems.

2.1.1 Minimizing a Function of One Variable

It is well known that if a function f (x) has a minimum value at $x = x_o$, then the two conditions

$$\frac{df}{dx}(x_o) = 0 \qquad (2.1.1)$$

$$\frac{d^2f}{dx^2}(x_o) \geq 0$$

must hold. To develop these conditions, assume that the function f has continuous derivatives of at least the second order and a bounded third derivative in the vicinity of the minimum point x_o, and expands f (x) about x_o in the truncated series

$$f(x) - f(x_o) = f_x \, \delta x + \frac{f_{xx} \, \delta x^2}{2} + \theta(\delta x^3) \qquad (2.1.2)$$

where $\delta x = x - x_o$ and $\theta(\delta x^3)$ denotes terms of order δx^3. If f has a local minimum at x_o, then, for any x "near" x_o, the right hand side of (2.1.2) must be greater than or equal to zero; that is,

$$f(x) - f(x_o) \geq 0 \qquad (2.1.3)$$

For δx small, the sign of the left hand side of Eq. (2.1.2) is determined by the sign of the first term with the result that

$$f_x \, \delta x \geq 0 \qquad (2.1.4)$$

Since δx can take positive or negative values, it follows that (2.1.4) will hold only if

$$f_x(X_o) = 0$$

With this condition, the sign of the left hand side of (2.1.2) is determined from the second order terms, and for (2.1.3) to be satisfied

$$f_{xx} \frac{\delta x^2}{2} \geq 0$$

from which the second condition in Eq. (2.1.1) follows.

Note that it has been necessary to assume that f has continuous second and bounded third derivatives in some region containing X_o in order to develop the series expression of (2.1.2). This condition can sometimes be relaxed. However, it is a relatively weak assumption and is inevitably satisfied in engineering applications.

2.1.2 Minimizing a Function of Several Variables

The extension of the previous results to the case in which X is a vector with n components is straightforward. Let X denote the vector

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

with the function $f(x) = f(x_1 x_2, \ldots, x_n)$ having a local minimum at x_o where

$$X_o = \begin{pmatrix} x_{10} \\ x_{20} \\ \vdots \\ x_{no} \end{pmatrix}$$

Again, assuming that f possesses continuous second and bounded third derivatives in all its arguments, and expanding f(x) about x_o provides

$$^* f(x) - f(x_o) = f_{x_i}\, \delta x_i + \frac{1}{2} f_{x_i x_j}\, \delta x_i\, \delta x_j + \Theta(\delta x^3) \geq 0 \qquad (2.1.5)$$

or in the vector notation

$$f(x) - f(x_o) = f_x\, \delta x + \frac{1}{2}\, \delta x^T f_{xx}\, \delta x + \Theta(\delta x^3) \geq 0 \qquad (2.1.5a)$$

where δx is the vector

---

* The tensor notation is used in which repeated subscripts indicate summation.

$$\delta x = \begin{pmatrix} x_1 - x_{1_0} \\ x_2 - x_{2_0} \\ x_3 - x_{3_0} \end{pmatrix}$$

and the superscript ^{T} denotes the transpose.

Reasoning as in the preceding section, it follows that

$$f_x \, \delta x = f_{x_i} \, \delta \, x_i \geq 0$$

and since the δx_i are arbitrary and independent, the above result becomes

$$f_x = \begin{pmatrix} f_{x_1} \\ f_{x_2} \\ \vdots \\ f_{x_n} \end{pmatrix} = 0 \qquad (2.1.6)$$

This leaves, on the left hand side of (2.1.5a), the second order terms which must be positive if x_o is to be a minimum point. Hence,

$$\frac{1}{2} \, \delta x^T f_{xx} \, \delta x = \frac{1}{2} f_{x_i x_j} \, \delta x_i \, \delta x_j \geq 0 \qquad (2.1.7)$$

The expression $\delta x^T f_{xx} \, \delta x$ is quadratic in the variables δx_i and is therefore called a quadratic form. Since the δx_i are arbitrary, condition (2.1.7) will be satisfied only if the matrix

$$(f_{xx}) = \begin{pmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_n} \\ f_{x_2x_1} & f_{x_2x_2} & \cdots & f_{x_2x_n} \\ \vdots & & & \\ f_{x_nx_1} & f_{x_nx_2} & \cdots & f_{x_nx_n} \end{pmatrix}$$

is positive semi-definite; that is, if all the eigenvalues of (f_{xx}) are positive or zero. Hence, to determine if condition (2.1.7) is satisfied, it is necessary to either directly compute the eigenvalues of (f_{xx}) or draw upon any one of several theorems used in the study of quadratic forms which assure that (f_{xx}) is positive semi-definite. One of these theorems (for example, see Ref. (1), page 260) requires that for (f_{xx}) to be positive semi-definite, the n determinant conditions

$$
\begin{aligned}
& f_{x_1x_1} \geq 0 \\
& \begin{vmatrix} f_{x_1x_1} & f_{x_1x_2} \\ f_{x_1x_2} & f_{x_2x_2} \end{vmatrix} \geq 0 \\
& \begin{vmatrix} f_{x_1x_1} & f_{x_1x_2} & f_{x_1x_3} \\ f_{x_1x_2} & f_{x_2x_2} & f_{x_2x_3} \\ f_{x_3x_1} & f_{x_3x_2} & f_{x_3x_3} \end{vmatrix} \geq 0 \\
& \quad \vdots \\
& \begin{vmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_n} \\ f_{x_1x_2} & f_{x_2x_2} & & f_{x_2x_n} \\ \vdots & & & \\ f_{x_nx_1} & f_{x_nx_2} & \cdots & f_{x_nx_n} \end{vmatrix} \geq 0
\end{aligned}
\qquad (2.1.8)
$$

must hold. Note that the conditions (2.1.8) and (2.1.6) above which must be satisfied at the minimum point x_o are the n dimensional analogue of conditions (2.1.1) in the one dimensional problem.

2.1.3 Minimizing a Function Subject to Equality Constraints

Quite often, problems arise in which a function, $f(x)$, is to be minimized, subject to a subsidiary condition $g(x) = 0$. To illustrate the treatment of such problems, it will be assumed to begin with that x is a two dimensional vector

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and that the minimum point occurs at x_o where

$$x_o = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}$$

One possible approach to the problem would be to solve the constraint equation

$$g(x_1, x_2) = 0$$

for one of the variables, say x_2 in terms of x_1, and then substitute this expression into f. The minimum point could then be determined using the techniques of Section 2.1.1. While this approach is straightforward, it has the major disadvantage that it is not always possible to solve the constraint equation for one of the variables in terms of the other. An alternate approach, and the one to be considered here, was developed by Lagrange in the early part of the 18th century. By this method, the constrained minimization problem can be handled in much the same manner as the unconstrained problem.

Since $g(x_1,x_2)$ is zero, minimizing the function $f(x_1,x_2)$ is equivalent to minimizing the augmented function

$$F(x_1, x_2) = f(x_1,x_2) + \lambda g(x_1,x_2) = f(x_1,x_2) \quad (2.1.9)$$

where λ is some constant to be determined. Expanding F in a truncated series about x_0 provides

$$F(x) - F(x_0) = F_{x_1}\delta x_1 + F_{x_2}\delta x_2 + \frac{1}{2}\left\{F_{x_1x_1}\delta x_1^2 + 2F_{x_1x_2}\delta x_1 \delta x_2 + F_{x_2x_2}\delta x_2^2\right\} + \theta(\delta x^3) \quad (2.1.10)$$

Now, the left hand side of (2.1.10) must be greater or equal to zero for all x and x_0, satisfying

$$g(x) = g(x_0) = 0$$

Consider x_2 (and hence δx_2) as the independent variable and x_1 as the dependent variable (x_1 is dependent through the constraint equation $g(x_1x_2) = 0$). But since λ in Equation (2.1.9) can be any constant, it will be chosen so that $F_{x_1} = 0$ at $x = x_0$; that is,

$$F_{x_1} = f_{x_1} + \lambda g_{x_1} = 0 \quad .$$

With this condition, Eq. (2.1.10) becomes

$$F(x) - F(x_0) = F_{x_2}\delta x_2 + \frac{1}{2}\left\{F_{x_1x_1}\delta x_1^2 + 2F_{x_1x_2}\delta x_1\delta x_2 + F_{x_2x_2}\delta x_2^2\right\} + \theta(\delta x^3) \quad (2.1.11)$$

Now since δx_2 is arbitrary (i.e., it has been chosen as the independent variable), it can be taken small enough so that the sign of the right hand side of (2.1.11) is determined by the first term. Hence, it follows that

$$F_{x2}\,\delta x_2 \geq 0$$

for a minimum to occur. Further, since the sign δx_2 is arbitrary, it must be that $F_{x_2} = 0$. But,

$$F_{x_2} = f_{x_2} + \lambda\, g_{x_2} = 0$$

Thus, for a minimum value to occur at x_0, the three conditions

$$\begin{aligned} & f_{x_1} + \lambda g_{x_1} = 0 \\ & f_{x_2} + \lambda g_{x_2} = 0 \\ & g(x_1, x_2) = 0 \end{aligned} \tag{2.1.12}$$

must be satisfied by selecting the values of x_{10}, x_{20} and λ. Note that the device of introducing the constant λ and forming the function $F = f + \lambda g$ has allowed the constrained minimum problem to be treated as an unconstrained problem as far as the first necessary condition is concerned. However, the condition (defining a maximum or minimum) on the second order terms differs somewhat from that developed in the unconstrained problem (i.e., Eqs. (2.1.1) and (2.1.8)) as will be shown below.

Since the first order terms vanish, the sign of the right hand side of (2.1.10) is determined by the second order terms which must be greater than or equal to zero for a minimum to occur. However, unlike the unconstrained problem, the variables δx_1 and δx_2 are not both arbitrary, but must satisfy the constraint equation which, to first order, provides

$$g_{x_1}\,\delta x_1 + g_{x_2}\,\delta x_2 = 0$$

Hence, the condition on the second order terms takes the form

$$F_{x_1x_1}\,\delta x_1^2 + 2F_{x_1x_2}\,\delta x_1\,\delta x_2 + F_{x_2x_2}\,\delta x_2^2 \geq 0 \tag{2.1.13}$$

for δx_1 and δx_2, satisfying

$$g_{x_1}\,\delta x_1 + g_{x_2}\,\delta x_2 = 0\ .$$

The extension of the analysis to the case in which x is an n dimensional vector and in which m constraints $g_j(x) = 0$, $j = 1, m$ are imposed, follows a procedure similar to that used in the two dimensional case. It is a simple matter to show that the necessary conditions in this case are

$$F_x = \begin{pmatrix} F_{x_1} \\ F_{x_2} \\ \vdots \\ F_{x_n} \end{pmatrix} = 0 \qquad (2.1.14)$$

$$\delta x^T F_{xx} \delta x \geq 0 \ , \ g_{j_x} \delta x = 0 \qquad ; j = 1, m \qquad (2.1.15)$$

where F is given by

$$F = f + \lambda_j \quad g_j \qquad (2.1.16)$$

To illustrate the application of these techniques, consider the example problem where

$$f = x_1{}^2 + x_2{}^2 = \text{minimum}$$

$$g = x_1 \ x_2 - 1 = 0$$

This problem consists of finding the point on the hyperbola $g = 0$ which is closest to the origin.

Following the previously outlined procedure, the function F is formed where

$$F = f + \lambda g$$

$$= x_1^2 + x_2^2 + \lambda(x_1 x_2 - 1)$$

The necessary conditions corresponding to Eqs. (2.1.12) are

$$2x_1 + \lambda x_2 = 0$$

$$2x_2 + \lambda x_1 = 0$$

$$x_1 x_2 = 1$$

from which it follows that a minimum point occurs at

$$\lambda = -2$$

$$x_{10} = x_{20} = \pm 1$$

that is, the function f has two minimums.

The second order test requires that

$$2\,\delta x_1^2 + 2\,\delta x_2^2 + 4\,\delta x_1\,\delta x_2 \geq 0 \qquad \text{for } \delta x_1 + \delta x_2 = 0$$

which is seen to be satisfied by both of the solutions.

2.1.4 Minimizing a Function Subject to Inequality Constraints

Often, the constraints entering a minimization problem are inequalities of the form

$$g_j(x) \leq 0 \qquad ; j = 1, m \tag{2.1.17}$$

The standard technique for handling such constraints is to reduce them to equality conditions through the introduction of additional variables.

Consider the m real variables η_j and rewrite conditions (2.1.17) in the equivalent form

$$g_j(x) + \eta_j^2 = 0 \qquad ; j = 1, m \tag{2.1.18}$$

Note that $\eta_j = 0$ if an equality condition holds in (2.1.17), while $\eta_j = 0$ indicates a definite inequality.

With the constraints written in equality form, the methods of the preceding section can be used to determine the minimum point x_o. Proceeding as in Eqs. (2.1.14) to (2.1.16), the function F is formed where

$$F = F(x, \eta) = f(x) + \lambda_j (g_j(x) + \eta_j^2) \equiv f$$

Thus differentiating with respect to X and η and equating the first derivative to zero provides

$$\left.\begin{aligned} F_{x_i} &= f_{x_i} + \lambda_j\, g_{j\,x_i} = 0 \qquad ; i = 1, n \\ F_{\eta_2} &= \lambda_1 \eta_1 = 0 \\ F_{\eta_2} &= \lambda_2 \eta_2 = 0 \\ &\vdots \\ F_{\eta_m} &= \lambda_m \eta_m = 0 \end{aligned}\right\} \tag{2.1.19}$$

while the condition on the second order terms is

$$(f+\lambda_j g_j)_{x_i x_\ell}\,\delta x_i\,\delta x_\ell + 2\lambda_j\,\delta\eta_j^2 \geq 0 \tag{2.1.20}$$

for δx_i and $\delta\eta_j$, satisfying

$$\begin{aligned} g_{1x_i}\,\delta x_i + 2\eta_1\,\delta\eta_1 &= 0 \\ g_{2x_i}\,\delta x_i + 2\eta_2\,\delta\eta_2 &= 0 \\ &\vdots \\ g_{mx_i}\,\delta x_i + 2\eta_m\,\delta\eta_m &= 0 \end{aligned} \tag{2.1.21}$$

Note from Eq. (2.1.19) that either η_j or λ_j is zero for $j = 1, m$. The case $\eta_j = 0$ corresponds to the minimum point lying on the boundary of the admissible region $g_j(x_o) = 0$, while the case $\lambda_j = 0$ corresponds to x_o in the interior $g(x_o) < 0$. Also note that Eq. (2.1.20) can be used to show that

$$\lambda_j \geq 0 \qquad ; j = 1, m \tag{2.1.22}$$

as follows. First, if $\eta_j \neq 0$, then $\lambda_j = 0$ and (2.1.22) holds. If $\eta_j = 0$ and all δx_i are set to zero, Eqs. (2.1.21) are satisfied identically, while Eq. (2.1.20) provides

$$2\lambda_j\,\delta\eta_j^2 \geq 0$$

for $\delta\eta_j$ arbitrary. Thus, (2.1.22) follows immediately.

2.2 CALCULUS OF VARIATIONS

The review of pertinent material will continue in order to provide the necessary background for subsequent discussion. In this section some of the concepts used in the Calculus of Variations will be reintroduced.

The Calculus of Variations is very much like maxima-minima theory except that the domain or space over which the minimization is to be performed is more complex. Instead of determining a point x_o at which a function $f(x)$ has a minimum value, it is desired that the time history of a function $x(t)$ be determined for which a functional J, given by

$$J(x(t)) = \int_{x^o, t^o}^{x^f, t^f} f(t, x, \dot{x})\, dt^* \tag{2.2.1}$$

is minimized be determined. The integrand f is a specified function of the variables t, x, and $\dot{x}$. However, the particular dependence of x on t, and the value of J (J varies as this dependence varies) are unspecified.

The procedure introduced in maxima-minima theory was to expand the function in a truncated Taylor series about the minimum point, and to conclude that the first order terms in the series must vanish while the second order terms must be greater than, or equal to, zero. (See Eqs. (2.1.1) and (2.1.6) to (2.1.8).) The extremum was then determined by setting the first order terms to zero and solving the resultant algebraic equation. The second order inequality was then tested to determine if the point selected is a true minimum and not a maximum or stationary point.

A similar procedure is used in the Calculus of Variations. It is assumed that the functional J has a minimum value at $x(t) = x_o(t)$, and then $J(x(t))$ is expanded about $J(x_o(t))$ in a truncated Taylor series. By a reasoning process analogous to that used in maxima-minima theory, it is a relatively simple matter to show that the first order terms in the series expansion must be zero for $x_o(t)$ to be the minimizing function, and that the second order terms must be greater than, or equal to, zero. However, the process of equating the first order terms to zero provides a differential equation (rather than an algebraic equation) which the minimizing function $x_o(t)$ must satisfy. This differential equation is referred to as the Euler equation. Again the second order condition serves as a test that the solution is minimizing and may also resolve certain ambiguities which arise in solving the Euler equation.

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* Dots over variables indicate differentiation with respect to time.

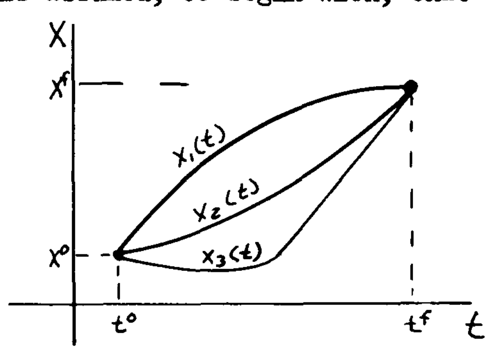
In the series expansion process by which the conditions on the first and second order terms are developed, it is necessary to assume that the solution x(t) is "near" to the solution $x_o(t)$, just as it was necessary to assume that the point x was near to the point x_o in the maxima-minima development. However, unlike the maxima-minima case, the concept "near" in the Variational Calculus can have several meanings. The particular meaning needed in the series expansion process is somewhat restricitve in that it limits the type of comparison solution, x(t), to a rather narrow class. A less restrictive interpretation of the concept "near" led the mathematician Weierstrass to another necessary condition which now bears his name and is somewhat stronger than the Euler condition. In the following paragraphs, both of these conditions, along with a third - the Legendre condition - will be developed, first for the one-dimensional problem and then for n-dimensional problem with constraints imposed. A fourth condition, the Jacobi condition, will be discussed only slightly since it is rather difficult to apply in most engineering problems.

2.2.1 One-Dimensional Lagrange Problem

The one-dimensional Lagrange problem consists of determining the time history of the variable X(t) such that the functional

$$J(x(t)) = \int_{t^o, x^o}^{t^f, x^f} f(t, x, \dot{x})\, dt \tag{2.2.2}$$

is a minimum. The point (t^o, x^o) denotes the lower limit of integration, while (t^f, x^f) denotes the upper limit. It is assumed, to begin with, that these points are fixed. The problem is represented graphically in the sketch to the right where three possible time histories, $x_1(t)$, $x_2(t)$ and $x_3(t)$ are shown.



Many well-known problems in analysis can be put in the form of Eq. (2.2.2). For example, consider the problem of determining the curve of shortest arc length connecting the points (t^o, x^o) and (t^f, x^f). In this case, Eq. (2.2.2) takes the form

$$J = \int_{t^o, x^o}^{t^f, x^f} \sqrt{1 + \dot{x}^2}\, dt$$

where the integrand represents the differential arc length along the curve.

Returning to the general problem, assume that $x_o(t)$ represents the minimizing function, and let x(t) represent a neighboring solution in which

$$x(t) = x_o(t) + \delta x(t)$$

where $\delta x(t)$ is the difference in x and x_o at the time t. Expanding $J(x(t))$ in a Taylor series about $J(x_o(t))$ provides

$$J(x(t)) - J(x_o(t)) = \int_0^f \left(\frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial \dot{x}} \delta \dot{x} \right) dt$$
$$+ \frac{1}{2} \int_0^f \frac{\partial^2 f}{\partial x^2} \delta x^2 + 2 \frac{\partial^2 f}{\partial x \partial \dot{x}} \delta x \delta \dot{x} + \frac{\partial^2 f}{\partial \dot{x}^2} \delta \dot{x}^2 \Big) dt + \int_0^f \Theta(dx^3) dt \tag{2.2.3}$$

where again $\Theta(\delta x^3)$ indicates terms of order δx^3. Since δx and $\delta \dot{x}$ are arbitrary, it will be required (as in maxima-minimal theory) that δx and $\delta \dot{x}$ be sufficiently small so that the sign of Eq. (2.2.3) is determined from the first order terms. Hence, if $x_o(t)$ is minimizing

$$J(x(t)) - J(x_o(t)) \geq 0$$

and

$$\int_0^f \left(\frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial \dot{x}} \delta \dot{x} \right) dt \geq 0 \tag{2.2.4}$$

But the sign of (2.2.4) can be reversed by considering a new solution $x_1(t) = x_o(t) + \delta x_1(t)$ such that $\delta x_1(t) = -\delta x$ and $\delta \dot{x}_1 = -\delta \dot{x}$, from which it follows that

$$\int_0^f \left(\frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial \dot{x}} \delta \dot{x} \right) dt = 0 \tag{2.2.5}$$

and, since the first order terms vanish, the second order terms must satisfy the inequality

$$\int_0^f \left(\frac{\partial^2 f}{\partial x^2} \delta x^2 + 2 \frac{\partial^2 f}{\partial x \partial \dot{x}} \delta x \delta \dot{x} + \frac{\partial^2 f}{\partial \dot{x}^2} \delta \dot{x}^2 \right) dt \geq 0 \tag{2.2.6}$$

for δx and $\delta \dot{x}$ sufficiently small.

Eq. (2.2.5) can, however, be put in a more usable form by integrating the second term by parts to provide

$$\frac{\partial f}{\partial \dot{x}} \delta x \Big]_0^f + \int_0^f \left(\frac{\partial f}{\partial x} - \frac{d}{dt} \frac{\partial f}{\partial \dot{x}} \right) \delta x \, dt = 0$$

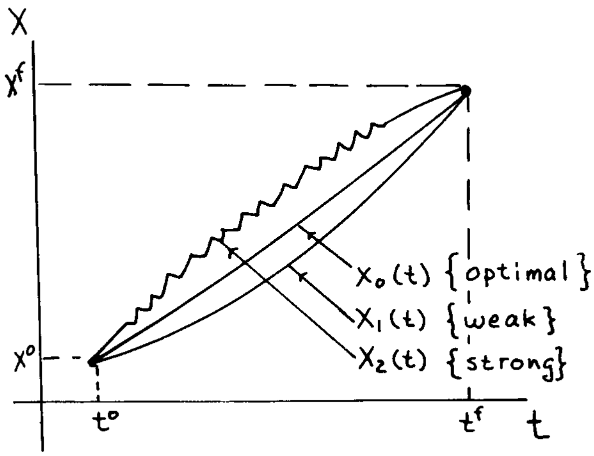
Since δx is zero at both ends of the interval (t°, t^f) and arbitrary in the interior, this expression reduces to the well-known Euler differential equation

$$\frac{\partial f}{\partial x} - \frac{d}{dt} \frac{\partial f}{\partial \dot{x}} = 0 \tag{2.2.7}$$

This equation, along with the boundary conditions

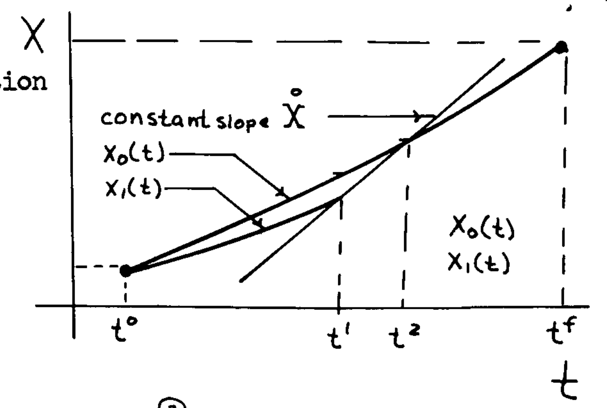
$$x = x^\circ \; ; \; t = t^\circ$$
$$x = x^f \; ; \; t = t^f$$

is used to compute the arc $x_o(t)$.



The assumption that δx and $\delta \dot{x}$ are small (which is necessary to develop Eqs. (2.2.6) and (2.2.7) restricts the class of comparison solutions. This fact is shown in the sketch to the right. In this sketch $x_1(t)$ represents a neighboring solution for which δx and $\delta \dot{x}$ are small, and $x_2(t)$ represents a solution for which δx is small but $\delta \dot{x}$ is large. A variation $\delta x = x - x_o$ such that both δx and $\delta \dot{x}$ are small is called a weak variation, while a variation for which only δx is small is called a strong variation. Thus, conditions (2.2.6) and (2.2.7) can be used to determine a minimizing function $x_o(t)$ only on the class of functions $x(t)$ which represent weak variations of $x_o(t)$. Another condition is needed to determine the minimizing function on the class of strong variations. Such a condition was developed by Weierstrass in the 1870's.

Let $x_o(t)$ represent the optimal solution and $x_1(t)$ a neighboring solution which differs from x_o on the interval $[t^\circ, t^2]$, which has a constant slope $\overset{\circ}{\dot{X}}$ on the interval $[t^1, t^2]$. (See sketch). Since $x_o(t)$ is minimizing,



$$J(x_1(t)) - J(x_o(t)) = \int_0^{②} f(x_1, \dot{x}_1, t)\,dt - \int_0^{②} f(x_o, \dot{x}_o, t)\,dt \geq 0$$

But this expression can be rewritten as

$$\int_0^{①} \left\{ f(x_1, \dot{x}_1, t) - f(x_o, \dot{x}_o, t) \right\} dt + \int_{①}^{②} \left\{ f(x_1, \dot{x}_1, t) - f(x_o, \dot{x}_o, t) \right\} dt \geq 0 \tag{2.2.8}$$

Under the assumptions that the interval $[t^1, t^2]$ is small and equal to the differential dt, and that δx and $\delta \dot{x}$ are small on the interval $[t^\circ, t^1]$, condition (2.2.8) becomes, to first order

$$\int_0^{①} \left(\frac{\partial f}{\partial x}(x_o, \dot{x}_o, t)\,\delta x + \frac{\partial f}{\partial \dot{x}}(x_o, \dot{x}_o, t)\,\delta \dot{x} \right) dt + \left\{ f(x_o, \dot{X}, t) - f(x_o, \dot{x}_o, t) \right\} dt \Big]^{t=t'} \geq 0 \tag{2.2.9}$$

Now, integrating the first term by parts and noting that x_o must satisfy the Euler condition of Eq. (2.2.7), provides

$$\left\{ \frac{\partial f}{\partial \dot{x}}(x_o, \dot{x}_o, t)\,dx + \left[f(x_o, \dot{X}, t) - f(x_o, \dot{x}_o, t) \right] dt \right\}^{t=t'} \geq 0$$

But

$$\begin{aligned} \delta x(t') &= x_1(t') - x_o(t') \\ &= \left\{ x_1(t^2) - x_o(t^2) \right\} - \left\{ x_1(t^2) - x_1(t') \right\} + \left\{ x_o(t^2) - x_o(t') \right\} \\ &= 0 - \dot{X}\,dt + \dot{x}_o\,dt \Big]^{t=t'} \end{aligned}$$

Substituting this result into (2.2.9) and noting that the time t^1 can be any time in the interval (t°, t^f) provides the final result

$$E\{\dot{X}, \dot{x}_o, x_o, t\} = f(x_o, \dot{X}, t) - f(x_o, \dot{x}_o, t) - (\dot{X} - \dot{x}_o) \frac{\partial f}{\partial \dot{x}}(x_o, \dot{x}_o, t) \geq 0 \tag{2.2.10}$$

which must hold along the optimal solution $x_o(t)$. This inequality is referred to as the Weierstrass condition.

A short summary at this point seems appropriate. Eqs. (2.2.6) and (2.2.7), which must hold along the optimal solution, provide a function $x_o(t)$ which yields a smaller value for the functional J than any other function $x(t)$ for which both $\delta x = x(t) - x_o(t)$ and $\delta \dot{x}$ are small. A larger class of functions are those for which δx is small while $\delta \dot{x}$ may be large, and the Weierstrass condition in (2.2.10) is a condition which must be satisfied by the function which is minimizing on this larger class.

It is apparent that if $x_o(t)$ is minimizing on a certain class of functions $C(x)$, then it is also minimizing on every subset of this class. For example, if

$$J(x_o(t)) < J(x(t)) \quad \text{for} \quad x(t) \in C(x)$$

Then

$$J(x_o(t)) < J(x(t)) \quad \text{for} \quad x(t) \in C_1(x)$$

where C_1 is contained in C. Likewise, for $x_o(t)$ to be minimizing on a certain class, it must also be minimizing on every subset of the class. <u>Hence, any necessary condition which must be satisfied for $x_o(t)$ to be minimizing on a subset of the class is also a necessary condition for $x_o(t)$ to be minimizing on the entire class.</u> This property will be used again and again to develop additional necessary conditions which the minimizing arc must satisfy. From this property, it is a simple matter to conclude that the conditions on the first and second order terms of the series expansion, Eqs. (2.2.6) and (2.2.7), which were developed for $x_o(t)$ to be optimal on the class of weak variations, must also hold for $x_o(t)$ to be optimal on the class of strong variations. For convenience, these necessary conditions are summarized below:

(1) Weierstrass Condition

$$E\{\dot{X}, \dot{x}_o, x_o, t\} = f(x_o, \dot{X}, t) - f(x_o, \dot{x}_o, t) - (\dot{X} - \dot{x}_o) \frac{\partial f}{\partial \dot{x}} \geq 0 \tag{2.2.11}$$

(2) Euler Condition

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}} = 0 \qquad (2.2.12)$$

(3) Second Order Condition

$$\int_0^f \left\{ \frac{\partial^2 f}{\partial x^2}\delta x^2 + \frac{2\,\partial^2 f}{\partial x\,\partial \dot{x}}\delta x\,\delta \dot{x} + \frac{\partial^2 f}{\partial \dot{x}^2}\delta \dot{x}^2 \right\} dt \geq 0 \qquad (2.2.13)$$

for δx , $\delta \dot{x}$ sufficiently small.

The second order condition in the form shown in Eq. (2.2.13) is generally too complicated to apply to most engineering problems of interest. Fortunately, it can be reduced to two simpler conditions which are equivalent to Eq.(2.2.13). The first of these conditions is the Legendre condition which can be developed rather simply from Eq. (2.2.11). The second is the Jacobi conjugate point condition.

From Eq. (2.2.11), it follows that

$$E\left\{\dot{X}, \dot{x}_o, x_o, t\right\} \geq 0$$

which must hold for any $\dot{X}$ if $x_o(t)$ is optimal. Specifically, it must hold for

$$\dot{X} = \dot{x}_o + \delta \dot{x}$$

where $\delta \dot{x}$ is small. Thus, the E function has a minimum value of zero at $\dot{X} = \dot{x}_o$ and, hence, its first derivative, $\partial E/\partial \dot{x}$ must vanish at $\dot{X} = \dot{x}_o$ and

$$\left(\frac{\partial^2 E}{\partial \dot{X}^2}\right)_{\dot{X}=\dot{x}_o} \delta \dot{x}^2 = \frac{\partial^2 f}{\partial \dot{x}^2}\delta \dot{x}^2 \geq 0$$

Since $\delta \dot{x}$ is arbitrary, except for the requirement that it be small, it follows immediately that along the minimizing curve $x_o(t)$

$$\frac{\partial^2 f}{\partial \dot{x}^2}(x_o, \dot{x}_o, t) \geq 0 \qquad (2.2.14)$$

This inequality is referred to as the Legendre condition.

The development of the Jacobi condition is somewhat more involved. Let K denote the value of the terms on the left hand side of (2.2.13); that is,

$$K = \int_0^f \left(\frac{\partial^2 f}{\partial x^2} \delta x^2 + \frac{2\,\partial^2 f}{\partial x \partial \dot{x}} \delta \dot{x} \delta x + \frac{\partial^2 f}{\partial \dot{x}^2} \delta \dot{x}^2 \right) dt \quad . \tag{2.2.15}$$

It follows that K can never be less than zero if $x_o(t)$ is the minimizing arc, and since the value of K varies with $\delta x(t)$ (also $\delta \dot{x}(t)$), there is some value of δx for which K takes on a minimum value, say K_{min}. The Jacobi condition is said to be satisfied if $K_{min} \geq 0$. If $K_{min} < 0$, the Jacobi condition is violated and the arc $x_o(t)$ is <u>not</u> minimizing.

To show these facts, it will be necessary to minimize K. Note that the coefficients

$$\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial \dot{x}}, \frac{\partial^2 f}{\partial \dot{x}^2}$$

are known functions of time, once the arc $x_o(t)$ is given. Hence, K is a function only of $\delta x(t)$, and that particular $\delta x(t)$ which minimizes K must satisfy the Euler condition of Eq. (2.2.12). Note also that if $\delta x(t)$ is zero over the entire arc, then K = 0 and the Euler condition is satisfied identically. To rule out this degenerate case in minimizing K, it will be required that $\delta x(t)$ satisfy the integral condition

$$G(\delta x) = \epsilon^2 - \int_0^f \delta x^2(t)\,dt = 0 \tag{2.2.16}$$

where ϵ^2 is some real small quantity. Also, since $G(\delta x) = 0$, minimizing the quantity K is equivalent to minimizing the quantity $\bar{K}$ where

$$\bar{K} = K + \lambda G \equiv K$$

and where λ is an arbitrary constant to be chosen so that Eq. (2.2.16) is satisfied.

At this point, $\bar{K}$ will be written as:

$$\bar{K} = \int_0^f \left\{ f_{xx}\,\delta x^2 + 2f_{x\dot{x}}\,\delta x\,\delta\dot{x} + f_{\dot{x}\dot{x}}\,\delta\dot{x}^2 - \lambda\,\delta x^2 \right\} dt + \lambda\varepsilon^2 \equiv K$$

and the Euler condition will be applied [Eq. (2.2.12)] to determine the minimizing $\delta\, x(t)$. The resultant relationship is

$$\frac{d}{dt}\left\{ f_{\dot{x}\dot{x}}\,\delta\dot{x} + f_{x\dot{x}}\,\delta x \right\} = \left(f_{xx} + \lambda \right) \delta x + f_{x\dot{x}}\,\delta\dot{x} \qquad (2.2.17)$$

The $\delta x(t)$ satisfying Eq. (2.2.17) provides the minimum value for the quantity K given in Eq. (2.2.15). This solution can be determined by multiplying (2.2.17) by $\delta\, x$ and integrating the first term by parts, to yield

$$\delta x \left[f_{\dot{x}\dot{x}}\,\delta\dot{x} + f_{x\dot{x}}\,\delta x \right]_0^{f} - \int_0^f \left(f_{\dot{x}\dot{x}}\,\delta\dot{x}^2 + 2f_{x\dot{x}}\,\delta x\,\delta\dot{x} + f_{xx}\,\delta x^2 \right) dt + \lambda \int_0^f \delta x^2 dt = 0$$

Since $\delta\, x$ is zero at both the initial and terminal points of the integral, a comparison of this expression with Eqs. (2.2.15) and (2.2.16) provides

$$K_{min} = \lambda\,\varepsilon^2 \qquad (2.2.18)$$

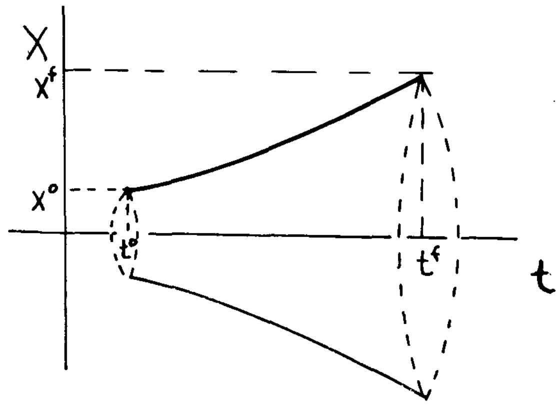
Thus, if λ is negative, the Jacobi condition is violated and the arc is not minimizing. If λ is positive, the Jacobi condition is satisfied. To determine the sign of λ, Eq. (2.2.17) is rewritten as

$$\delta\ddot{x} + \delta\dot{x}\,\frac{\frac{d}{dt} f_{\dot{x}\dot{x}}}{f_{\dot{x}\dot{x}}} + \delta x \left[\frac{-\lambda - f_{xx} + \frac{d}{dt} f_{x\dot{x}}}{f_{\dot{x}\dot{x}}} \right] = 0$$

Assuming the strong Legendre conditions of Eq. (2.2.14) is satisfied with $f_{\dot{x}\dot{x}} > 0$ over the entire interval, and noting that this equation is a form of the Sturm-Liouville equation for which a rather general theory has been developed, the theory of the solution to the Sturm-Liouville equation can be applied to determine the sign of the quantity λ .

The Jacobi and Legendre conditions together are sufficient for the second order terms in Eq. (2.2.13) to be greater than, or equal to, zero. However, while the Legendre condition is rather easy to apply, the Jacobi condition is not. For this reason, the Jacobi condition is seldom employed in analyzing engineering problems. Hence, in the treatment which follows, no further consideration will be given to the Jacobi condition or its implications.

To illustrate the applications of the necessary conditions, consider the minimum surface of revolution problem in which it is desired to determine the curve connecting the two points $[t^\circ, x^\circ]$ and $[t^f, x^f]$ such that the surface formed by rotating the curve about the t axis is a minimum. In this case, the quantity to be minimized (the surface area) takes the form



$$J = 2\pi \int_{o}^{f} x \sqrt{1+\dot{x}^2}\, dt$$

The Euler condition requires that the minimizing curve satisfy the differential equation

$$\frac{d}{dt}\left\{\frac{x\dot{x}}{\sqrt{1+\dot{x}^2}}\right\} = \sqrt{1+\dot{x}^2}$$

By direct substitution, it can be shown that the extremal is a catenary of the form

$$x = a \cosh \left(\frac{t-b}{a}\right)$$

where a and b are constants selected so that the boundary conditions

$$x = x^\circ, \quad t = t^\circ$$
$$x = x^f, \quad t = t^f$$

are satisfied.

The Legendre condition requires that

$$\frac{X}{\left(\sqrt{1+\dot{X}^2}\right)^3} = \frac{a}{\cosh^2\left(\frac{t-b}{a}\right)} \geq 0$$

If x(t) > 0 along the arc, then it follows that a > 0, and that this condition is satisfied. The Weierstrass condition yields

$$\frac{X}{\sqrt{1+\dot{x}^2}}\left\{\sqrt{1+\dot{X}^2}\sqrt{1+\dot{x}^2} - \left(1+\dot{X}\dot{x}\right)\right\} \geq 0$$

where $\dot{x}$ denotes the derivative along the extremizing curve (i.e., the catenary) and $\dot{X}$ denotes any other derivative. Again, it is a simple matter to show that the inequality holds provided x > 0 along the optimizing arc.

2.2.2 N-Dimensional Lagrange Problem

The n-dimensional Lagrange problem is concerned with minimizing a functional of the form

$$J = \int_0^f f\,(t,\ x_1,x_2,\ldots,x_n,\dot{x}_1,\ \dot{x}_2,\ldots\dot{x}_n)\ dt$$

or in the vector notation

$$J = \int_0^f f\,(t,\ x,\ \dot{x})\ dt$$

where x is the n dimensional vector

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \dot{x}_n \end{pmatrix}$$

The necessary conditions which the extremizing arc must satisfy are essentially the vector equivalent of the scalar conditions developed in the preceding section for the one dimensional problem. The method of development is exactly the same.

The condition that the first variation (i.e., the first order terms in a series expansion of the function J about its minimum value) vanish for the minimizing arc to be minimizing on the class of weak variations leads to the n Euler equations

$$\frac{\partial f}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}_i}\right) = 0 \quad ; \; i = 1, n \tag{2.2.19}$$

Note that the form of the Euler equations above is the same as that given in Eq. (2.2.12) for the one dimensional problem. The Legendre condition, which must be satisfied for the second variation to be greater than or equal to zero, is expressed as

$$\left(\frac{\partial^2 f}{\partial \dot{X}^2}\right) = \begin{pmatrix} \frac{\partial^2 f}{\partial \dot{X}_1^2} & \frac{\partial^2 f}{\partial \dot{X}_1 \partial \dot{X}_2} & \cdots & \frac{\partial^2 f}{\partial \dot{X}_1 \partial \dot{X}_n} \\ \vdots & & & \\ \frac{\partial^2 f}{\partial \dot{X}_n \partial \dot{X}_1} & \frac{\partial^2 f}{\partial \dot{X}_n \partial \dot{X}_2} & \cdots & \frac{\partial^2 f}{\partial \dot{X}_n^2} \end{pmatrix} \geq 0 \tag{2.2.20}$$

that is, the matrix $\left(\frac{\partial^2 f}{\partial \dot{X}^2}\right)$ must be positive semi-definite along the extremizing arc. Finally, the Weierstrass condition becomes

$$E = f(\dot{X}, x_o, t) - f(\dot{x}_o, x_o, t) - \sum_{i=1}^{n} \left(\dot{X}_i - \dot{x}_{i_o}\right) \frac{\partial f}{\partial \dot{X}_i}(\dot{x}_o, x_o, t) \geq 0 \tag{2.2.21}$$

where $\dot{x}_o$ denotes the derivative along the extremal and $\dot{X}$ denotes any other value of the derivative.

A fourth condition, the Jacobi condition, can also be developed. However, as pointed out in the previous section, this condition is usually too difficult to apply to be of any use.

In the treatment of both the one-dimensional and n-dimensional Lagrange problem it has been assumed that the initial and terminal points are fixed; that is, that the value of x at t = t° as well as the value of x at t = t^f have been specified. Frequently this is not the case, and the limits of integration are allowed to vary over specified surfaces in the (x,t) space. This case will be treated next.

2.2.3 Boundary and Corner Conditions

Consider the n-dimensional Lagrange problem in which the integral

$$J = \int_{0}^{f} f\,(x, \dot{x}, t)dt \tag{2.2.22}$$

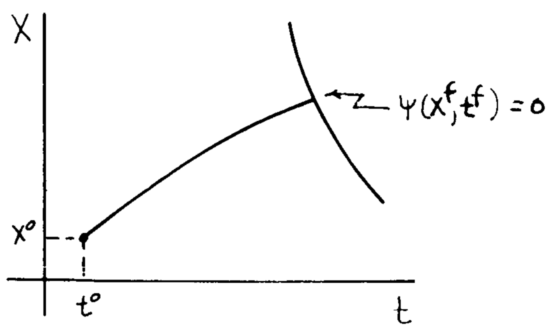
is to be minimized subject to the boundary conditions

$$x = x^{\circ} = \begin{pmatrix} x_1^{\circ} \\ x_2^{\circ} \\ \vdots \\ x_n^{\circ} \end{pmatrix} \text{ at } t = t^{\circ}$$

and

$$\psi_i(x^f, t^f) = 0 \qquad ; \; j = 1, m \le n \tag{2.2.23}$$

In this case, the upper limit of integration (x^f, t^f) is not specified. Rather, the terminal state is required to be on the surface defined by the m constraint equations $\psi_j(x^f, t^f) = 0; j = 1, m$. If the problem is one dimensional, then the surface, $\psi(x^f, t^f) = 0$, on which the upper limit must lie becomes a curve in the (x,t) space (see sketch), and the problem consists of determining both the arc $x_0(t)$ and the terminal point (x^f, t^f) for which the functional J has a minimum value.



Since the ψ_j of Eq. (2.2.23) are zero, minimizing the functional J is equivalent to minimizing the functional $\bar{J}$ where

$$\bar{J} = \int_{0}^{f} f(x, \dot{x}, t)dt + \mu_j \psi_j^{*}$$

* Repeated subscripts indicate summation.

and where the μ_j are constants to be determined. Setting the first variation of $\bar{J}$ to zero, a condition which must hold for J to be minimizing, provides

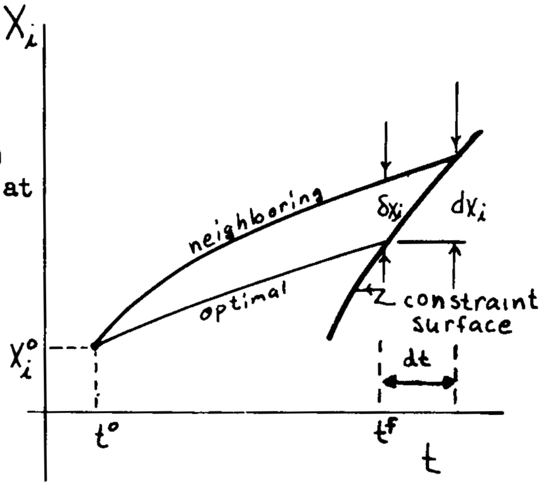
$$\delta \bar{J} = \int_0^f \left(f_{\dot{x}_i}\,\delta\dot{x}_i + f_{x_i}\,\delta x_i\right) dt + f\,dt \Big]^f + \mu_j\left(\frac{\partial\psi_j}{\partial x_i}\, dx_i + \frac{\partial \psi_j}{\partial t}\, dt\right)\Big]^f = 0$$

where the fdt term is due to the fact that the upper limit of integration can vary. Integrating the first term in the integrand by parts now yields

$$\delta\bar{J} = \int_0^f \left[\left(f_{x_i} - \frac{d}{dt}\left(f_{\dot{x}_i}\right)\right)\delta x_i\, dt + f\,dt\right] + \left[f_{\dot{x}_i}\,\delta x_i + \mu_j\left(\frac{\partial\psi_j}{\partial x_i}\, dx_i + \frac{\partial\psi_j}{\partial t}\, dt\right)\right]^f = 0 \quad (2.2.24)$$

Note that the variation $\delta\, x(t^f)$ is the difference between a neighboring arc and the optimal arc at the time t^f; that is

$$\delta x(t^f) = x\,(t^f) - x_o(t^f)$$



while the differential, dx_i, in Eq. (2.2.24) is the difference between the two arcs, but at different times, since the terminal times along the two solutions are not the same. From the sketch to the right it is a simple matter to show that to first order, δx and dx are related by

$$\delta x = dx - \dot{x}\, dt$$

or in the scalar notation

$$\delta x_i = dx_i - \dot{x}_i\, dt \quad (2.2.25)$$

where x_i denotes the slope along the optimal solution at the terminal point. Substituting (2.2.25) into (2.2.24) provides

$$\delta\bar{J} = \int_0^f \left\{f_{x_i} - \frac{d}{dt}\left(f_{\dot{x}_i}\right)\right\}\delta x_i\, dt + \left[\left(f - f_{\dot{x}_i}\dot{x}_i + \mu_j\frac{\partial\psi_j}{\partial t}\right) dt^f + \left(f_{\dot{x}_i} + \mu_j\frac{\partial\psi_j}{\partial x_i}\right) dx_i^f\right] = 0 \quad (2.2.26)$$

Consider a special subclass of neighboring solutions, all of which pass through the same terminal point as the optimal solution. In this case, the quantities dt^f and dx_i^f in (2.2.26) are zero with the first variation reducing to

$$\delta \bar{J} = \int_0^f \left\{ f_{x_i} - \frac{d}{dt}\left(f_{\dot{x}_i}\right) \right\} \delta x_i \, dt \tag{2.2.27}$$

Since the δx_i are arbitrary, it follows immediately that the Euler equations

$$f_{x_i} - \frac{d}{dt}\left(f_{\dot{x}_i}\right) = 0; \; i = 1, n \tag{2.2.28}$$

must hold along the optimal solution.

The expression for the first variation in Eq. (2.2.26) reduces to Eq. (2.2.27) only for a special subclass of weak variations; namely, that class which passes through the same terminal point as the optimal solution. Hence, the Euler conditions of (2.2.28) are necessary only for this special subclass. However, as pointed out in the previous section (Eq. 2.2.11), any necessary condition which the extremizing arc $x_o(t)$ must satisfy to be minimizing on a subset of the class of possible variations is also necessary for $x_o(t)$ to be minimizing on the entire class. Hence, the Euler equations of (2.2.28) must hold for <u>all</u> weak variations of the minimizing arc $x_o(t)$ whether these variations pass through the same terminal point as x_o or not.

Returning to the general case in which the neighboring and optimal solutions do not go through the same terminal point, it follows from the above arguments and Eq. (2.2.28) that the first variation of (2.2.26) takes the form

$$\delta \bar{J} = \left\{ f - f_{\dot{x}_i} \dot{x}_i + \mu_j \frac{\partial \psi_j}{\partial t} \right\} dt + \left\{ f_{\dot{x}_i} + \mu_j \frac{\partial \psi_j}{\partial \dot{x}_i} \right\} dx_i \Big]^f = 0 \tag{2.2.29}$$

The n + 1 differentials dt and d_{x_i}, i = 1,n are not all independent because of the m terminal constraints of Eq. (2.2.23). Consider the first m differentials dx_1, dx_2... dx_m as dependent through the constraint equations $\psi_j = 0$; j = 1,m and the remaining dx_{m+1}, dx_{m+2}, ...dx_n as independent. Now, since the μ_j are arbitrary, select the μ_j, j = 1, m, so that the coefficients of dx_1, dx_2,..., dx_m in Eq. (2.2.29) vanish; that is, so that

$$f_{\dot{x}_i} + \mu_j \frac{\partial \psi_j}{\partial x_i} = 0 \quad ; \quad i = 1, m \tag{2.2.30}$$

Then (2.2.29) becomes

$$\delta \bar{J} = \left(f - f_{\dot{x}_i} \dot{x}_i + \mu_j \frac{\partial \psi_j}{\partial t}\right) dt + \left(f_{\dot{x}_i} + \mu_j \frac{\partial \psi_j}{\partial x_i}\right) dx_i \Big]^f = 0$$

where the subscript on dx_i ranges from m+1 to n. But since dt and dx_i are independent for these variables, it follows that $\delta \bar{J}$ will vanish only if

$$f - f_{\dot{x}_i} \dot{x}_i + \mu_j \frac{\partial \psi_j}{\partial t} = 0$$

$$f_{\dot{x}_i} + \mu_j \frac{\partial \psi_j}{\partial x_i} = 0 \quad ; \; i = m+1, n$$

Collecting the results, the arc $x_o(t)$ which minimizes the functional in Eq. (2.2.22) subject to the terminal constraints of Eq. (2.2.23) must satisfy the n differential equations

$$f_{x_i} - \frac{d}{dt}(f_{\dot{x}_i}) = 0; \; i = 1, n \tag{2.2.31}$$

and the boundary or transversality conditions at the terminal point

$$\begin{aligned} &\psi_j = 0 \quad ; \; j = 1, m \\ &f_{\dot{x}_i} + \mu_j \frac{\partial \psi_j}{\partial x_i} = 0 \quad ; \; i = 1, n \\ &f - f_{\dot{x}_i} \dot{x}_i + \mu_j \frac{\partial \psi_j}{\partial t} = 0 \end{aligned} \tag{2.2.32}$$

Note that Eqs. (2.2.32) are consistent in that they constitute a set of n+1+m equations in the n+1+m unknowns x_i^f, t^f, and μ_j .

Two observations regarding the development of this set of equations can be made. First, the differential constraints, Eqs. (2.2.31), and the boundary conditions, Eqs. (2.2.32), are uncoupled in the sense that the same set of differential equations must be satisfied by the minimizing arc irregardless of how the terminal constraints may vary (that is, the same set of equations holds whether the terminal point is fixed or moves along a surface in the space). From this fact, it follows directly that all differential constraints necessary for the fixed end point problem are also necessary for the variable end point problem. Thus, in addition to the Euler equations, the minimizing arc must also satisfy the Legendre and Weierstrass conditions of Eqs. (2.2.20) and (2.2.21).

A second observation concerns the role of the multipliers μ_j. These multipliers are used in much the same way as the constants λ_j in the constrained maxima-minima problem of Section 2.1.4. By introducing these multipliers, the constrained variation in the terminal conditions (constrained by means of the equations $\psi_j = 0$) can be treated as an unconstrained variation. This technique will be used again and again throughout the report.

As an example, consider the problem of finding the shortest distance from a point (x^o, t^o) to a curve $\psi\ (x^f, t^f) = 0$. In this case the integral is

$$J = \int_o^f \sqrt{1+\dot{x}^2}\, dt$$

with the minimizing arc consisting of a straight line along which $\dot{x}$ is a constant. The boundary conditions of (2.2.32) become

$$\psi(x^f, t^f) = 0$$

$$\frac{\dot{x}}{\sqrt{1+\dot{x}^2}} + \mu \frac{\partial \psi}{\partial x} = 0$$

$$\frac{1}{\sqrt{1+\dot{x}^2}} + \mu \frac{\partial \psi}{\partial t} = 0$$

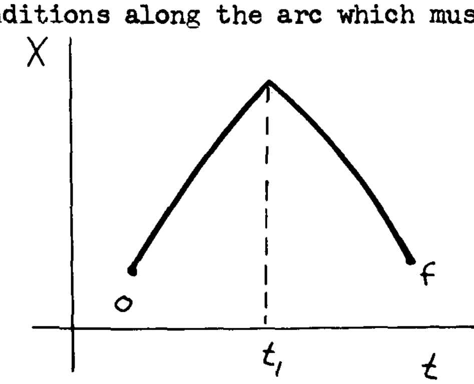
and reduce to

$$\dot{x} = \left(\frac{dx}{dt}\right)_{optimal} = \frac{\dfrac{\partial \psi}{\partial x}}{\dfrac{\partial \psi}{\partial t}}$$

through the elimination of the multiplier μ.

Since the slope along the curve $\psi = 0$ is given by $-(\partial\psi/\partial t)/(\partial\psi/\partial x)$, this expression proves that the minimizing curve and the constraint $\psi = 0$ are orthogonal at the point of intersection.

In addition to the boundary conditions which the optimal solution must satisfy, there are also intermediate conditions along the arc which must hold if the arc is to be minimizing. Specifically, certain conditions must hold at a corner point where the derivative $\dot{x}$ is discontinuous. These conditions will be developed below.



In developing the Euler conditions of Eqs. (2.2.31) and (2.2.12), it has been tacitly assumed that the derivative $\dot{x}$ is continuous along the extremal arc, a condition that will not hold if the arc has corners (see preceding sketch) resulting from sudden control changes, etc. At a corner point, say $t = t_1$, the derivative $\dot{x}$ is not defined. Hence, the functional J which is to be minimized is rewritten as

$$J = \int_0^{t_1^-} f(x,\dot{x},t)dt + \int_{t_1^+}^{f} f(x, \dot{x},t)dt \quad .$$

Now forming the first variation and equating it to zero yields

$$\delta J = \int_0^{t^-} \left(\frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial \dot{x}}\delta \dot{x}\right)dt + f(x,\dot{x}^-,t)\,dt$$
$$- f(x,\dot{x}^+,t)\,dt \quad + \int_{t^+}^{t^f}\left(\frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial \dot{x}}\delta \dot{x}\right)dt \quad = 0$$

but in each interval $[t^\circ, t^-]$ and $[t^+, t^f]$ the second term in the integrand can be integrated by parts as

$$\int \frac{\partial f}{\partial \dot{x}}\delta \dot{x}\,dt = \frac{\partial f}{\partial \dot{x}}\delta x - \int \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right)\delta x\,dt$$

Thus, the variation δJ reduces to

$$\delta J \equiv 0 = \int_0^{t^-}\left[\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right)\right]\delta x\,dt$$
$$\int_{t^+}^{t^f}\left[\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right)\right]\delta x\,dt$$
$$+\left[f\,dt + \frac{\partial f}{\partial \dot{x}}\delta x\right]\Big|^{t^-} - \left[f\,dt + \frac{\partial f}{\partial \dot{x}}\delta x\right]\Big|^{t^+}$$

But the integrands are clearly zero along an extremizing arc since they are, in fact, the Euler equations. Thus,

$$\delta J = \left\{f(x,\dot{x}^-,t)dt + \frac{\partial f}{\partial \dot{x}_i}(x,\dot{x}^-,t)\,\delta x_i\right\}\Big]^{t_1^-} -$$
$$\left\{f(x,\dot{x}^+,t)dt + \frac{\partial f}{\partial \dot{x}_i}(x,\dot{x}^+,t)\,\delta x_i\right\}\Big]^{t_1^+} = 0$$

Now since

$$\delta x_i = dx_i - \dot{x}_i \, dt$$

this expression becomes

$$\delta J = \left\{ \left(f(x,\dot{x}^-,t) - \frac{\partial f}{\partial \dot{x}_i}(x,\dot{x}^-,t)\,\dot{x}_i^- \right) - \left(f(x,\dot{x}^+,t) - \frac{\partial f}{\partial \dot{x}_i}(x,\dot{x}^+,t)\,\dot{x}_i^+ \right) \right\} dt$$

$$+ \left\{ \frac{\partial f}{\partial \dot{x}_i}(x,\dot{x}^-,t) - \frac{\partial f}{\partial \dot{x}_i}(x,\dot{x}^+,t) \right\} dx_i = 0 \qquad (2.2.33)$$

If the corner is unconstrained (i.e., the corner is not required to be on some specified surface) then dt and dx_i are independent (the corner can occur any place), and Eq. (2.2.33) reduces to the two conditions

$$\frac{\partial f^+}{\partial \dot{x}_i} = \frac{\partial f^-}{\partial \dot{x}_i} \qquad (2.2.34a)$$

$$f^+ - \frac{\partial f^+}{\partial \dot{x}_i}\dot{x}_i^+ = f^- - \frac{\partial f^-}{\partial \dot{x}_i}\dot{x}_i^- \qquad (2.2.34b)$$

These conditions are usually referred to as the Weierstrass-Erdman corner conditions.

As an example of the application of these conditions, consider a one dimensional problem of the form

$$J = \int_0^f (1+\dot{x}^2)^{1/2}\, G(x,t)\, dt$$

where G(x,t) is any function of x and t. If the minimizing solution has a corner, then at the corner Eqs. (2.2.34a) and (2.2.34b) become

$$\frac{\dot{x}^-}{\sqrt{1+\dot{x}^{2-}}} = \frac{\dot{x}^+}{\sqrt{1+\dot{x}^{2+}}} \quad ; \quad \frac{G(x,t)}{\sqrt{1+\dot{x}^{2-}}} = \frac{G(x,t)}{\sqrt{1+\dot{x}^{2+}}}$$

from which it follows that

$$\dot{x}^- = \dot{x}^+$$

and that the optimal solution can not have any corners.

2.2.4. General Treatment of Equality Constraints

Often problems arise in which a functional J is to be minimized subject to subsidary constraint conditions. These constraints generally take one of three forms.

(1) Integral Constraints

The functional J is to be minimized on the class of functions x (t) satisfying the integral condition

$$\int_0^f G_1(x, \dot{x}, t)\, dt - k = 0 \qquad (2.2.35)$$

where k is a specified constant.

(2) Surface Constraints

The functional J is to be minimized on the class of functions x(t) lying on the surface

$$G_2\ (x,t) = 0 \qquad (2.2.36)$$

(3) Differential Constraints

The functional J is to be minimized on the class of functions x(t) satisfying the differential equation

$$G_3(x,\dot{x},t) = 0 \qquad (2.2.37)$$

In many cases more than one constraint of a certain type will be imposed, and frequently problems arise in which all three types are present. However, it should be noted that the number of surface and differential constraints combined must be less than the dimensions of the vector x in the problem; that is, if the problem is n dimensional, then at most n-1 subsidiary constraints of the surface and differential type can be imposed. For example, in a one dimensional problem, a surface constraint

$$G_2(x,t) = 0$$

could not be included, since the curve connecting the initial and terminal points would be completely specified by this constraint and there would remain no degree of freedom for minimizing the functional J. On the other hand, the integral type constraint is a weak constraint; and any number of these can be included regardless of the problem's dimensions, provided they are not contradictory.

By a slight amount of algebraic manipulation, both integral and surface constraints can be put in the form of differential constraints. Hence, in the formulation, one need only consider the inclusion of differential constraints.

To convert the integral constraint to differential form, introduce a new dimension into the problem and let

$$X_{n+1} = \int_0^t G_1(X,\dot{X},t)\,dt$$

The vector X now has n+1 dimensions. Differentiating this expression provides

$$\dot{X}_{n+1} = G_1(X,\dot{X},t) \tag{2.2.38}$$

which, along with the boundary conditions

$$\begin{aligned} X_{n+1} &= 0 \quad ; \quad t = t^0 \\ X_{n+1} &= k \quad ; \quad t = t^f \end{aligned} \tag{2.2.39}$$

is equivalent to the integral condition of equation (2.2.35).

The process of converting the surface constraint to differential form consists simply in differentiating it. Hence

$$\frac{dG_2}{dt} = \frac{\partial G_2}{\partial X_i}\dot{X}_i + \frac{\partial G_2}{\partial t} = 0 \tag{2.2.40}$$

Equation (2.2.40) is now in differential form, and it is only a matter of imposing one of the two-boundary conditions

$$G_2(X^0; t^0) = 0$$

or (2.2.41)

$$G_2(X^f, t^f) = 0$$

to insure that $G_2(X,t)$ is zero over the entire arc.

Since all equality constraints can be put in differential form, the following n-dimensional problem will be formulated: Determine the arc X(t) for which the functional

$$J = \int_0^f f(X,\dot{X},t)\,dt \tag{2.2.42}$$

is minimized subject to the boundary conditions

$$X(t^\circ) = X^\circ$$
$$\psi_j(X^f, t^f) = 0 \quad ; \quad j = 1, m \leq n \tag{2.2.43}$$

and the differential constraints

$$G_k(X, \dot{X}, t) = 0 \quad ; \quad k = 1, r < n \tag{2.2.44}$$

Since both ψ_j and G_k are zero, minimizing the functional J is again equivalent to minimizing $\bar{J}$ where

$$\bar{J} = \int_o^f \left\{ f + P_k G_k \right\} dt + \mu_j \psi_j \equiv J$$

where μ_j (j=1,m) and P_k (k=1,r) are multipliers to be determined. Letting

$$F = f + P_k G_k$$

and setting the first variation of $\bar{J}$ to zero provides

$$\delta \bar{J} = \int_o^f \left(\frac{\partial F}{\partial \dot{X}_i} \delta \dot{X}_i + \frac{\partial F}{\partial X_i} \delta X_i \right) dt + \left\{ F dt + \mu_j \left(\frac{\partial \psi_j}{\partial X_i} dX_i + \frac{\partial \psi_j}{\partial t} dt \right) \right\}^f$$

Integrating the first term in the integral by parts, and noting that

$$\delta X_i = dX_i - \dot{X}_i \, dt$$

yields

$$\delta \bar{J} = \int_o^f \left\{ \frac{\partial F}{\partial X_i} - \frac{d}{dt} \left(\frac{\partial F}{\partial \dot{X}_i} \right) \right\} \delta X_i \, dt + \left(F - F_{\dot{X}_i} \dot{X}_i + \mu_j \frac{\partial \psi_j}{\partial t} \right)^f dt + \left(F_{\dot{X}_i} + \mu_j \frac{\partial \psi_j}{\partial X_i} \right)^f dX_i \tag{2.2.45}$$

As in the preceding section, a weak variation in which dX_i and dt are zero (with the neighboring solution going through the same terminal point as the optimal solution) can be considered to conclude that for a general weak variation, both the integral expression in equation (2.2.45) and the boundary conditions must be zero independently. Thus, the vanishing of the first variation reduces to the two conditions

$$\int_o^f \left(F_{X_i} - \frac{d}{dt} F_{\dot{X}_i} \right) \delta X_i \, dt = 0 \tag{2.2.46}$$

where

$$F = f + P_k G_k$$

and

$$\left(F - F_{\dot{X}_i} \dot{X}_i + \mu_j \frac{\partial \psi_j}{\partial t}\right) dt + \left(F_{\dot{X}_i} + \mu_j \frac{\partial \psi_j}{\partial X_i}\right) dX_i \Big]^f = 0 \qquad (2.2.47)$$

The δX_i in equation (2.2.46) are not all independent since $X_i(t)$ must satisfy the r differential constraints of equation (2.2.44). If the first r of the δX_i(δX_1, $\delta X_2 \cdots \delta X_r$) are considered as dependent with the multipliers P_i* selected so that the coefficients of δX_1, $\delta X_2, \ldots \delta X_r$ in the integrand in (2.2.46) are zero, then the remaining coefficients must also vanish due to the independence of δX_{r+1}, $\delta X_{r+2}, \ldots \delta X_n$. Thus, the n Euler equations result where

$$\frac{\partial F}{\partial X_i} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{X}_i}\right) = 0 \quad ; \quad i = 1, n \qquad (2.2.48)$$

By a similar reasoning process, it can be shown that the multipliers μ_j can be selected so that the coefficients of dX_i and dt in equation (2.2.45) all vanish yielding the n+1 boundary conditions

$$\begin{aligned} F - F_{\dot{X}_i} \dot{X}_i + \mu_j \frac{\partial \psi_j}{\partial t} &= 0 \\ F_{\dot{X}_i} + \mu_j \frac{\partial \psi_j}{\partial X_i} &= 0 \quad ; \quad i = 1, n \end{aligned} \qquad (2.2.49)$$

Note that the Euler equations, equation (2.2.48), constitute a system of n equations in n+r unknowns (the $X_i(i=1,n)$ and the $P_k(k=1,r)$). This system, together with the r constraints of equation (2.2.44) is sufficient to determine the X_i and the P_k provided the coefficient determinant of the highest derivatives ($\ddot{X}_i$ and $\dot{P}_k$)

$$\begin{pmatrix} F_{\dot{X}\dot{X}} & G_{\dot{X}} \\ G_{\dot{X}} & 0 \end{pmatrix} = \begin{pmatrix} F_{\dot{X}_1\dot{X}_1}, & F_{\dot{X}_1\dot{X}_2}, & \cdots & F_{\dot{X}_1\dot{X}_n}, & G_{1\dot{X}_1}, & \cdots & G_{r\dot{X}_1} \\ \vdots & & & & & & \\ F_{\dot{X}_n\dot{X}_1} & F_{\dot{X}_n\dot{X}_2} & \cdots & F_{\dot{X}_n\dot{X}_n} & G_{1\dot{X}_n} & \cdots & G_{r\dot{X}_n} \\ G_{1\dot{X}_1}, & G_{1\dot{X}_2}, & & G_{1\dot{X}_n} & G_1 & & \\ \vdots & & & & & & 0 \\ G_{r\dot{X}_1} & G_{r\dot{X}_2} & \cdots & G_{r\dot{X}_n} & & & \end{pmatrix} \qquad (2.2.50)$$

*While the multipliers μ_j are constants, the multipliers P_i are functions of time and vary from point to point along the trajectory: that is, $P_i = P_i(t)$

is not zero. If the coefficient determinant is zero at a point, then a unique solution to the Euler and constraint equations does not exist at the point. The reason for this is that at any point where t, x and $\dot{x}$ are known a unique value for $\ddot{x}$ and $\dot{P}$ can be computed only if the coefficient determinant does not vanish.

In addition to the Euler equations, a Weierstrass condition can also be developed by a reasoning process similar to that used in the derivation of equation (2.2.10). For this problem, the Weierstrass condition takes the form

$$E(x_o, \dot{x}_o \dot{X}, t) = F(\dot{X}, x_o, t) - F(\dot{x}_o, x_o, t) - \left(\dot{X}_i - \dot{x}_{o_i}\right)\frac{\partial F}{\partial \dot{x}_i}(x_o, \dot{x}_o, t) \geq 0 \quad (2.2.51)$$

where $\dot{x}_o$ denotes the derivative along the minimizing solution and $\dot{X}$ denotes any other value of the derivative that satisfies the constraint equations

$$G_k(x_o, \dot{X}, t) = 0 \quad ; k = 1, r \quad (2.2.52)$$

Finally, using the Weierstrass condition, the Legendre condition can be developed for the weak variation case in which

$$\dot{X} = \dot{x}_o + \delta\dot{x}$$

where $\delta\dot{x}$ is small. Since, from equation (2.2.51), the E function is greater than or equal to zero, this function has a minimum value at $\dot{X}=\dot{x}_o$ and the second order terms in a series expansion about the x_o point must not be negative. Hence, it follows that

$$\frac{\partial^2 F}{\partial \dot{x}_i \partial \dot{x}_j} \delta\dot{x}_i \delta\dot{x}_j \geq 0 \quad (2.5.53)$$

where the $\delta\dot{x}_i$ must satisfy the constraint equations to first order which become

$$\frac{\partial G_k}{\partial \dot{x}_i} \delta\dot{x}_i = 0 \;, k = 1, r \quad (2.2.54)$$

Collecting results, for the functional J

$$J = \int_0^f f(x, \dot{x}, t)\, dt$$

to be a minimum subject to the terminal conditions

$$\psi_j(x^f, t^f) = 0 \qquad ; j = 1, m$$

and the differential constraints

$$G_k(x, \dot{x}, t) = 0 \qquad ; k = 1, r$$

the minimizing arc must satisfy

(1) The Euler and constraint equations

$$\begin{aligned} &G_k(x, \dot{x}, t) = 0 && ; k = 1, r \\ &\frac{\partial F}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}_i}\right) = 0 && ; i = 1, n \end{aligned} \qquad (2.2.55)$$

where

$$F = f + p_k G_k$$

(2) The Weierstrass condition

$$E(\dot{x}_o, \dot{X}, x_o, t) = F(\dot{X}, x_o, t) - F(\dot{x}_o, x_o, t) - (\dot{X}_i - \dot{x}_{o_i}) \frac{\partial F}{\partial \dot{x}_i}(\dot{x}_o, x_o, t) \geq 0 \qquad (2.2.56)$$

for $\dot{x}_o$ and $\dot{X}$ satisfying the constraint equations

$$G_k(x_o, \dot{x}_o, t) = G_k(x_o, \dot{X}, t) = 0 \qquad ; k = 1, r \qquad (2.5.57)$$

(3) The Legendre condition

$$\frac{\partial^2 F}{\partial \dot{x}_i \partial \dot{x}_j} \delta \dot{x}_i \delta \dot{x}_j \geq 0 \qquad (2.2.58)$$

for $\delta \dot{x}_i$ satisfying

$$\frac{\partial G_k}{\partial \dot{x}_i} \delta \dot{x}_i = 0 \qquad ; k = 1, r \qquad (2.2.59)$$

(4) The Boundary or transversality conditions

$$F - F_{\dot{X}_i}\dot{X}_i + \mu_j \frac{\partial \psi_j}{\partial t} = 0$$
$$F_{\dot{X}_i} + \mu_j \frac{\partial \psi_j}{\partial X_i} = 0 \qquad (2.2.60)$$

2.2.5 Inequality Constraints

As in the case of equality constraints, inequalities usually take one of three forms:

(1) Integral Inequalities

$$\int_0^f G_1(X, \dot{X}, t)\,dt - k \leq 0 \qquad (2.2.61)$$

(2) Surface Inequalities

$$G_2(X, t) \leq 0 \qquad (2.2.62)$$

(3) Differential Inequalities

$$G_3(X, \dot{X}, t) \leq 0 \qquad (2.2.63)$$

The procedure for handling inequalities in the Calculus of Variations is essentially the same as that used in maxima-minima theory (see section 2.1.4) and consists of converting the inequality to an equality through the introduction of additional variables.

The integral inequality (2.2.61) is put in the differential form

$$\dot{X}_{n+1} = G_1(X, \dot{X}, t) \qquad (2.2.64)$$

with the boundary conditions

$$X_{n+1} = 0 \quad ; \; t = t^{\circ}$$
$$X_{n+1} + \eta_1^2 = k \text{ at } t = t^f . \qquad (2.2.65)$$

For η_1 real, the inequality (2.2.61) will always be satisfied. Equations (2.2.64) and (2.2.65) are then adjoined to the original problem through the introduction of additional multipliers in exactly the same way that equality constraints and boundary conditions are adjoined, and the standard necessary conditions are applied (i.e., Euler, Weierstrass, etc.)

The differential inequality is converted to the equality

$$G_3(X, \dot{X}, t) + \eta_3^2 = 0$$

and adjoined to the original problem. The solution consists of arcs along which $G_3 < 0$ ($\eta_3 \neq 0$) plus arcs along which $G_3 = 0$ ($\eta_3 = 0$)

For the surface constraint, let

$$G_2 (X, t) + \eta_2^2 = 0$$

Differentiating provides

$$\frac{\partial G_2}{\partial X_i} \dot{X}_i + \frac{\partial G_2}{\partial t} + 2 \eta_2 \dot{\eta}_2 = 0 \qquad (2.2.66)$$

with this differential form of the constraint adjoined to the original problem and with variable $\eta = 0$ when a definite equality holds in (2.2.62) and arbitrary otherwise.

From this discussion, it is concluded that inequality constraints can be treated in much the same manner as equality constraints. It should be noted, however, that when the standard necessary conditions to the reformulated problem (with inequalities represented as equalities) are supplied, the form of these conditions (Euler, Weierstrass, Legendre, corner and transversality) does reflect the inequality nature of the constraints.* Further, the computational procedure of generating a solution becomes more involved when inequalities are present. Precisely how the computational procedure and necessary conditions are affected will be discussed in later sections.

2.2.6 Discussion

From the preceding sections, it is apparent that the methods used in the Calculus of Variations are very similar to those used in maxima-minima theory. It is further apparent that most of the information about the extremizing arc is derived from the process of equating the first variation of the functional J or the modified functional $\bar{J}$=zero, where $\bar{J}$ consists of J plus additional terms that are equal to zero. Of course, it must be remembered that the first variation is zero only on the class of weak variations where δ x and $\delta \dot{x}$ are both small. For convenience, the conditions resulting from the first variation are listed below for the problem of minimizing the

*In some cases, the form of the Legendre and corner conditions are modified due to the inclusion of the inequality. Thus, care should be taken to see that these conditions are correctly stated.

functional

$$J = \int_0^f f(x, \dot{x}, t)\, dt \qquad (2.2.67)$$

subject to the terminal conditions

$$\psi_j(x^f, t^f) = 0 \qquad ; \; j = 1, m \leq n \qquad (2.2.68)$$

and the differential constraints

$$G_k(x, \dot{x}, t) = 0 \qquad ; \; k = 1, r < n \qquad (2.2.69)$$

(1) **Euler Condition**

$$\frac{\partial F}{\partial x_i} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}_i} = 0 \qquad ; \; i = 1, n \qquad (2.2.70)$$

where

$$F = f + p_k G_k \qquad (2.2.71)$$

(2) Boundary or Transversality Conditions

$$\begin{aligned} &\psi_j = 0 \quad ; \; j = 1, m \\ &F_{\dot{x}_i} + \mu_j \frac{\partial \psi_j}{\partial x_i} = 0 \quad ; \; i = 1, n \\ &F - F_{\dot{x}_i}\dot{x}_i + \mu_j \frac{\partial \psi_j}{\partial t} = 0 \end{aligned} \qquad (2.2.72)$$

(3) Corner Conditions (i.e., conditions which must hold across a discontinuity in $\dot{x}$)

$$\begin{aligned} &\frac{\partial F^-}{\partial \dot{x}_i} = \frac{\partial F^+}{\partial \dot{x}_i} \quad ; \; i = 1, n \\ &F^- - \frac{\partial F^-}{\partial \dot{x}_i}\dot{x}_i^- = F^+ - \frac{\partial F^+}{\partial \dot{x}_i}\dot{x}_i^+ \end{aligned} \qquad (2.2.73)$$

where the superscript $^+$ denotes one side of the corner and the superscript $^-$ denotes the other side.

The condition that the first variation be zero results from a truncated Taylor series expansion of the functional J about the minimizing solution $x_o(t)$. Hence, the truncated expansion must include at least the second-order terms of the series. This requirement in turn dictates that the functions f, Ψ_j and G_k must possess at least continuous second derivatives in all their arguments. This restriction is not severe since, in almost all problems of interest, these functions possess many more continuous derivatives than the second. Often these functions are, in fact, analytic.

For the strong variation in which δx is small but $\delta \dot{x}$ is not, the first variation is not necessarily zero, but greater than or equal to zero; that is, if $J(x_o)$ is minimizing, then $J(x)$ is greater than $J(x_o)$, and the first order difference between these two quantities [as developed for example in equations (2.2.8) to (2.2.10)] must be non-negative. This fact leads to the Weierstrass condition

$$F(X_o, \dot{X}, t) - F(X_o, \dot{X}_o, t) - (\dot{X}_i - \dot{X}_{o_i}) \frac{\partial F}{\partial \dot{X}_i}(X_o, \dot{X}_o, t) \geq 0 \qquad (2.2.74)$$

where, again

$$F = f + P_k G_k$$

In many cases, the Weierstrass condition is used simply as a test that the extremizing arc, as developed from equations (2.2.70) to (2.2.73), is indeed minimizing. However, it should be noted that the satisfying of the Weierstrass condition does not guarantee that the arc X_o does minimize the functional J.

Finally, the Weierstrass condition can be used to develop the Legendre condition which requires that

$$\frac{\partial^2 F}{\partial \dot{X}_i \partial \dot{X}_j} \delta \dot{X}_i \delta \dot{X}_j \geq 0 \qquad (2.2.75)$$

for $\delta \dot{X}_i$ satisfying

$$\frac{\partial G_k}{\partial \dot{X}_i} \delta \dot{X}_i = 0 \quad ; \quad k = 1, r \qquad (2.2.76)$$

These conditions are necessary, but not sufficient, for $X_o(t)$ to be minimizing; that is, $X_o(t)$ might satisfy these conditions.
and still not be minimizing. The sufficient conditions for a minimum have not been developed, since, in most engineering problems, sufficiency is very difficult to prove. Rather, physical reasoning is generally employed to deduce that the problem as formulated does have a true minimum solution and that the solution resulting from an application of the necessary conditions is, indeed,

the desired solution. Such logic can, of course, lead to incorrect results.

The treatment of the preceding sections has centered on the Lagrange problem where the functional to be minimized is an integral of the form

$$J = \int_0^f f(X, \dot{X}, t)$$

There are, however, two other well known problems in the Calculus of Variations: the Bolza problem and the problem of Mayer. The Mayer problem consists of minimizing a function ϕ of the terminal state

$$J = \phi(X^f, t^f) \qquad = \text{minimum}$$

subject to boundary and differential constraints of the form in equations (2.2.68) and (2.2.69). In the Bolza problem, the sum of an integral and a function of the terminal state is to be minimized

$$J = \phi(X^f, t^f) + \int_0^f f(X, \dot{X}, t)\, dt$$

subject to the same type of boundary and differential constraints as in the problem of Mayer. It is shown in Reference (2) (page 189) that all three problems are equivalent; that is, by an appropriate transformation, the Bolza and Mayer problems can be put in the form of a Lagrange problem.

In modern trajectory and control analysis, most problems are cast in the Mayer form. Hence, for convenience in what follows, the Lagrange problem of equations (2.2.67) to (2.2.69) will be reformulated as a Mayer problem and the corresponding necessary conditions listed.

Let

$$X_{n+1} = \int_0^t f(X, \dot{X}, t)$$

with

$$\dot{X}_{n+1} = f(x, \dot{x}, t)$$

and

$$X_{n+1} = 0 \qquad ; \quad t = t^{o}$$

Then the functional to be minimized is

$$J = X_{n+1}(t^f)$$

subject to the constraints of equations (2.2.68) and (2.2.69) and the additional constraint

$$G_0 = \dot{X}_{n+1} - f(X, \dot{X}, t) = 0 \quad .$$

To accomplish the minimization, the revised functional $\bar{J}$ is formed

$$\bar{J} \equiv J = X_{n+1}(t^f) + \int_0^f (P_0 G_0 + P_k G_k)\, dt + \mu_j \psi_j \tag{2.2.77}$$

and the first variation is set to zero to provide

(1) Euler Conditions

$$\frac{\partial \hat{F}}{\partial X_i} - \frac{d}{dt}\frac{\partial \hat{F}}{\partial \dot{X}_i} = 0 \qquad ; \; i = 1, n \tag{2.2.78}$$

where

$$\hat{F} = P_0 G_0 + P_k G_k \tag{2.2.79}$$

(2) Boundary Conditions

$$\begin{aligned} &\psi_j = 0 && ; \; j = 1, m \\ &\hat{F}_{\dot{X}_i} + \mu_j \frac{\partial \psi_j}{\partial X_i} = 0 && ; \; i = 1, n+1 \\ &\hat{F} - \hat{F}_{\dot{X}_i} \dot{X}_i + \mu_j \frac{\partial \psi_j}{\partial t} = 0 && \end{aligned} \tag{2.2.80}$$

(3) Corner Conditions

$$\begin{aligned} &\frac{\partial \hat{F}^-}{\partial \dot{X}_i} = \frac{\partial \hat{F}^+}{\partial \dot{X}_i} \quad ; \; i = 1, n+1 \\ &\hat{F}^- - \hat{F}^-_{\dot{X}_i} \dot{X}^-_i = \hat{F}^+ - F^+_{\dot{X}_i} \dot{X}^+_i \end{aligned} \tag{2.2.81}$$

The Weierstrass and Legendre conditions are the same as before, but with the function $\hat{F}$ replacing F in equations (2.2.74) and (2.2.75), respectively. These conditions for the Mayer problem will be used extensively in the sections that follow. In concluding this brief review, it is noted that all of the necessary conditions used in the Calculus of Variations result from comparing the minimizing arc X_o and the corresponding value of the functional $J(x_o)$ with a neighboring arc infinitesimally removed from x_o. Hence, in in applying the Calculus of Variations to engineering problems, it is not required to draw on the standard necessary condition listed in the literature. Rather, it is possible to use the comparison technique to generate a set of conditions corresponding to the particular problem. This latter approach is usually safer since it avoids the misapplication of standard equations to non-standard problems. In all cases, with the possible exception of the Jacobi condition, the development is both conceptually and algebraically straightforward.

2.3 THE PONTRYAGIN MAXIMUM PRINCIPLE

The Maximum Principle was developed by L. S. Pontryagin and his colleagues at the Steklov Mathematical Institute in Moscow. The principle came to the attention of scientists and engineers in the country through the translation of a series of Russian articles in the late 1950's, the most important of which were three articles by L. I. Rozonoer in Automation and Remote Control. [See Reference (3)]. Since then, the principle has been extensively documented in both the open literature and in many mathematical and engineering texts, with the most complete treatment given by the originators of the principle in Reference (4).

The Maximum Principle is very much like the Calculus of Variations. In fact, it is essentially a generalization of the classical Weierstrass condition. Both the Maximum Principle and the Calculus of Variations lead to the same set of governing equations which the extremizing solution must satisfy. Hence, there is no mathematical preference for using one method as opposed to the other for solving an engineering problem. However, since the Maximum Principle is a clearer and more concise statement of how the optimization is to be conducted (particularly when certain types of inequality constraints are present), its use often facilitates the construction of proofs, and leads to new theorems and numerical methods.

2.3.1 Problem Statement

In the classical Calculus of Variations, the problem of minimizing a functional involves the selection of an appropriate time history for the n-dimensional vector X. During this selection process, no special attention is focussed on the individual components of X as to, say, their physical significance. Rather, the emphasis is on the mathematical structure with the components of X serving merely as dependent variables.

In modern control and trajectory problems this procedure is not followed; the vector X is separated into two sets of components. The first set of components represents the state variables of the problem and the second set the control variables. The reason for this separation is that the state components play a different role in the problem than do the control components; thus, it serves to clarify physical interpretation of the results to distinguish between the two. Mathematically, the distinction is not at all necessary.

The state of a system is usually described as the least amount of information required at the present time to predict the system's behavior at some future instant. For example, the state of a point mass moving in a specified force field is its position and velocity vectors. These two vectors in conjunction with Newton's second law determine the position and velocity at any future time. The control variables are those variables that directly affect the forces acting on the system. For example, the control variables in a chemical rocket could be the steering angle (angle which the thrust vector makes with some reference line) and the throttle setting. A variation in these variables causes a variation in the direction and magnitude of the thrust force acting on the rocket. Thus, the control and state components are counterparts in a cause-effect relationship with the optimization problem consisting in determining the control (cause or input) so that the state (effect or output) evolves in some optimum fashion. The separation of the variables so as to establish this cause-effect relationship usually proves convenient in both problem formulation and in interpretation of results.

In modern control terminology, the optimization problem is usually cast in the following form: Determine the control action u where u is a r-dimensional vector

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{pmatrix}$$

from the set $\mathcal{V}$, where $\mathcal{V}$ is a compact set in the r-dimensional control space*, so that a function of the terminal state is minimized

$$J = \phi(x^f, t^f) = \text{MIN} \tag{2.3.1}$$

subject to the terminal conditions

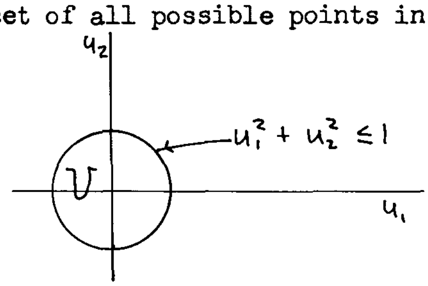
$$\psi_j(x^f, t^f) = 0 \quad ; \quad j = 1, m \leq n \tag{2.3.2}$$

and the differential constraints

$$\dot{x}_i = f_i(x, u) \quad ; \quad i = 1, N \tag{2.3.3}$$

---

*The r-dimensional control space is a Euclidean space and consists of all possible values which the r-dimensional vector $\begin{pmatrix} u_1 \\ u_2 \\ \vdots \end{pmatrix}$ can take with each u_i ranging from $-\infty$ to $+\infty$. For example, if r is 2 and u is a two dimensional vector, the control space is the set of all possible points in the u_1, u_2 plane (see sketch). The set $\mathcal{V}$ denotes a compact region in the control space. For example, in the two-dimensional case, the control vector might be required to lie on or in the interior of the unit circle. In this case $\mathcal{V}$ is the set of vectors $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ for which $u_1^2 + u_2^2 \leq 1$



The term compact is equivalent to the requirement that $\mathcal{V}$ be closed and bounded.

The n-dimensional vector X is the state of the system with the initial state specified by

$$x = x^{\circ} = \begin{pmatrix} x_1^{\circ} \\ x_2^{\circ} \\ \vdots \\ x_n^{\circ} \end{pmatrix} \quad at \quad t = t^{\circ} \tag{2.3.4}$$

Note that this problem is very similar to the Mayer problem treated in Section 2.2.6 but with the state vector (X) replaced by the combined state and control **vectors** $\binom{X}{u}$ and the dimensions of the problem increased from n to n+r.

The set of admissible controls, U, is some specified compact set in the r-dimensional control space from which the optimal control must be selected. In general, most physical problems are such that the set U can be represented as an inequality constraint or as a series of inequality constraints of the type

$$u \in U \Longleftrightarrow g(u) \leq 0 \tag{2.3.5}$$

or

$$u \in U \Longleftrightarrow g_k(u) \leq 0 \qquad ; k = 1, s \tag{2.3.6}$$

In such cases, however, it has been shown that the inequalities can be converted to equalities through the introduction of additional variables, η_k to allow the optimization problem to be treated as a Mayer problem in the Calculus of Variations. The mathematical process would consist of forming the modified functional $\bar{J}$

$$\bar{J} = \phi + \mu_j \psi_j + \int_{t^{\circ}}^{t^f} \left\{ P_i(\dot{x}_i - f_i(x, u)) + \lambda_k (g_k(u) + \eta_k^2) \right\} dt$$

and then applying the standard necessary conditions developed in the preceding section. Such a procedure will be carried out in a subsequent section. For the present, however, the problem will be formulated using the Maximum Principle.

2.3.2 Maximum Principle Formulation

The problem under consideration is the determination of the control u from the set U such that J is minimized

$$J = \phi(x^f, t^f) = MIN \tag{2.3.7}$$

subject to the differential constraints

$$\dot{x}_i = f_i(x, u) \qquad ; \; i = 1, N \tag{2.3.8}$$

and the boundary conditions

$$\begin{aligned} &x = x^\circ \quad at \quad t = t^\circ \\ &\psi_j(x^f, t^f) = 0 \qquad ; \; j = 1, M \end{aligned} \tag{2.3.9}$$

Before attempting to solve or formulate this problem, however, it is necessary to place certain mathematical restrictions on the functions appearing in the problem statement. Assume that

(i) The functions ϕ and ψ_j have continuous first derivatives and bounded second derivatives with respect to all their arguments.

(ii) The functions f_i and $\partial f_i / \partial x_j$ are continuous with respect to all their arguments, while the second derivatives $\frac{\partial^2 f_i}{\partial x_j \partial x_k}$ are bounded.

(iii) The derivatives $\frac{\partial f_i}{\partial u_j}$ exist.

(iv) The derivatives $\frac{\partial f_i}{\partial x_j}$ satisfy a Lipschitz condition with respect to the variables u_k; that is,

$$\left| \frac{\partial f_i}{\partial x_j}(x, \hat{u}) - \frac{\partial f_i}{\partial x_j}(x, u) \right| \le C_{ij} \sum_{k=1}^{r} \left| \hat{u}_k - u_k \right|$$

where C_{ij} are finite positive constants.

These conditions are not severe and will be satisfied in most engineering problems. Also, it will be required that the individual components of the control vector u, in addition to lying in the set $\mathcal{U}$, be piecewise continuous; that is, the class of possible controls on which the search for an optimal is to be conducted must

(i) lie in the set $\mathcal{U}$

(ii) be piecewise continuous

Again, these restrictions are rather weak.

To state the Maximum Principle, two additional quantities must be introduced. The first of these quantities is a function called the Hamiltonian and given by

$$H = P_i f_i (x, u) = H(p, x, u) \tag{2.3.10}$$

where the P_i are variables satisfying the differential equations

$$\dot{P}_i = -\frac{\partial H}{\partial x_i} = -P_j \frac{\partial f_j}{\partial x_i} \tag{2.3.11}$$

and the boundary conditions

$$\left.\begin{aligned} P_i + \mu_j \frac{\partial \psi_j}{\partial x_i} + \frac{\partial \phi}{\partial x_i} &= 0 \qquad ; \; i = 1, n \\ P_i f_i = H = \mu_j \frac{\partial \psi_j}{\partial t} &+ \frac{\partial \phi}{\partial t} \end{aligned}\right\} \text{at } t = t^f \, ^{\neq} \tag{2.3.12}$$

and where the μ_j are constants determined so that both equation (2.3.12) and the boundary conditions of equation (2.3.9) are satisfied. For any given X and P, the Hamiltonian in equation (2.3.10) is a function of the control u only.

Let M(P, X) denote the maximum value of H for u in the set $\mathcal{U}$ and for fixed X and P; that is,

$$M(p, x) = \max_{u \in \mathcal{U}} H(p, x, u)$$

With these definitions, the Maximum Principle can now be stated.

Theorem: Let $u_o(t)$ be an admissible control and $x_o(t)$ the corresponding state of the system with initial value x^o and terminal value in the set $\psi_j(x^f, t^f) = 0$; $j = 1, M$. If u minimizes the functional $J = \phi(x^f, t^f)$, then it is necessary that there exist a non-zero continuous vector function p

$$p(t) = \begin{pmatrix} P_1(t) \\ P_2(t) \\ \vdots \\ P_n(t) \end{pmatrix}$$

$\neq$ The final time t^f may or may not be specified. However, if t^f is set at some value, a , then a terminal constraint of the form

$$\psi(x^f, t^f) = t^f - a = 0$$

must be included.

satisfying the differential equations

$$\dot{P}_i = \frac{-\partial H}{\partial x_i} = -P_j \frac{\partial f_j}{\partial x_i} \qquad ; i = 1, n \tag{2.3.13}$$

and the terminal conditions

$$P_i + \mu_j \frac{\partial \psi_j}{\partial x_i} + \frac{\partial \phi}{\partial x_i} = 0 \tag{2.3.14a}$$

$$H = \frac{\partial \phi}{\partial t} + \mu_j \frac{\partial \psi_j}{\partial t} \tag{2.3.14b}$$

such that for all t, $t^o \leq t \leq t^f$

$$H(p, x_o, u_o) = M(p, x_o) \quad . \tag{2.3.15}$$

that is, the optimal control is that value of u with maximizes the Hamiltonian at each instant along the trajectory. Furthermore, the Hamiltonian is a constant with

$$H(x_o, p, u_o) = C \, . \tag{2.3.16}$$

This constant is observed to be given by equation (2.3.14b) with

$$C = \frac{\partial \phi}{\partial t} + \mu_j \frac{\partial \psi_j}{\partial t} \tag{2.3.17}$$

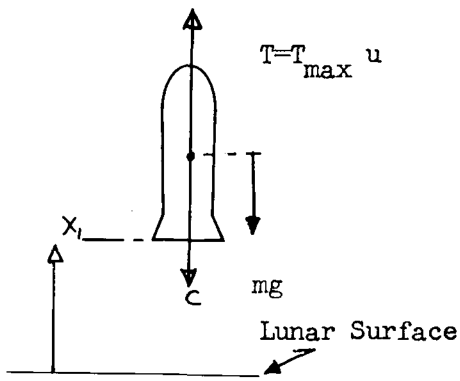
This theorem may appear formidable at first sight, though in fact, it is rather simple to apply. The application requires that the n state constraints of equation (2.3.8) and the n-differential equations governing the P vector, equations (2.3.11), be formed. Then, the boundary conditions of equations (2.3.9) and (2.3.12) are just sufficient to determine a solution to these 2 n equations while the control is selected so that at each point the Hamiltonian is maximized with

$$H(x, p, u_o) \geq H(x, P, \hat{u})$$

where u_o is the optimal control and $\hat{u}$ is any other control (both u_o and $\hat{u}$ are contained in the set $\mathcal{V}$).

Before proving the Maximum Principle, an example will be considered. Suppose a vehicle is to make a vertical descent, soft-landing on the surface of the moon. Let T_{MAX} denote the maximum value of the thrust and let u denote the throttle setting which can vary between zero and unity. Then, the thrust at any instant is given by

$$T = T_{max}\, u = -C\dot{m}$$



where $\dot{m}$ is the time rate of change of mass of the vehicle and C is the exhaust velocity (assumed to be constant). Under the assumption that the vehicle is close enough to the lunar surface for the flat moon, (uniform gravitational field is assumed) the governing equations become

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{T_{MAX}\, u}{x_3} - g \\ \dot{x}_3 &= -\frac{T_{MAX}\, u}{C} \end{aligned} \tag{2.3.18}$$

where

X_1 = distance above the lunar surface

X_2 = vertical velocity

X_3 = mass of the vehicle

g = gravitational acceleration

u = throttle setting ($0 \leq u \leq 1$)

Now, the problem is to determine the control action u from the set U where

$$u \in U \Longleftrightarrow 0 \leq u \leq 1 \tag{2.3.19}$$

so that the fuel expended during the maneuver is a minimum, or alternately, so that the final mass X_3^f is maximized ($-X_3^f$ is minimized)

$$J = \phi(x^f, t^f) = -x_3^f = MIN \tag{2.3.20}$$

The boundary conditions for this problem are

$$
\left.\begin{aligned} X_1 &= X_1^\circ \\ X_2 &= X_2^\circ \\ X_3 &= X_3^\circ \end{aligned}\right\} \quad \text{at} \quad t = t^\circ \tag{2.3.21}
$$

that is, the initial X vector is completely specified. At the terminal time (which is not specified), the conditions are

$$
\left.\begin{aligned} X_1 &= 0 \\ X_2 &= 0 \\ X_3 &= \text{MAXIMUM} \end{aligned}\right\} \quad \text{at} \quad t = t^f \tag{2.3.22}
$$

The conditions that X_1 and X_2 are zero at the final time are the softlanding conditions.

Following the Maximum Principle, the variables P_i, i=1, 3 are introduced which satisfy the equations

$$
\begin{aligned} \dot{P}_1 &= -\frac{\partial H}{\partial X_1} = 0 \\ \dot{P}_2 &= -\frac{\partial H}{\partial X_2} = -P_1 \\ \dot{P}_3 &= -\frac{\partial H}{\partial X_3} = \frac{P_2 T_{MAX} u}{X_3^2} \end{aligned} \tag{2.3.23}
$$

where

$$
H = P_1 X_2 + P_2 \left(\frac{T_{MAX} u}{X_3} - g \right) - \frac{P_3 T_{MAX} u}{c} \tag{2.3.24}
$$

But, the boundary constraints are

$$
\psi_1(X^f, t^f) = X_1^f = 0
$$

$$
\psi_2(X^f, t^f) = X_2^f = 0
$$

therefore, the boundary conditions on the P vector [from equation (2.3.12)] are

$$
P_1 + \mu_1 = 0 \tag{2.3.25a}
$$

$$
P_2 + \mu_2 = 0 \quad \text{at} \quad t = t^f \tag{2.3.25b}
$$

$$
P_3 - 1 = 0 \tag{2.3.25c}
$$

$$
H = 0 \tag{2.3.25d}
$$

Note that equations (2.3.25a and b) provide no additional information since the multipliers μ_1 and μ_2 are unspecified; that is P_1 and P_2 equal certain multipliers which are themselves unspecified. Since the Hamiltonian H is a constant, it follows from equation (2.3.25d) that

$$H(p, x, u) = p_i f_i = 0 \tag{2.3.26}$$

over the entire trajectory.

Rewriting the Hamiltonian as

$$H = T_{MAX} u \left\{ \frac{P_2}{X_3} - \frac{P_3}{c} \right\} + p_1 x_2 - P_2 g \quad ,$$

it follows that the control u which maximizes the Hamiltonian at each instant is given by

$$u = \begin{cases} 0, & \theta < 0 \\ 1, & \theta > 0 \\ \text{arbitrary}, & \theta = 0 \end{cases} \tag{2.3.27}$$

where

$$\theta = \frac{P_2}{X_3} - \frac{P_3}{c} \tag{2.3.28}$$

The variable θ is termed the switching function since its sign determines whether the throttle setting is on the "off" or "full-on" position (i.e., u=zero or unity). If θ is zero over a finite time interval, then the throttle can take some intermediate setting between zero and unity. In this case, the Maximum Principle itself does not provide any information as to the nature of the setting. Such arcs (arcs on which θ is zero) occur infrequently in trajectory and control problems and are termed singular arcs. If no such arc exists, the control is said to be "bang-bang" since it jumps discontinuously from one extreme setting (u=0) to another (u=1).

Collecting results, the minimum fuel soft landing problem requires the solution of the six differential equations

$$\begin{aligned} \dot{X}_1 &= X_2 \\ \dot{X}_2 &= \frac{T_{MAX} u}{X_3} - g \\ \dot{X}_3 &= -\frac{T_{MAX} u}{c} \\ \dot{P}_1 &= 0 \\ \dot{P}_2 &= -P_1 \\ \dot{P}_3 &= \frac{P_2 T_{MAX} u}{X_3^2} \end{aligned} \tag{2.3.29}$$

and the boundary conditions

$$\begin{aligned} x_1 &= x_1^{\circ} \\ x_2 &= x_2^{\circ} \quad \text{at } t = t^{\circ} \\ x_3 &= x_3^{\circ} \\ x_1 &= 0 \\ x_2 &= 0 \quad \text{at } t = t^{f} \\ P_3 &= 1 \end{aligned} \tag{2.3.30}$$

with the Hamiltonian zero over the trajectory

$$P_1 x_2 + P_2\left(\frac{T_{MAX}u}{x_3} - g\right) - P_3 \frac{T_{MAX}u}{c} = 0 \tag{2.3.31}$$

The control action is selected at each point so that

$$\begin{aligned} u_1 &= 0, \quad \theta < 0 \\ u_1 &= 1, \quad \theta > 0 \\ u_1 &\text{ arbitary}, \quad \theta = 0 \\ \theta &= \frac{P_2}{x_3} - \frac{P_3}{c} \end{aligned} \tag{2.3.32}$$

Note that a solution can be rather easily generated provided no singular arcs occur along which $\theta = 0$. The fact that no such arcs are possible for this problem is the subject of the following paragraphs.

If $\theta = 0$ over a finite time interval then $\dot{\theta}$ must also be zero over this interval. But

$$\dot{\theta} = -\frac{P_1}{x_3} \tag{2.3.33}$$

and so P_1 (which is a constant from equation (2.3.29) must be identically zero. However, if P_1 is zero, then P_2 must be a constant from equation (2.3.23) and that constant must be zero for the Hamiltonian in equation (2.3.31) to vanish; thus, it follows from equation (2.3.29) that P_2 is identically zero over the entire trajectory. But substitution of this result into the sixth of equation (2.3.29) along with the boundary condition of equation (2.3.30) provides the result $P_3 = 1$. Therefore, for θ to be zero over a segment of finite length $P_1 = P_2 = 0$, $P_3 = 1$.

But then

$$\theta = \frac{P_2}{x_3} - \frac{P_3}{c} = 0 - \frac{1}{c} = \frac{1}{c}$$

which is no longer zero and a contradiction results. Thus no singular arcs exist.

Since the solution contains no singular arcs, the next question of interest concerns the number of zero thrust and full thrust arcs. It can be shown that the optimal solution contains at most one switch; that is, if the control switches from the zero to unity position, or from the unity to the zero position, it does so no more than once. If P_1 is zero [note from equation (2.3.29) that P_1 is a constant] then from (2.3.33) $\dot{\theta}$ is zero and θ is a constant over the trajectory. Thus, the solution consists of one arc on which u is either unity or zero. The zero case can be ruled out on physical grounds since a soft landing can not be made without expenditure of fuel. If P_1 is not zero, then $\dot{\theta}$ is never zero and the switching function is monotonic in the interval $[t^{\circ}, t^{f}]$. Therefore there can be at most one point where $\theta = 0$. Hence, the solution contains only one switch. Again, it is obvious on physical grounds that if there is a switch, the engine must go from the "full-off" to the "full-on" position. The optimal solution, then, consists of a free fall arc with $u = 0$ followed by a maximum thrust arc along which $u = 1$.

The full solution to the soft landing problem can be obtained graphically by a simultaneous solution of equations (2.3.29) subject to the boundary condition of equations (2.3.30) and (2.3.31) and the control law of equation (2.3.32). However, since the purpose of this example has been to illustrate the application of the Maximum Principle, there is no need to produce the solution here. The interested reader should consult Reference (5) for the detailed solution.

2.3.3 Proof of the Maximum Principle

To minimize the functional J where

$$J = \phi\,(x^f,\ t^f)$$

subject to the differential constraints

$$\dot{x}_i = f\ \ (x,u)$$

and the boundary conditions

$$x = x^\circ \text{ at } t = t^\circ$$

$$\psi_j\,(x^f,\ t^f) = 0; \qquad j = 1,\ m \leq n \tag{2.3.34}$$

the multipliers P_i are introduced which satisfy the equations

$$\dot{P}_i = -P_j \frac{\partial f_j}{\partial x_i}\ ; \qquad i = 1, N \tag{2.3.35}$$

and

$$\left.\begin{aligned} &P_i + \mu_j \frac{\partial \psi_j}{\partial x_i} + \frac{\partial \phi}{\partial x_i} = 0\ ;\ i = 1, n \\ &P_i f_i = \mu_j \frac{\partial \psi_j}{\partial t} + \frac{\partial \phi}{\partial t} \end{aligned}\right\} \quad \text{at } t = t^f \tag{2.3.36}$$

Next, it is noted that the Hamiltonian H given by

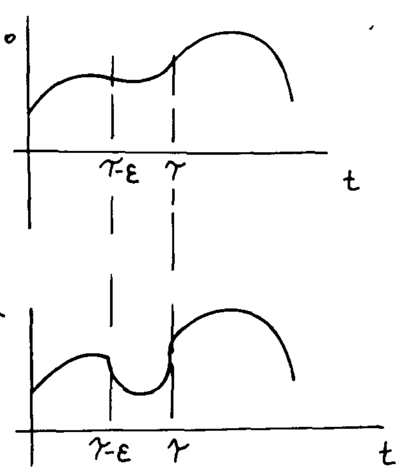
$$H = P_i\ f_i$$

is a constant and obeys the equation

$$H(p, x_o, u_o) = C = \mu_j \frac{\partial \psi_j}{\partial t} + \frac{\partial \phi}{\partial t} \tag{2.3.37}$$

Finally, the control u is selected from the set $\mathcal{U}$ so that H(x,p,u) is maximized for each value of x and p along the optimal solution.

The proof the Maximum Principle proceeds as follows. Let u_o and x_o denote the optimal control and state vectors and let u and x denote any other control and state where it is assumed that u_o and u are contained in the set $\mathcal{U}$ and that x and x_o satisfies the specified boundary conditions of equations (2.3.34). Now, require that u and u_o be identical on $[t^\circ, t^f]$ except for a small subinterval $[\tau - \varepsilon,\ \tau]$ (shown on the sketch to the right) where ε is a small quantity. Next, the modified functional J is formed



$$\bar{J} = \phi + \mu_j \psi_j + \int_{t^o}^{t^f} p_i \{ \dot{x}_i - f_i(x, u) \} \, dt$$

where μ_j are constants and where the p vector satisfies equations (2.3.35) and (2.3.36).

If u_o is minimizing, then

$$\bar{J}(u) - \bar{J}(u_o) \geq 0$$

and this inequality can be written

$$\bar{J}(u) - \bar{J}(u_o) = \phi(x^f, t^f) - \phi(x_o^f, t_o^f) + \mu_j \{ \psi_j(x^f, t^f) - \psi_j(x_o^f, t_o^f) \}$$
$$+ \int_{t^o}^{t^f} p_i(\dot{x}_i - f_i(x, u)) \, dt_o - \int_{t^o}^{t_o^f} p_i(\dot{x}_{o_i} - f_i(x_o, u_o)) \, dt \geq 0 \quad (2.3.38)$$

where the subscript zero indicates the optimal solution. Thus, since both integrals in (2.3.38) are zero and since the Hamiltonian obeys the equation $H = p_i f_i$, it follows that

$$\bar{J}(u) - \bar{J}(u_o) = \phi(x^f, t^f) - \phi(x_o^f, t_o^f) + \mu_j \{ \psi_j(x^f, t^f) - \psi_j(x_o^f, t_o^f) \}$$

$$+ \int_{t^o}^{t_o^f} p_i(\dot{x}_i - \dot{x}_{o_i}) \, dt - \int_{t^o}^{t_o^f} \{ H(p, x, u) - H(p, x_o, u_o) \} \, dt \geq 0 \quad (2.3.39)$$

Now, under the condition that u and u_o differ only in the small interval $[\tau-\varepsilon, \tau]$, it follows that the solution X(t) can be written as*

$$x(t) = x_o(t) + \varepsilon\, \delta x(t) + 0(\varepsilon) \tag{2.3.40}$$

where o (ε) indicates higher order terms in ε (i.e., $\lim_{\varepsilon \to 0} (o(\varepsilon))/\varepsilon = o$). Note that u and u_o may differ drastically on $[\tau-\varepsilon, \tau]$, but since ε is small, the effect of this difference on the solution x will be small. If this representation is substituted into equation (2.3.39), the boundary conditions become

$$\phi(x_j^f, t^f) - \phi(x_o^f, t_o^f) + \mu_j \left\{ \psi_j(x^f, t^f) - \psi_j(x_o^f, t_o^f) \right\} =$$

$$\varepsilon\left(\frac{\partial \phi}{\partial x_i} + \mu_j \frac{\partial \psi_j}{\partial x_i}\right)\delta x_i + \varepsilon\left\{\frac{\partial \phi}{\partial t} + \mu_j \frac{\partial \psi_j}{\partial t} + \dot{x}_{oi}\left(\frac{\partial \phi}{\partial x_i} + \mu_j \frac{\partial \psi_j}{\partial x_i}\right)\right\} dt + o(\varepsilon) \tag{2.3.41}$$

where the condition

$$\delta x_i = dx_i - \dot{x}_i\, dt$$

is employed.

The first integral in (2.3.39), in view of the continuity on the p vector and the fact that the initial state is specified, can now be written as

$$\begin{aligned}\int_{t^o}^{t^f} p_i(\dot{x}_i - \dot{x}_{oi})dt &= \varepsilon\, p_i\, \delta x_i\Big]^{t^f_o} - \varepsilon \int_{t^o}^{t^f} \delta x_i\, \dot{p}_i\, dt + o(\varepsilon) \\ &= \varepsilon\, p_i\, \delta x_i\Big]^{t^f_o} + \varepsilon \int_{t^o}^{t^f} \delta x_j\, p_i \frac{\partial f_j}{\partial x_i} dt + o(\varepsilon)\end{aligned} \tag{2.3.42}$$

At this point, if the representation of $x = x\ (t, \varepsilon)$ in equation (2.3.40) is substituted into the second integral in equation (2.3.39), the result is

$$\begin{aligned}\int_{t^o}^{t^f} \left\{H(x, p, u) - H(x_o, u_o, p)\right\} dt &= \int_{t^o}^{t^f} \left\{H(x_o, p, u) - H(x_o, p, u_o)\right\} dt \\ &+ \varepsilon \int_{t^o}^{t^f} \frac{\partial H}{\partial x_i}(x_o, p, u)\, \delta x_i\, dt + o(\varepsilon)\end{aligned} \tag{2.3.43}$$

* that is, x(t) approaches $x_o(t)$ uniformly as $\varepsilon \longrightarrow 0$.

Now substituting equations (2.3.41) to (2.3.43) into the inequality (2.3.39) provides

$$\bar{J}(u)-\bar{J}(u_o) = \varepsilon\left\{\frac{\partial\phi}{\partial x_i}+\mu_j\frac{\partial\psi_j}{\partial x_i}+p_i\right\}\delta x_i$$

$$+\varepsilon\left\{\frac{\partial\phi}{\partial t}+\mu_j\frac{\partial\psi_j}{\partial t}+\dot{x}_{o_i}\left(\frac{\partial\phi}{\partial x_i}+\mu_j\frac{\partial\psi_j}{\partial x_i}\right)\right\}dt\Bigg]^{t_o^f}$$

$$-\int_{t^o}^{t_o^f}\left\{H(x_o,u,p)-H(x_o,u_o,p)\right\}dt$$

$$-\varepsilon\int_{t^o}^{t_o^f}\frac{\partial}{\partial x_i}\left\{H(x_o,u,p)-H(x_o,u_o,p)\right\}\delta x_i\,dt+O(\varepsilon)\geq 0$$

Finally, in view of the boundary conditions of (2.3.36) and (2.3.37), this expression reduces to

$$\bar{J}(u)-\bar{J}(u_o)=\int_{t^o}^{t_o^f}\left\{H(x_o,u_o,p)-H(x_o,u,p)\right\}dt+\varepsilon\int_{t^o}^{t_o^f}\frac{\partial}{\partial x_i}\left\{H(x_o,u_o,p)-H(x_o,u,p)\right\}\delta x_i\,dt$$
$$+O(\varepsilon)\geq 0 \qquad (2.3.44)$$

At this point, recall that u and u_o differ only on a small interval $[\tau-\varepsilon,\tau]$. Hence (2.3.44) becomes

$$\bar{J}(u)-\bar{J}(u_o)=\int_{\tau-\varepsilon}^{\tau}\left\{H(x_o,u_o,p)-H(x_o u,p)\right\}dt+\varepsilon\int_{\tau-\varepsilon}^{\tau}\frac{\partial}{\partial x_i}\left\{H(x_o,u_o,p)-H(x_o,u,p)\right\}\delta x_i\,dt$$
$$+O(\varepsilon)\geq 0 \qquad (2.3.45)$$

Now, from the definition of the Hamiltonian

$$\frac{\partial}{\partial x_i}\left[H(x_o,u_o,p)-H(x_o,u,p)\right]=p_j\left(\frac{\partial f_j}{\partial x_i}(x_o,u_o,p)-\frac{\partial f_j}{\partial x_i}(x_o,u,p)\right) \qquad (2.3.46)$$

Thus, since $(\partial f_j/\partial x_i)$ satisfies a Lipschitz condition in u, the quantity in (2.3.46) is bounded. Also δx_i is bounded from which it follows that the integrand in the second integral in (2.3.45) is bounded; that is

$$\left|\frac{\partial}{\partial x_i}\left\{H(x_o,u_o,p)-H(x_o,u,p)\right\}\delta x_i\right|\leq K$$

where K is a finite number. Hence.

$$\varepsilon\int_{\tau-\varepsilon}^{\tau}\frac{\partial}{\partial x_i}\left[H(x_o,u_o,p)-H(x_o,u,p)\right]\delta x_i\,dt\leq K\varepsilon^2$$

and to o(ε) equation (2.3.45) becomes

$$\bar{J}(u) - J(u_o) = \int_{\tau-\varepsilon}^{\tau} \left\{ H(X_o, u_o, p) - H(X_o, u, p) \right\} dt + o(\varepsilon) \geq 0 \tag{2.3.47}$$

Further, the integral in (2.3.47) is first order in ε , so as $\varepsilon \longrightarrow 0$ with $x \longrightarrow x_o$ and $u \longrightarrow u_o$, the dominant term in (2.3.47) is the integral. Thus, it follows that

$$\int_{\tau-\varepsilon}^{\tau} \left\{ H(X_o, u_o, p) - H(X_o, u, p) \right\} dt \geq 0 \tag{2.3.48}$$

To prove the Maximum Principle, it must be shown that

$$H(X_o, u_o, p) \geq H(X_o, u, p) \tag{2.3.49}$$

at each point in the interval (t^o, t^f). The proof is developed by contradiction. Let τ in (2.3.48) be a regular point where both u_o and u are continuous and assume that

$$H(X_o(\tau), u_o(\tau), p(\tau)) - H(X_o(\tau), u(\tau), p(\tau)) = -a^2 < 0 \tag{2.3.50}$$

Since x and p are continuous and u and u_o are piecewise continuous, there will be some region surrounding the point τ for which the integrand in (2.3.48) will be negative. Let ε be sufficiently small so that the integrand is negative in the entire subinterval $[\tau - \varepsilon, \tau]$ from which it follows that

$$\int_{\tau-\varepsilon}^{\tau} \left\{ H(X_o, u_o, p) - H(X_o, u, p) \right\} dt < 0$$

a contradiction. Thus,

$$H(X_o(\tau), u_o(\tau), p(\tau)) - H(X_o(\tau), u(\tau), p(\tau)) \geq 0 \tag{2.3.51}$$

at every regular point τ in the interval $[t^o, t^f]$, that is every point in the interval except a finite number of points. At a discontinuous point τ , the piecewise continuity condition of u requires that $u(t) \longrightarrow u(\tau^-)$ continuously as $t \longrightarrow \tau$ from the right. Hence, if

$$H(X_o(\tau^-), u_o(\tau^-), p(\tau^-)) - H(X_o(\tau^-), u_o(\tau^-), p(\tau^-)) = -a^2 < 0$$

then it can be shown that the inequality in (2.3.48) is again reversed and a contradiction results. A similar situation holds at $\tau = \tau^+$. Thus, (2.3.49) must be valid on both sides of a discontinuity as well as at every regular point in $[t^o, t^f]$.

One final condition must be demonstrated to complete the proof of the Maximum Principle - the constancy of the Hamiltonian. This proof is accomplished by showing that the total derivative dH/dt is zero.
Since $H = p_i f_i$, it follows that

$$\frac{dH}{dt} = \frac{\partial H}{\partial x_i}\dot{x}_i + \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial u_k}\dot{u}_k$$

But the x and p vectors satisfy the differential equations

$$\dot{x}_i = \frac{\partial H}{\partial p_i}$$

$$\dot{p}_i = \frac{-\partial H}{\partial x_i}$$

Hence,

$$\frac{dH}{dt} = \frac{\partial H}{\partial u_k}\dot{u}_k \qquad (2.3.52)$$

Now, if the optimal control lies in the interior of $\mathcal{U}$, then for H to be maximized

$$\frac{\partial H}{\partial u_k} = 0 \quad ; \quad k = 1, r$$

from which it follows that dH/dt is zero. In contrast, if u lies on a section of the boundary of $\mathcal{U}$, let this boundary be represented by the constraint equation $g(u)=0$, and the two equations

$$\frac{\partial}{\partial u_k}(H + \lambda g) = 0 \qquad (2.3.53)$$

$$g(u) = 0$$

determine the optimal control. But if λ is zero, then so are $\partial H/\partial u_k$ and dH/dt and the proof is complete. So assume $\lambda \neq 0$, then from continuity considerations $\lambda \neq 0$ over some finite time interval with $g(u) = 0$ on this interval and with the derivative

$$\frac{dg}{dt} = \frac{\partial g}{\partial u_k}\dot{u}_k = 0 \qquad (2.3.54)$$

Combining equations (2.3.54) and (2.3.53) provides $\frac{\partial H}{\partial u_k} = 0$, it follows from (2.3.52) that,

$$\frac{dH}{dt} = \frac{\partial H}{\partial u_k}\dot{u}_k = 0$$

Hence, the Hamiltonian H is a constant over the entire trajectory. This completes the proof of the Maximum Principle.

2.3.4 The Orbital Transfer Problem

As an example of the application of the Maximum Principle, consider the problem of transferring a vehicle between two coplanar circular orbits in space. Let $T = T_{max}\, u_1$ denote the thrust (nondimensional) magnitude where u_1 is again the throttle setting with $0 \leq u_1 \leq 1$ and let u_2 denote the steering angle. The equations of motion can now be written in terms of these parameters. However, to simplify the formulation, the constants of the problem will be selected so as to simplify the equations of motion. Taking as the unit of length, the radius of the initial orbit; as the unit of velocity, the velocity of the initial orbit; as the unit of time, the period of the initial orbit over 2π; and as the unit of mass, the initial mass of the vehicle, the equations of motion become

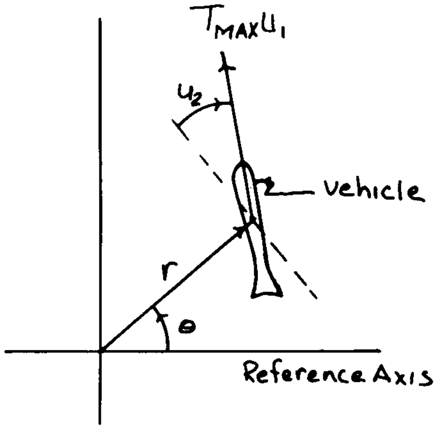
$$\frac{dr}{dt} = V$$

$$\frac{d\hbar}{dt} = \frac{r T_{MAX} u_1}{m} \cos u_2$$

$$\frac{dV}{dt} = \frac{T_{MAX} u_1 \sin u_2}{m} + \frac{\hbar^2}{r^3} - \frac{1}{r^2}$$

$$\frac{dm}{dt} = -\frac{T_{MAX} u_1}{C} \qquad (2.3.55)$$

$$\frac{d\theta}{dt} = \frac{\hbar}{r^2}$$



where

r = distance from the center of attraction

h = angular momentum (per unit mass)

v = radial velocity

θ = central range angle

T_{max} = nondimensional maximum thrust magnitude (thrust per g of the initial orbit)

u_1 = throttle setting

u_2 = steering angle

c = nondimensional exhaust velocity

With this choice of units, the boundary conditions for the circular orbit transfer are

$$\left.\begin{aligned} r &= 1 \\ h &= 1 \\ v &= 0 \\ m &= 1 \\ \theta &= 0 \end{aligned}\right\} \quad t = t^{o} \tag{2.3.56}$$

and

$$\begin{aligned} r &= r^{f} \text{ (specified)} \\ h &= h^{f} = \sqrt{r_f} \\ v &= v^{f} = 0 \end{aligned} \tag{2.3.57}$$

Note that the values of r^f, h^f and v^f determine the final circular orbit and must be specified as indicated in equation (2.3.57), while the values of m^f, θ^f and t^f may or may not be specified depending on the particular problem to be solved and the optimizing criterion.

The optimum orbital transfer problem has been extensively analyzed in the literature with the most complete treatments given in References (6) and (7). In the particular formulation used here, it is assumed that the vehicle is thrust limited with the thrusting engine allowed to take any magnitude between zero and T_{max}. An alternate formulation removes the thrust magnitude constraint and requires instead that the jet power of the vehicle have some finite upper bound. This power limited case is treated in References (8) and

(9) with an analytical approximation given in Reference (10).

To complete the statement of the problem, the optimizing criterian [the function ϕ in equation (2.3.1)] must be specified. Two different criteria and, hence, two different types of transfers will be considered here: the minimum time transfer and the minimum fuel transfer.

A. Minimum Time Transfer

To minimize the time to transfer, the performance function is set equal to the final time; that is,

$$J = \phi(x^f, t^f) = t^f. \tag{2.3.58}$$

Since the problem has five state variables, r, h, v, m and θ, the five multipliers P_r, P_h, P_v, P_m and P_θ are introduced which obey the equations

$$\begin{aligned}
\dot{P}_r &= -\frac{\partial H}{\partial r} = -P_h\left\{\frac{T_{MAX}u_1\cos u_2}{m}\right\} - P_v\left(\frac{2}{r^3} - \frac{3h^2}{r^4}\right) + P_\theta\frac{2h}{r^3} \\
\dot{P}_h &= -\frac{\partial H}{\partial h} = -P_v\frac{2h}{r^3} - P_\theta\frac{1}{r^2} \\
\dot{P}_v &= -\frac{\partial H}{\partial v} = -P_r \\
\dot{P}_m &= -\frac{\partial H}{\partial m} = P_h\left(\frac{rT_{MAX}u_1\cos u_2}{m^2}\right) + P_v\left(\frac{T_{MAX}u_1\sin u_2}{m^2}\right) \\
\dot{P}_\theta &= -\frac{\partial H}{\partial \theta} = 0
\end{aligned} \tag{2.3.59}$$

where

$$\begin{aligned}
H = P_r v + P_h\frac{rT_{MAX}u_1\cos u_2}{m} + P_v\left\{\frac{T_{MAX}u_1\sin u_2}{m} + \frac{h^2}{r^3} - \frac{1}{r^2}\right\} \\
- P_m\frac{T_{MAX}u_1}{c} + P_\theta\frac{h}{r^2}
\end{aligned} \tag{2.3.60}$$

From equation (2.3.57) there are at least three terminal constraints

$$\left.\begin{aligned}
\psi_1 &= r - r^f = 0 \\
\psi_2 &= h - h^f = 0 \\
\psi_3 &= v - v^f = 0
\end{aligned}\right\} t = t^f \tag{2.3.61}$$

where r^f, h^f and v^f are specified members as indicated in (2.3.57). In addition, there may be a fourth constraint if θ^f is specified; that is, if the angle in which the transfer is to be completed is specified, then a fourth constraint

$$\Psi_4 = \theta - \theta^f = 0; \qquad t = t^f \tag{2.3.62}$$

must be included. From equation (2.3.12), the terminal conditions on the p vector take the form

$$\begin{aligned} P_r^f + \mu_1 &= 0 \\ P_h^f + \mu_2 &= 0 \\ P_v^f + \mu_3 &= 0 \\ P_m^f &= 0 \\ P_\theta^f + \mu_4 &= 0 \end{aligned} \tag{2.3.63}$$

where μ_4 is zero ($P_\theta^f = 0$) if the final angle is not specified. Since the constants μ_i are uncoupled and do not appear elsewhere in the problem, equation (2.3.63) reduces to the statement that P_r^f, P_h^f and P_v^f are unspecified at the final time, while $P_m^f = 0$. Also, since t^f is unconstrained, equation (2.3.16) and (2.3.17) become

$$H = \text{const.} = 0 \tag{2.3.64}$$

The controls u_1 and u_2 are to be selected to maximize the Hamiltonian at each point along the trajectory with the set of admissible controls given by

$$U = \begin{cases} 0 \le u_1 \le 1 \\ u_2 \text{ arbitrary but finite} \end{cases}$$

To define these controls, rewrite equation (2.3.60) in the form

$$H = H_1(u) + H_2 = \frac{T_{MAX} u_1}{m} \left\{ r P_h \cos u_2 + P_v \sin u_2 - \frac{P_m m}{c} \right\}$$
$$+ P_r v + P_v \left(\frac{h^2}{r^3} - \frac{1}{r^3} \right) + P_\theta \frac{h}{r^2}$$

where H_2 does not depend on u_1 or u_2. Now, letting

$$\frac{r P_h}{\sqrt{P_v^2 + r^2 P_h^2}} = \cos Z$$

$$\frac{P_v}{\sqrt{P_v^2 + r^2 P_h^2}} = \sin Z$$

the quantity $H_1(u)$ reduces to

$$H_1(u) = \frac{T_{MAX}\, u_1}{m} \sqrt{P_v^2 + r^2 P_h^2} \left\{ \cos(Z - u_2) - \frac{P_m\, m}{C\sqrt{P_v^2 + P_h^2 r^2}} \right\}$$

from which it follows that u_2 must equal Z to maximize the Hamiltonian; i.e.,

$$\cos u_2 = \frac{r P_h}{\sqrt{P_v^2 + r^2 P_h^2}} \qquad\qquad (2.3.65)$$

$$\sin u_2 = \frac{P_v}{\sqrt{P_v^2 + r^2 P_h^2}}$$

Hence

$$H_1(u) = \frac{T_{MAX}\, u_1}{m} \left\{ \sqrt{P_v^2 + r^2 P_h^2} - \frac{P_m\, m}{C} \right\}$$

with the control u_1 which maximizes H, given by

$$u_1 = \begin{cases} 1.0, & \theta > 0 \\ 0.0, & \theta < 0 \\ \text{arbitrary}, & \theta = 0 \end{cases} \qquad\qquad (2.3.66)$$

where

$$\theta = \sqrt{P_v^2 + r^2 P_h^2} - \frac{P_m\, m}{C} \qquad\qquad (2.3.67)$$

Equation (2.3.66) indicates that the optimal transfer trajectory consists of arcs of null thrust ($u_1 = 0$), arcs of maximum thrust ($u_1 = 1$), and possibly intermediate thrusting arcs if the switching function θ vanishes over a finite time interval. In this last case, the arc is called a "singular arc" and the Maximum Principle fails to provide any information on the optimal control action.

It is a rather simple matter to show that no singular arcs or zero thrust arc can occur in a minimum time transfer maneuver. Substituting equation (2.3.65) into the fourth of equation (2.3.59) provides

$$\dot{P}_m = \frac{\sqrt{P_v^2 + r^2 P_\theta^2} \quad T_{MAX} u_1}{m^2} \tag{2.3.68}$$

Combining this equation with the condition that P_m^f is zero, equation (2.3.63), it follows that P_m is never positive. Hence, the switching function, Θ in equation (2.3.66), can never be negative or zero and the engine is on full during the entire maneuver (i.e., $u_1 = 1.0$, $T = T_{max}$).

Summarizing results, the minimum time transfer is accomplished by setting u_1 equal to unity and selecting u_2 to satisfy equation (2.3.65). The state and P vectors of equations (2.3.55) and (2.3.59) constitute a tenth-order system (ten first-order differential equations) and since the final time is not specified, eleven boundary conditions are needed to construct a solution. Five of these come from the initial conditions of equation (2.3.56). Three more come from the terminal orbit conditions of (2.3.57) while equations (2.3.63) and (2.3.64) provide the ninth and tenth conditions

$$P_m^f = 0$$
$$H = H^f = 0$$

The last condition is either

$$\theta = \theta_f \text{ (specified)}; \; t = t^f$$

or

$$P_\theta^f = 0$$

depending on whether the angle in which the transfer is to be completed is specified.

B. <u>Minimum Fuel Transfer</u>

The fuel used during the transfer maneuver is the difference between the initial mass (which is unity due to the choice of units) and the terminal mass. Hence, for a minimum fuel transfer the performance index is given by

$$J = \phi\,(X^f, t^f) = 1 - m^f$$

where in this case, the final time t^f is specified to be some value greater than the minimum time needed to complete the transfer.

The state equations and boundary conditions are again given by equations (2.3.55) to (2.3.57) with the P vector and Hamiltonian satisfying (2.3.59) and (2.3.60). The boundary conditions on the P vector are

$$P_m^f = 1.0$$
$$P_\theta^f = \begin{cases} 0, \text{ if } \theta_f \text{ unspecified} \\ \text{arbitrary, if } \theta_f \text{ specified} \end{cases} \tag{2.3.69}$$

As before, the optimal control action takes the form

$$\cos u_2 = \frac{r P_h}{\sqrt{P_v^2 + r^2 P_h^2}} \quad ; \quad \sin u_2 = \frac{P_v}{\sqrt{P_v^2 + P_h^2 r^2}} \tag{2.3.70}$$

$$U_1 = \begin{cases} 1.0, & \theta > 0 \\ 0.0, & \theta < 0 \\ \text{arbitrary}, & \theta = 0 \end{cases} \tag{2.3.71}$$

where

$$\theta = \sqrt{P_v^2 + P_h^2 r^2} - \frac{P_m m}{C} \tag{2.3.72}$$

In this case, however, neither null thrust arc ($u_1 = 0$) nor singular arc ($0 < u_1 < 1$) can be ruled out.

That null thrust arcs do occur is rather well established by the numerical results contained in the literature [see References (7) and (11) for example]. Whether singular arcs occur is not known at this writing. As yet no optimal solution has been found which contains a singular arc.

For a singular arc to occur, the switching function θ must be zero over a finite time interval. In this case, the Maximum Principle becomes degenerate (the Hamiltonian does not contain u_1 explicitly and hence u_1 can not be chosen to maximize H) and provides no information to aid in the determination of u_1. However, the condition that θ is to be zero over the interval can be used to compute u_1; that is, u_1 is chosen so that θ is identically zero over the arc.

Some recent work in regards to both the existence and minimality of singular arcs in the orbital transfer problem (References 10, 12, and 13) indicates that the occurrence of such arcs is highly unlikely. For θ to be zero over a finite interval, $\dot{\theta}$, $\ddot{\theta}$, $\dddot{\theta}$ ... etc. must also vanish. The mechanism for keeping θ zero is the control variable u_1. Hence, u_1 must be selected so that the first derivative of θ, which contains u_1 explicitly, is zero. It can be shown that the fourth derivative, $d^4\theta/dt^4$, is the first such derivative. Therefore, from the continuity condition on the P vector, it follows that to initiate a singular arc at some time t_k the four conditions

$$\theta = 0$$

$$\dot{\theta} = 0$$

$$\ddot{\theta} = 0$$

$$\dddot{\theta} = 0$$

must be simultaneously satisfied. At any point where θ is zero, it would be unusual for $\dot{\theta}$ to vanish let alone $\ddot{\theta}$ and $\dddot{\theta}$. For this reason, the existence of such arcs is seldom considered in the orbital transfer problem, a procedure which will be followed here.

From this discussion, it is assumed that the switching function will vanish only at a finite number of points and equation (2.3.71) is replaced by

$$u_1 = \begin{cases} 1.0, \theta > 0 \\ 0, \theta \leq 0 \end{cases} \tag{2.3.73}$$

Under this assumption, the optimal transfer trajectory is seen to consist of coasting arcs along which the thrust is zero and powered arcs along which the thrust is set at its maximum value.

Summarizing, the minimum fuel transfer is determined from the solution of the tenth-order system, equations (2.3.55) and (2.3.59), which satisfies the five initial conditions of equations (2.3.56) and the five terminal conditions of equations (2.3.57) and (2.3.69) with the final time specified. The optimal control action is computed from equations (2.3.70) and (2.3.73).

2.3.5 Maximum Principle and the Calculus of Variations

It has been mentioned previously that the Maximum Principle is very similar to the Calculus of Variations. To show the relationship between the two methods, the optimization problem used in the statement of the Maximum Principle will be reformulated within the variational framework. For convenience, this problem is restated as follows: Determine the control u from the set U such that a function of the terminal state is minimized

$$J = \phi(x^f, t^f) = MIN. \tag{2.3.74}$$

subject to the differential constraints

$$\dot{x}_i = f_i(x, u) \quad ; \quad i = 1, n \tag{2.3.75}$$

and the boundary conditions

$$\begin{aligned} x &= x^0 \; ; \quad t = t^0 \\ \psi_j(x^f, t^f) &= 0 \quad ; \quad j = 1, m \end{aligned} \tag{2.3.76}$$

To use the variational methods, it is necessary to assume that the functions ϕ, ψ, and f_i have continuous first derivatives and bounded second derivatives with respect to all their arguments. This condition is slightly stronger than that used in the Maximum Principle [below equation

(2.3.9)]. Also, it is required that the admissible control set U be represented by an inequality constraint of the form

$$u \in U \Longleftrightarrow g(u) \leq 0$$

where g(u) has continuous first derivatives and bounded second derivatives along its boundary. Such a representation is almost always possible in engineering problems. For example, in the orbital transfer problem the set U is given by

$$u \in U \text{ for } \begin{cases} 0 \leq u_1 \leq 1 \\ u_2 \text{ arbitrary} \end{cases}$$

with the equivalent inequality constraint taking the form

$$u \in U \Longleftrightarrow g(u) = u_1(u_1 - 1) \leq 0$$

In the variational treatment no distinction is made between control and state vectors. Both are simply considered to be dependent variables. However, an assumption in the variational formulation is that all variables, X_i and u_k are continuous - a situation that may not hold in regards to the control variables (as for example in the orbital transfer problem where u_1 goes discontinuously from zero to unity). To remove this difficulty the transformation*

$$u = \dot{z} \tag{2.3.77}$$

is used and $\dot{z}$ is substituted for u in the differential constraints

$$\dot{X}_i = f_i(X, \dot{z}) \tag{2.3.78}$$

and in the control region constraint

$$g(\dot{z}) \leq 0 \tag{2.3.79}$$

By this device the dependent variables X and Z are made continuous with discontinuities occuring only in the first and higher derivatives.

*Also, application of the Weierstrass condition runs into trouble without this transformation

To minimize J, the modified functional $\bar{J}$ is formed

$$\bar{J} = \phi(x^f, t^f) + \mu_j \psi_j(x^f, t^f) + \int_{t^o}^{t^f} F\, dt \tag{2.3.80a}$$

where

$$F \equiv 0 = p_i \left\{ \dot{x}_i - f_i(x, \dot{z}) \right\} + \lambda \left(g(\dot{z}) + \dot{\eta}^2 \right) \tag{2.3.80b}$$

and where η is a real variable introduced to convert the inequality in (2.3.79) to an equality. Setting the first variation of $\bar{J}$ to zero provides [starting with equation (2.2.78)]

(1) Euler Equation

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}_i}\right) - \frac{\partial F}{\partial x_i} = 0 \quad ; \quad i = 1, n \tag{2.3.81}$$

$$\frac{d}{dt} \frac{\partial F}{\partial \dot{z}_k} - \frac{\partial F}{\partial z_k} = 0 \quad ; \quad k = 1, r \tag{2.3.82a}$$

$$\frac{d}{dt} \frac{\partial F}{\partial \dot{\eta}} - \frac{\partial F}{\partial \eta} = 0 \tag{2.3.82b}$$

(2) Boundary Conditions

$$F_{\dot{x}_i} + \mu_j \frac{\partial \psi_j}{\partial x_i} + \frac{\partial \phi}{\partial x_i} = 0 \quad ; \quad i = 1, n \tag{2.3.83}$$

$$F_{\dot{z}_k} + \mu_j \frac{\partial \psi_j}{\partial z_k} + \frac{\partial \phi}{\partial z_k} = 0 \quad ; \quad k = 1, r$$

$$F_{\dot{\eta}} + \mu_j \frac{\partial \psi_j}{\partial \eta} + \frac{\partial \phi}{\partial \eta} = 0 \tag{2.3.84}$$

$$F - F_{\dot{x}_i}\dot{x}_i - F_{\dot{z}_k}\dot{z}_k - F_{\dot{\eta}}\dot{\eta} + \mu_j \frac{\partial \psi_j}{\partial t} + \frac{\partial \phi}{\partial t} = 0 \tag{2.3.85}$$

(3) Corner Conditions

$$\frac{\partial F^{(-)}}{\partial \dot{x}_i} = \frac{\partial F^{(+)}}{\partial \dot{x}_i} \tag{2.3.86}$$

$$\frac{\partial F^{(-)}}{\partial \dot{z}_k} = \frac{\partial F^{(+)}}{\partial \dot{z}_k}, \quad \frac{\partial F^{(-)}}{\partial \dot{\eta}} = \frac{\partial F^{(+)}}{\partial \dot{\eta}} \tag{2.3.87}$$

$$F^{(-)} - F_{\dot{x}_i}^{(-)}\dot{x}_i^{(-)} - F_{\dot{z}_i}^{(-)}\dot{z}_i^{(-)} - F_{\dot{\eta}}^{(-)}\dot{\eta}^{(-)} = F^{(+)} - F_{\dot{x}_i}^{(+)}\dot{x}_i^{(+)} - F_{\dot{z}_i}^{(+)}\dot{z}_i^{(+)} - F_{\dot{\eta}}^{(+)}\dot{\eta}^{(+)} \quad (2.3.88)$$

Application of the Weierstrass condition provides

(4) Weierstrass Condition

$$F(\dot{X},\dot{Z},\dot{\eta}) - F(\dot{x},\dot{z},\dot{\eta}) - (\dot{X}_i - \dot{x}_i)F_{\dot{x}_i} - (\dot{Z}_i - \dot{z}_i)F_{\dot{z}_i} - (\dot{\eta} - \dot{\eta})F_{\dot{\eta}} \geq 0 \quad (2.3.89)$$

where ($\dot{x}$, $\dot{z}$, $\dot{\eta}$) denote the optimal values and ($\dot{X}$ $\dot{Z}$ $\dot{\eta}$) denote any other values satisfying the constraints of equations (2.3.79) and (2.3.80).

Since the functions f_i, g, ϕ and ψ_j do not contain the variables z_i, it follows from (2.3.84) and (2.3.82) that

$$\frac{\partial F}{\partial \dot{z}_k} = 0 \quad (2.3.90)$$

Now, from the definition of F and the Hamiltonian H, and by substituting u for $\dot{z}$ this expression reduces to

$$\frac{\partial H}{\partial u} - \lambda \frac{\partial g}{\partial u} = 0 . \quad (2.3.91)$$

Also, equation (2.3.82b) reduces to

$$\frac{\partial F}{\partial \dot{\eta}} = 0 = \lambda \dot{\eta} \quad (2.3.92)$$

Thus, either λ is zero and $g(u) < 0$ with the control lying in the interior of U or $\dot{\eta} = 0$ and the control lies on the boundary of U with $g(u) = 0$. Using these results plus the fact that F is identically zero, the remaining Euler and boundary become

$$\dot{p}_i = -\frac{\partial H}{\partial x_i} = -p_j \frac{\partial f_j}{\partial x_i} \quad (2.3.93)$$

$$P_i + \mu_j \frac{\partial \psi_j}{\partial x_i} + \frac{\partial \phi}{\partial x_i} = 0 \quad ; \; i = 1, n$$

$$H = \mu_j \frac{\partial \psi_j}{\partial t} + \frac{\partial \phi}{\partial t} \qquad (2.3.94)$$

while the corner conditions of equations (2.3.86) and (2.3.88) take the form

$$P_i^{(-)} = P_i^{(+)}$$

$$H(p^{(-)}, u^{(-)}, x) = H(p^{(+)}, u^{(+)}, x) \; . \qquad (2.3.95)$$

Finally, the Weierstrass condition reduces to

$$\dot{x}_i P_i - \dot{X}_i P_i \geq 0$$

or

$$H(x, p, u_0) \geq H(x, p, \hat{u}) \qquad (2.3.96)$$

where u_0 is the optimal control and $\hat{u}$ is any other control in the set $\mathcal{U}$.

An examination of the necessary conditions of equations (2.3.93), (2.3.94), and (2.3.96) **shows** that they are the same as the necessary conditions arising from the Maximum Principle [see equations (2.3.13) to (2.3.17)]. Also, the corner conditions of equation (2.3.95) are consistent with the requirement of the Maximum Principle that the p vector and the Hamiltonian H be continuous. The two additional equations (2.3.91) and (2.3.92) require some explanation

For the optimal control to satisfy the Weierstrass condition of equation (2.3.96) it must be selected from the set $\mathcal{U}$ so that the Hamiltonian is maximized at each point along the trajectory. The procedure for maximizing H subject to a constraint

$$u \in \mathcal{U} \Longleftrightarrow g(u) \leq 0$$

or the equivalent equality condition

$$g(u) + \dot{\eta}^2 = 0$$

has been treated in Section 2.1.4 and consists of setting the first derivative of the modified function $\bar{H} = H + \lambda_1 (g + \dot{\eta}^2)$ to zero. Hence

$$\frac{\partial \bar{H}}{\partial u} = 0 = \frac{\partial H}{\partial u} + \lambda_1 \frac{\partial g}{\partial u}$$

$$\frac{\partial H}{\partial \dot{\eta}} = 2 \lambda_1 \dot{\eta} = 0$$

Setting λ_1 equal to $-\lambda$ in these two equations provides the two additional conditions of equations (2.3.91) and (2.3.92). Thus, these conditions are merely a consequence of the fact that the optimal control maximizes the Hamiltonian.

From this discussion, it is apparent that the variational treatment of an optimization problem leads to essentially the same results as the Maximum Principle. However, the development of the necessary conditions via the Calculus of Variations employs assumptions which are somewhat more restrictive. For example, it is necessary to assume that the set U has an analytical representation $g(u) \leq 0$ where g has a continuous first derivative and bounded second derivative along its boundary. Though such restrictions are rather weak and are usually satisfied in most problems, the Maximum Principle avoids the problem. The primary advantage of the Maximum Principle, however, is that it represents a more concise statement of how the optimization is to be performed. The device of introducing the additional variables $\dot{\eta}$ and λ, and solving equations (2.3.91) and (2.3.92), a procedure inevitably followed in the variational formulation prior to the development of the Maximum Principle, is no longer necessary. Rather, the Weierstrass condition is used directly to compute the optimal control.

2.3.6 Methods of Solution

Formulation of an optimization problem, can be accomplished by either the Maximum Principle or the Calculus of Variations with the development of the governing equations rather straightforward in both cases. Unfortunately, the formulation is only a minor part of the analysis. The major portion of the problem involves the generation of numerical solutions.

As indicated in the previous sections, the determination of a solution to an optimization problem involves the selection of a function or set of functions which satisfy five conditions:

1. differential state equations

$$\dot{x}_i = f_i(x, u) \tag{2.3.97}$$

2. differential adjoint equations

$$\dot{p}_i = -\frac{\partial H}{\partial x_i} = -p_j \frac{\partial f_j}{\partial x_i} \tag{2.3.98}$$

3. control optimization condition

$$H(x, p, u_o) \geq H(x, p, \hat{u}) \tag{2.3.99}$$

where u_o denotes the optimal control and $\hat{u}$ denotes any other control, both of which are contained in the set $\mathcal{U}$.

4. state boundary conditions

$$X = X^{o} \quad ; \quad t = t^{o}$$
$$\psi_j(X^f, t^f) = 0 \tag{2.3.100}$$

5. tranversality conditions (adjoint boundary conditions)

$$\left.\begin{aligned} p_i + \mu_j \frac{\partial \psi_j}{\partial X_i} + \frac{\partial \phi}{\partial t} &= 0 \\ H = \frac{\partial \phi}{\partial t} + \mu_j \frac{\partial \psi_j}{\partial t} & \end{aligned}\right\} \quad at \quad t = t^f \tag{2.3.101}$$

Taken together, these five conditions constitute a boundary value problem of the two point type; that is, some of the boundary conditions are specified at the initial point and some at the terminal point. At present, only linear boundary value problems can be solved directly. Hence, a solution to the nonlinear system above must be effected iteratively.

The iterative techniques employed in nonlinear boundary value problems are linear in the sense that one starts with a solution which satisfies some of the five conditions and then uses the linear theory to correct this solution in a direction which tends to satisfy the other conditions. However, while all iterative techniques are linear, not all are of the same order; that is, some take into consideration only the first order terms in a series representation about a nominal condition while others account for both first and second order terms.

In this section, three numerical techniques will be outlined: the gradient or steepest descent method, which is a first order theory; the neighboring extremal method; and quasilinearization. Both of these latter methods are second order. Since all numerical methods used in optimization problems are essentially extrapolations into a function space of techniques used in maxima-minima theory, the analysis is begun by considering the numerical procedures available for finding a vector X which minimizes a function f(x).

A. Minimizing a Function

Consider the problem of determining the value of the n-dimensional vector X which minimizes the function f(x). For f sufficiently differentiable, the minimum point satisfies the equations

$$\frac{\partial f}{\partial X_i} = 0 \quad ; \quad i = 1, n \tag{2.3.102}$$

$$\left(\frac{\partial^2 f}{\partial X_i \, \partial X_j}\right) = \text{positive semi-definite}$$

If equations (2.3.102) have an explicit solution, the problem is solved. If not, the solution must be effected iteratively. Two iterative techniques in common usage are presented below.

(1) Gradient Method

Let $X^{(1)}$ denote the first guess of the minimum point and let f(x) be approximated by the truncated Taylor series.

$$f(x) \cong f(x^{(1)}) + \frac{\partial f}{\partial X_i}(X_i - X_i^{(1)}) \tag{2.3.103}$$

Require that a second guess satisfy the magnitude constraint

$$\left| X - X^{(1)} \right|^2 = \sum_{i=1}^{n} (X_i - X_i^{(1)})^2 \leq k^2 \tag{2.3.104}$$

and select X so that the first order approximation to f(x) given in equation equation (2.3.103) is minimized. By this selection, the function f evaluated at the new value of X will be less than $f(x^{(1)})$ and hence, nearer its minimum value. Following the procedure developed in section 2.1.4, the modified function F(x) is formed with

$$F = f(x^{(1)}) + \frac{\partial f}{\partial X_i}(X_i - X_i^{(1)}) + \lambda \left\{ \sum_{i=1}^{n} (X_i - X_i^{(1)})^2 + \eta^2 - k^2 \right\} \tag{2.3.105}$$

where λ is a constant multiplier and the variable η is used to convert the inequality in (2.3.104) to the equality condition

$$\sum_{i=1}^{n} (X_i - X_i^{(1)})^2 + \eta^2 = k^2$$

Now, differentiating F with respect to X_i and η and setting the first derivative to zero provides

$$2\lambda(X_i - X_i^{(1)}) + \frac{\partial f(x^{(1)})}{\partial X_i} = 0 \quad ; \quad i = 1, n \tag{2.3.106}$$

and

$$2\lambda\eta = 0 \tag{2.3.107}$$

while the second derivative condition requires (in part) that

$$\frac{\partial^2 F}{\partial \eta^2} = \lambda \geq 0$$

Since the equality condition in the second partial is a degenerate case, it follows that, in general, $\lambda > 0$ and, thus, that $\eta = 0$. Equation (2.3.104) then becomes

$$\sum_{i=1}^{n} (X_i - X_i^{(1)})^2 = k^2 \qquad (2.3.108)$$

Combining (2.3.106) and (2.3.108) yields

$$4\lambda^2 \sum_{i=1}^{n} (X_i - X_i^{(1)})^2 = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial X_i}\right)^2 = 4\lambda^2 k^2 \quad \Rightarrow \quad \lambda = \frac{1}{2K}\sqrt{\sum_{i=1}^{n}\left(\frac{\partial f}{\partial X_i}\right)^2}$$

and

$$X_i - X_i^{(1)} = -\frac{1}{2\lambda}\frac{\partial f}{\partial X_i} = K \frac{\partial f}{\partial X_i} \Big/ \left(\sum_{i=1}^{n}\left(\frac{\partial f}{\partial X_i}\right)^2\right) \qquad (2.3.109)$$

Note from (2.3.109) that the change in the value of X is proportional to the negative gradient of f with respect to X. For this reason, the iterative technique is called the gradient method. The new value of X, denoted by $X^{(2)}$ and satisfying

$$X_i^{(2)} = X_i^{(1)} - \frac{1}{2\lambda}\frac{\partial f}{\partial X_i} = X_i^{(1)} - K \frac{\frac{\partial f}{\partial X_i}}{\sum_{i=1}^{n}\left(\frac{\partial f}{\partial X_i}\right)^2}$$

is used to compute the new value for the function f. The process is repeated until the minimum value is found.

From this discussion, it follows that the second guess of the minimum point is determined so as to minimize a first order approximation to the function [equation (2.3.105)] subject to the magnitude constraint [equation (2.3.104)] which requires that the second guess be sufficiently close to the first guess. Since only the zeroth and first terms in the series expansion are used, the process is referred to as first order.

(2) Second Order Approach (Newton-Raphson Method)

Let $X^{(1)}$ denote the first guess of the minimum point and approximate $f(x)$ by the second order truncated series expansion

$$f(x) \cong f(X^{(1)}) + \frac{\partial f}{\partial X_i}(X_i - X_i^{(1)}) + \frac{1}{2}\frac{\partial^2 f}{\partial X_i \partial X_j}(X_i - X_i^{(1)})(X_j - X_j^{(1)}) \qquad (2.3.110)$$

If $X^{(1)}$ is reasonably close to the minimum point, then selecting $X^{(2)}$ to minimize this approximation of f(x) should provide a smaller value for the function f. Thus, setting

$$\frac{\partial}{\partial X_k}\left\{f(X^{(1)}) + \frac{\partial f}{\partial X_i}(X_i - X_i^{(1)}) + \frac{1}{2}\frac{\partial^2 f}{\partial X_i \partial X_j}(X_i - X_i^{(1)})(X_j - X_j^{(1)})\right\} = 0 \ ; \ k = 1, n$$

yields

$$X_i - X_i^{(1)} = -\left(\frac{\partial^2 f}{\partial X_j \partial X_k}\right)^{-1} \frac{\partial f}{\partial X_i} \qquad ; \ i = 1, n \tag{2.3.11}$$

from which the second guess is computed. The process is repeated until the minimum value is found. Since the second terms in the series approximation are used, the approach is second order and, as indicated in Reference (14), is accompanied by a rate of convergence which is at least quadratic with

$$\left|X^{(n)} - X^{(n-1)}\right| \sim \left|X^{(n-1)} - X^{(n-2)}\right|^2$$

The effectiveness of this technique and the gradient method, as far as rate and radius of convergence is concerned, is primarily a function of the particular problem under consideration and the point at which the iteration is begun. If the first guess of X is "close" to the minimum point, then the second-order approach will provide the most rapid convergence since it gives both a magnitude and direction of correction. If the starting point is not "close", the second order technique will diverge. On the other hand, the gradient technique can be made to converge from points "far removed" from the minimum point due to the fact that the magnitude of correction is controlled with the technique itself providing only a direction of correction. In the vicinity of the solution, however, the derivatives $\frac{\partial f}{\partial X_i}$ approach zero and the correction mechanism in the gradient method breaks down. Thus, as a general rule, the first order method should be used at the start of the iteration until the starting point has been moved sufficiently "close" to the optimal point, at which time a switch to the second-order method should be made.

B. Minimizing a Functional

The method used to effect solutions to variational problems are very similar to those used in maxima-minima theory. In the following paragraphs, three techniques will be presented, two of which are second order and one first order. For convenience, the conditions to be satisfied by the minimizing solution are restated:

(1) differential state equations

$$\dot{X}_i - f_i(X, u) \ ; \ i = 1, n \tag{2.3.112}$$

(2) differential adjoint equations

$$\dot{p}_i = -p_j \frac{\partial f_j}{\partial x_i} \quad ; \quad i = 1, n \tag{2.3.113}$$

(3) control optimization condition

$$H(x, p, u_o) \geq H(x, p, \hat{u}) \tag{2.3.114}$$

(4) state boundary conditions

$$\begin{aligned} x &= x^o \quad ; \quad t = t^o \\ \psi_j(x^f, t^f) &= 0 \quad ; \quad j = 1, M \end{aligned} \tag{2.3.115}$$

(5) transversality conditions

$$p_i + \frac{\partial \phi}{\partial x_i} + \mu_j \frac{\partial \psi_j}{\partial x_i} = 0 \quad ; \quad i = 1, N \tag{2.3.116}$$

at $t = t^f$

$$H = \frac{\partial \phi}{\partial t} + \mu_j \frac{\partial \psi_j}{\partial t} \tag{2.3.117}$$

Also, to simplify the development, two assumptions regarding the form of the optimization problem will be made:

(i) the admissible control set U is the entire control space with the control that maximizes H satisfying

$$\frac{\partial H}{\partial u_k} = 0 \quad ; \quad k = 1, r \tag{2.3.118}$$

(ii) the terminal time t^f is specified and therefore the transversality condition in equation (2.3.117) provides no additional information (it contains an arbitrary unknown constant) and can be eliminated.

These assumptions are somewhat restrictive. However, they are made only to facilitate the presentation of the methods. The treatment of more general cases is straightforward and is given in Reference (15).

(1) Gradient Method

In the gradient method, which is a first order technique, each successive iteration satisfies conditions (1), (2), (4) and (5) but does not satisfy condition (3). As in the gradient technique used in maxima-minima theory, the iteration process consists of minimizing the first order approximation to the functional $J = \phi(x^f,t^f)$ subject to a magnitude constraint.

Take as a starting solution a control $u = u^{(1)}$ which drives the system

$$\dot{x}_i = f_i(x, u)$$

from the initial point

$$x = x^0 \quad ; \quad t = t^0$$

to the target set

$$\psi_j(x^f, t^f) = 0 \quad ; \quad t = t^f \ (\text{specified})$$

but which does not necessarily minimize the performance index

$$J = \phi(x^f, t^f)$$

Admittedly, such a control policy may be difficult to find, but not as difficult as one which satisfies all the boundary conditions and minimizes $\phi(x^f,t^f)$ (i.e., the optimal control). Now, form the modified functional $\bar{J}$

$$\bar{J} = \phi + \mu_j \psi_j + \int_{t^0}^{t^f} p_i(\dot{x}_i - f_i(x,u))\,dt \equiv J$$

and note that minimizing $\bar{J}$ is equivalent to minimizing J. Approximating $\bar{J}$ by its first order Taylor series expansion about the nominal value $\bar{J}(u^{(1)})$ provides

$$\bar{J}(u) = \bar{J}(u^{(1)}) + \delta\bar{J}$$

where

$$\bar{J}(u^{(1)}) = \phi(x^{(1)f}, t^f)$$

and

$$\dagger\ \delta\bar{J} = \left(\phi_{x_i} + \mu_j \psi_{j x_i}\right)\delta x_i\Big]^{t^f} + \int_{t^0}^{t^f} p_i\left\{\delta\dot{x}_i - \frac{\partial f_i}{\partial x_j}\delta x_j - \frac{\partial f_i}{\partial u_k}\delta u_k\right\}dt \qquad (2.3.119)$$

†Note: Terms such as $\psi_j\,\delta\mu_j\Big]^{t^f}$ and $\int_{t^0}^{t^f}\delta p_i(\dot{x}_i - f_i(x,u))\,dt$ are omitted in (2.3.119) because they vanish on account of the trial solution satisfying conditions (1), (2) and (4) on pages 77 and 78.

Integrating the first term in the integrand by parts and requiring that the adjoint equations

$$\dot{P}_i = -P_j \frac{\partial f_j}{\partial X_i} \tag{2.3.120}$$

be satisfied provides

$$\delta \bar{J} = \left(\frac{\partial \phi}{\partial X_i} + \mu_j \frac{\partial \psi_j}{\partial X_i} + P_i \right) \delta X_i \Big]^{t^f} - \int_{t^o}^{t^f} P_i \frac{\partial f_i}{\partial u_k} \delta u_k \, dt \tag{2.3.121}$$

If the amount by which the control action can change is limited, minimizing the first order approximation to $\bar{J}$ should provide a smaller value for the performance index ϕ. Hence, require that

$$\int_{t^o}^{t^f} \sum_{k=1}^{r} \delta u_k^2 \, dt \le k^2 \tag{2.3.122}$$

where k^2 is some small quantity and choose the change in the control $\delta u(t)$ to minimize $\delta \bar{J}$ under the constraint (2.3.122).

Proceeding formally, the multiplier ν is introduced and the modified functional $\delta \bar{\bar{J}}$ formed

$$\delta \bar{\bar{J}} = \delta \bar{J} + \nu \left\{ \int_{t^o}^{t^f} \sum \delta u_k^2 \, dt + \eta^2 - k^2 \right\}$$

$$= \left(\frac{\partial \phi}{\partial X_i} + \mu_j \frac{\partial \psi_j}{\partial X_i} + P_i \right) \delta X_i \Big]^{t^f} - \int_{t^o}^{t^f} \left(P_i \frac{\partial f_i}{\partial u_k} - \nu \delta u_k \right) \delta u_k \, dt + \nu(\eta^2 - k^2)$$

Now, requiring the first variation of $\delta \bar{\bar{J}}$ to vanish provides

$$\delta u_i = \frac{1}{2\nu} P_j \frac{\partial f_j}{\partial u_i} \tag{2.3.123a}$$

$$\nu \eta = 0 \tag{2.3.123b}$$

and the boundary conditions

$$\frac{\partial \phi}{\partial X_i} + \mu_j \frac{\partial \psi_j}{\partial X_i} + P_i = 0 \tag{2.3.124}$$

Since $\nu = 0$ in **(2.3.2.3b)** is a degenerate case, it follows that $\eta = 0$ and that an equality holds in (2.3.122).

A review at this point seems appropriate. The first iteration $u^{(1)}(t)$ is selected so that the state

$$\dot{x}_i = f_i(x, u)$$

is driven from the initial point X° to the target set $\psi_j(X^f, t^f) = 0$. Once the state is computed, the adjoint equations

$$\dot{p}_i = -p_j \frac{\partial f_j}{\partial x_i} \quad ; \quad i = 1, n$$

are integrated backwards with the boundary conditions

$$p_i + \frac{\partial \phi}{\partial x_i} + \mu_j \frac{\partial \psi_j}{\partial x_i} = 0$$

The second iteration on the control is computed from

$$u^{(2)} = u^{(1)} + \delta u$$

where equation (2.3.123a) is used to evaluate δ u. The constant ν in this equation is determined from the magnitude constraint

$$\int_{t^o}^{t^f} \sum_k \delta u_k^2 \, dt = h^2 = \frac{1}{4\nu^2} \int_{t^o}^{t^f} \sum_k \left(p_j \frac{\partial f_j}{\partial u_k} \right)^2 dt \Rightarrow \nu = \frac{1}{2K} \sqrt{\int_{t^o}^{t^f} \sum_K p_j^2 \frac{\partial f_j^2}{\partial u_K} dt} \quad (2.3.123c)$$

Note that the integration of the adjoint equations requires a knowledge of the M multipliers μ_j used in the boundary conditions. These constants are to be selected so that the new terminal state $X^{(2)}(t^f)$ resulting from the control $u^{(2)}(t)$ satisfies the terminal constraints

$$\psi_j(x^f, t^f) = 0$$

or, to the first order

$$\frac{\partial \psi_j}{\partial x_i} \delta x_i^f = 0 \; ; \qquad t = t^f \text{(specified)}$$

where $\delta X_i^f = X_i^{(2)}(t^f) - X_i^{(1)}(t^f)$ and the derivatives $\partial\Psi_j/\partial X_i$ are evaluated at the point $X^{(1)}(t^f)$. The process by which the correct μ_j are selected is treated next.

Integrate the system

$$\dot{P}_i = -\frac{\partial f_j}{\partial \chi_i} P_j$$

from t^f to t° m+1 times with the m+1 different sets of boundary conditions

$$\left.\begin{aligned} P_i^{[1]} &= -\frac{\partial \Psi_1}{\partial X_i} \quad ; i = 1, n \\ P_i^{[2]} &= -\frac{\partial \Psi_2}{\partial X_i} \quad ; i = 1, n \\ &\vdots \\ P_i^{[M]} &= -\frac{\partial \Psi_M}{\partial X_i} \quad ; i = 1, n \\ P_i^{[M+1]} &= -\frac{\partial \phi}{\partial X_i} \quad ; i = 1, n \end{aligned}\right\} t = t^f \tag{2.3.125}$$

and let

$$P_i = P_i^{[M+1]} + \sum_{j=1}^{M} \mu_j P_i^{[j]} \quad ; i = 1, N \tag{2.3.126}$$

Since the disturbed state equations are given by

$$\delta \dot{X}_i = \frac{\partial f_i}{\partial X_j} \delta X_j + \frac{\partial f_i}{\partial u_k} \delta u_k \quad ,$$

it is a simple matter to show that

$$P_i^f \delta X_i^f = \dot{P}_i^{\,0} \delta X_i^\circ + \int_{t^\circ}^{t^f} P_i \frac{\partial f_i}{\partial u_k} \delta u_k \, dt \, .$$

With $\delta X_i^\circ = 0$ (the initial point is fixed) this expression becomes

$$\left[P_i^{[j]} \delta X_i \right]^{t = t^f} = \int_{t^\circ}^{t^f} P_i^{[j]} \frac{\partial f_i}{\partial u_k} \delta u_k \, dt \quad ; j = 1, M+1 \, . \tag{2.3.127}$$

Thus, substitution of the terminal values from equation (2.3.125) provides

$$d\Psi_i = \frac{\partial \Psi_i}{\partial X_j} \delta X_j \Big]^{t^f} = -\int_{t^\circ}^{t^f} P_j^{[i]} \frac{\partial f_j}{\partial u_k} \delta u_k \, dt = 0 \quad ; i = 1, M \, . \tag{2.3.128}$$

Using the expression for the P vector given in (2.3.126) and the control change in (2.3.123), equation (2.3.128) reduces to

$$\int_{t^o}^{t^f} P_i^{[s]} \frac{\partial f_i}{\partial u_k} \left\{ \left(P_j^{[M+1]} + \sum_{\ell=1}^{M} \mu_\ell P_j^{[\ell]} \right) \frac{\partial f_j}{\partial u_k} \right\} dt = 0 \;\; ; \; s = 1, M. \quad (2.3.129)$$

These M equations are used to compute the values of the μ_j for which the conditions

$$\frac{\partial \Psi_i}{\partial x_j} \delta x_j^f = 0 \; ; \; i = 1, m \qquad (2.3.130)$$

are satisfied. Using equations (2.3.129), (2.3.127) and the last of equation (2.3.125) it follows that

$$d\phi = - p_i^f \delta x_i^f = - \frac{1}{2V} \int_{t^o}^{t^f} \sum_k \left(p_i \frac{\partial f_i}{\partial u_k} \right)^2 dt \qquad (2.3.131)$$

where the quantity V is determined from equation (2.3.123c) and can be shown to be positive from the second variation test on the quantity $\overline{\overline{\delta J}}$.

The step by step calculation procedure used in the gradient method is as follows:

(i) Select $u^{(1)}(t)$ so that $X = X^o$ and $\Psi_j(X^f, t^f) = 0$

(ii) Integrate the state system

$$\dot{x}_i = f_i(x, u)$$

from t^o to t^f (recall that it has been assumed that t^f is fixed).

(iii) Integrate the adjoint system

$$\dot{P}_i = - P_j \frac{\partial f_j}{\partial x_i}$$

from t^f to t^o m+1 times with the m+1 terminal conditions

$$\left.\begin{aligned} P_i^{[1]} &= - \frac{\partial \Psi_1}{\partial x_i} \\ P_i^{[2]} &= - \frac{\partial \Psi_2}{\partial x_i} \\ &\;\vdots \\ P_i^{[M]} &= - \frac{\partial \Psi_M}{\partial x_i} \\ P_i^{[M+1]} &= - \frac{\partial \phi}{\partial x_i} \end{aligned}\right\} t = t^f$$

(iv) Compute the μ_j from the m conditions

$$\int_{t^o}^{t^f} P_i^{[s]} \frac{\partial f_i}{\partial u_k} \left\{ \left(P_j^{[m+1]} + \sum_{\ell=1}^{m} \mu_\ell P_j^{[\ell]} \right) \frac{\partial f_j}{\partial u_k} \right\} dt = 0 \quad ; \quad s = 1, m$$

(v) Compute P and ν from

$$P_i = P_i^{[m+1]} + \sum_{j=1}^{m} \mu_j P_i^{[j]}$$

$$\frac{1}{4\nu^2} \int_{t^o}^{t^f} \sum_k \left(P_i \frac{\partial f_i}{\partial u_k} \right)^2 dt = k^2$$

(vi) Set $u^{(2)} = u^{(1)} + \delta u$ where

$$\delta u_i = \frac{1}{2\nu} P_j \frac{\partial f_j}{\partial u_i} \quad ; \quad i = 1, r$$

(vii) Go to step (ii)

The iteration continues until changes in the control policy produce no changes in the performance index ϕ. At this point, the iteration is stopped. The first order approximation to the change in ϕ given in (2.3.131) (which is negative since ν is positive) indicates that each successive iteration should reduce ϕ until the condition

$$P_i \frac{\partial f_i}{\partial u_k} = 0 \quad ; \qquad k = 1, r$$

is reached. From assumption (1) or equation (2.3.118) this condition is the optimizing condition, and the solution corresponding to this iteration will be the optimal solution.

(2) Neighboring Extremal and Quasilinearization

Since both of these techniques are second order, (i.e., second order terms in a series expansion are used) they will be presented together. The presentation, however, will be abbreviated since both techniques are rather complex algebraically. For a full development, the reader should consult the literature [References (15) to (18)].

Proceeding as in the gradient method, the functional $\bar{J}$ is formed

$$\bar{J} = \phi + \mu_j \psi_j + \int_{t^o}^{t^f} P_i \left(\dot{x}_i - f_i(x, u) \right) dt$$

and expanded in a truncated series about some first guess at the optimal solution $(x^{(1)}, u^{(1)})$

$$\bar{J} \cong \bar{J}(x^{(1)}, u^{(1)}) + \delta \bar{J} + \delta^2 \bar{J} \tag{2.3.132}$$

where

$$\delta \bar{J} = \overbrace{\left(\frac{\partial \phi}{\partial x_i} + \mu_j \frac{\partial \psi_j}{\partial x_i} \right) \delta x_i}^{+\ \psi_j \delta \mu_j} \Bigg]^{t^f} + \int_{t^o}^{t^f} \delta P_i (\dot{x}_i - f_i)\, dt + \int_{t^o}^{t^f} P_i \left(\delta \dot{x}_i - \frac{\partial f_i}{\partial x_j} \delta x_j - \frac{\partial f_i}{\partial u_k} \delta u_k \right) dt$$

and where (2.3.133)

$$\delta^2 \bar{J} = \left(\frac{\partial^2 \phi}{\partial x_i \partial x_k} + \mu_j \frac{\partial^2 \psi_j}{\partial x_i \partial x_k} \right) \frac{\delta x_i \, \delta x_k}{2} + \delta \mu_j \frac{\partial \psi_j}{\partial x_i} \delta x_i \Bigg|^{t^f} \tag{2.3.134}$$

$$+ \int_{t^o}^{t^f} \delta P_i \left\{ \delta \dot{x}_i - \frac{\partial f_i}{\partial x_j} \delta x_j - \frac{\partial f_i}{\partial u_k} \delta u_k \right\} dt - \int_{t^o}^{t^f} \frac{P_i}{2} \left\{ \frac{\partial^2 f_i}{\partial x_j \partial x_k} \delta x_j \delta x_k + \frac{2 \partial^2 f_i}{\partial x_j \partial u_k} \delta x_j \delta u_k \right.$$

$$\left. + \frac{\partial^2 f_i}{\partial u_k \partial u_l} \delta u_k \delta u_l \right\} dt$$

In both neighboring extremal and quasilinearization, the corrections δX_i, $\delta \mu_j$, and δu_k are selected so as to minimize the second order approximation to $\bar{J}$; that is, to minimize the right hand side of equation (2.3.132). The difference in the two techniques lies in the selection of the starting solution $(x^{(1)}, u^{(1)})$.

In the neighboring extremal method, a starting solution consisting of a control and state history, $X^{(1)}$ and $u^{(1)}$ is assumed which satisfies equations (2.3.112), (2.3.113) and (2.3.114) but which does not satisfy equations (2.3.115), (2.3.116) or (2.3.117).

These last two conditions ≠

$$\psi_j (x^f, t^f) = 0 \quad ; \qquad j = 1, m \tag{2.3.135}$$

$$P_i + \mu_j \frac{\partial \psi_j}{\partial x_i} + \frac{\partial \phi}{\partial x_o} = 0 \quad ; \qquad i = 1, n$$

≠ Since the final time t^f is specified, the last equation in condition (5) provides no information since it contains an arbitrary constant.

constitute a set of n+m boundary constraints and can be reduced to n boundary constraints through elimination of the multipliers μ_j. The process of generating the starting solution consists of guessing an initial value for the P vector $[P^\circ = P(t^\circ)]$ and integrating the 2nd system

$$\dot{x}_i = f_i(x, u)$$

$$\dot{p}_i = -p_j \frac{\partial f_j}{\partial x_i}$$

from t° to t^f where the control u is selected to satisfy the optimizing condition

$$H(x^{(1)}, p^{(1)}, u^{(1)}) \geq H(x^{(1)}, p^{(1)}, \hat{u})$$

If the correct P° vector has been guessed, the terminal constraints of equation (2.3.135) will be satisfied. When these conditions are not met, the neighboring extremal technique of minimizing the second approximation of the functional J provides a correction to the P° vector, δP°; and if the first solution is "close" to the optimal solution, then the second solution resulting from the updated value of the P° vector will provide a better matching of the terminal conditions.

Since the conditions

$$\dot{x}_i = f_i(x, u)$$

$$\dot{p}_i = -p_j \frac{\partial f_j}{\partial x_i}$$

$$x(t = t^o) = x^o$$

$$\frac{\partial H}{\partial u_k} = 0$$

are satisfied by the starting solution, minimizing the second order approximation to J reduces to minimizing the quantity

$$\left(\frac{\partial \phi}{\partial x_i} + \mu_j \frac{\partial \psi_j}{\partial x_i} + p_i \right) \delta x_i + \delta\mu_j \left(\psi_j + \frac{\partial \psi_j}{\partial x_k} \delta x_k \right) + \left(\frac{\partial^2 \phi}{\partial x_i \partial x_k} + \mu_j \frac{\partial^2 \psi_j}{\partial x_i \partial x_j} \right) \frac{\delta x_i \, \delta x_k}{2} \Bigg|^{t^f}$$

$$- \int_{t^o}^{t^f} \frac{p_i}{2} \left\{ \frac{\partial^2 f_i}{\partial x_j \partial x_k} \delta x_j \delta x_k + \frac{2 \partial^2 f_i}{\partial x_j \partial u_k} \delta x_j \delta u_k + \frac{\partial^2 f_i}{\partial u_k \partial u_l} \delta u_k \delta u_l \right\} dt$$

$$+ \int_{t^o}^{t^f} \delta p_i \left\{ \delta \dot{x}_i - \frac{\partial f_i}{\partial x_j} \delta x_j - \frac{\partial f_i}{\partial u_k} \delta u_k \right\} dt$$

Setting the variation of the above expression to zero with respect to the variables δP_i, δu_k, $\delta X_i f$, and $\delta \mu_j$ provides the differential equations

$$\delta \dot{X}_i = \frac{\partial f_i}{\partial X_j} \delta X_j + \frac{\partial f_i}{\partial u_k} \delta u_k \quad ; \quad i = 1, n$$

$$\delta \dot{P}_i = -\frac{\partial^2 H}{\partial P_j \partial X_i} \delta P_j - \frac{\partial^2 H}{\partial X_i \partial X_k} \delta X_k - \frac{\partial^2 H}{\partial X_i \partial u_l} \delta u_l \quad ; \quad i = 1, n$$

$$\frac{\partial^2 H}{\partial u_i \partial P_k} \delta P_k + \frac{\partial^2 H}{\partial u_i \delta X_k} \delta X_k + \frac{\partial^2 H}{\partial u_i \partial u_l} \delta u_l = 0 \quad ; \quad i = 1, r$$

and the boundary conditions

$$\psi_j + \frac{\partial \psi_j}{\partial X_i} \delta X_i = 0 \quad ; \quad j = 1, m$$

$$\frac{\partial \phi}{\partial X_i} + (\mu_j + \delta \mu_j) \frac{\partial \psi_j}{\partial X_i} + P_i + \delta P_i + \frac{\partial^2 \phi}{\partial X_i \partial X_k} \delta X_k + \mu_j \frac{\partial^2 \psi_j}{\partial X_i \partial X_k} \delta X_k = 0 \; ; \quad i = 1, n$$

This system of equations and boundary conditions, along with the requirement that $\delta X = 0$ at $t = t°$, is a two point boundary value problem. However, since the equations and boundary conditions are linear, the problem can be solved directly (without iteration) to give the correction in the initial P vector, $\delta P°$. The details of the solution are contained in References (15) and (16).

In quasilinearization, a starting solution is selected which satisfies (2.3.114), (2.3.115) and (2.3.116) but not equations (2.3.112) and (2.3.113); that is, the starting solution satisfies the boundary conditions of the problem but not the governing differential equations. The control u is again determined from the optimizing condition

$$\frac{\partial H}{\partial u_k} = 0$$

The iteration process then consists of determining corrections $[\delta P(t), \delta X(t)]$ to the starting solution $[\bar{X}^{(1)}(t), P^{(1)}(t)]$ so that the second iteration

$$X^{(2)}(t) = X^{(1)}(t) + \delta X(t) \quad ; \quad P^{(2)}(t) = P^{(1)}(t) + \delta P(t)$$

more nearly satisfies the governing equations.

Since the starting solution satisfies the equations

$$\frac{\partial H}{\partial u_i} = 0 \qquad i = 1, n$$

$$\psi_j(x^f, t^f) = 0 \qquad j = 1, M$$

$$\frac{\partial \phi}{\partial x_i} + \mu_j \frac{\partial \psi_j}{\partial x_i} + p_i = 0 \qquad i = 1, N$$

minimizing the second order approximation to $\bar{J}$ reduces to minimizing the expression

$$\left(\frac{\partial^2 \phi}{\partial x_i \partial x_k} + \mu_j \frac{\partial^2 \psi_j}{\partial x_i \partial x_k}\right) \frac{\delta x_i \delta x_k}{2} + \delta \mu_j \frac{\partial \psi_j}{\partial x_i} \delta x_i$$

$$- \int_{t^o}^{t^f} \left(\dot{p}_i + \frac{\partial f_j}{\partial x_i} p_j\right) \delta x_i \, dt + \int_{t^o}^{t^f} \delta p_i (\dot{x}_i - f_i) \, dt + \int_{t^o}^{t^f} \delta p_i \left\{ \delta \dot{x}_i - \frac{\partial f_i}{\partial x_j} \delta x_j - \frac{\partial f_i}{\partial u_k} \delta u_k \right\} dt$$

$$- \int_{t^o}^{t^f} \frac{p_i}{2} \left\{ \frac{\partial^2 f_i}{\partial x_j \partial x_k} \delta x_j \delta x_k + \frac{2 \partial^2 f_i}{\partial x_j \partial u_k} \delta x_j \delta u_k + \frac{\partial^2 f_i}{\partial u_k \partial u_l} \delta u_k \delta u_l \right\} dt$$

Thus, setting the first variation to zero provides

$$\left.\begin{aligned} &\frac{\partial \psi_j}{\partial x_i} \delta x_i = 0 \\ &\frac{\partial^2 \phi}{\partial x_i \partial x_k} \delta x_k + \mu_j \frac{\partial^2 \psi_j}{\partial x_i \partial x_k} \delta x_k + \delta \mu_j \frac{\partial \psi_j}{\partial x_i} + \delta p_i = 0 \end{aligned}\right\} \quad t = t^f$$

$$\delta \dot{x}_i = \frac{\partial f_i}{\partial x_j}\,\delta x_j + \frac{\partial f_i}{\partial u_k}\,\delta u_k + f_i(X^{(1)}, u^{(1)}) - \dot{X}_i^{(1)}$$

$$\delta \dot{p}_i = -\,\delta p_j \frac{\partial f_j}{\partial x_i} - \frac{\partial^2 H}{\partial x_i \partial x_j}\,\delta x_j - \frac{\partial^2 H}{\partial x_i \partial u_k}\,\delta u_k - p_j^{(1)} \frac{\partial f_j^{(1)}}{\partial x_i} - \dot{p}_i^{(1)}$$

$$\frac{\partial^2 H}{\partial u_i \partial u_k}\,\delta u_k + \frac{\partial^2 H}{\partial u_i \partial x_j}\,\delta x_j + \frac{\partial^2 H}{\partial u_i \partial p_j}\,\delta p_j = 0$$

The solution to this system provides the corrections $\delta P(t)$ and $\delta X(t)$. The details of the solution are presented in References (15), (17) and (18).

Both quasilinearization and neighboring extremal are the variational analog of the second order approach used in maxima-minima theory. Both provide a magnitude and direction of correction (the gradient technique provides only direction). Further, as in maxima-minima theory, both are accompanied by a rate of convergence which is at least quadratic. However, the starting solution must be "close" to the optimal solution for the iteration process to converge. How close depends on the particular problem under consideration.

2.3.7 Some Generalizations

The problem of selecting the control u from the set $\mathcal{U}$ to minimize the functional

$$J = \phi(x^f, t^f) = MIN \tag{2.3.136}$$

subject to the differential constraints

$$\dot{x}_i = f_i(x, u) \tag{2.3.137}$$

and the boundary conditions

$$\begin{aligned} &x - x^o \quad ; \quad t = t^o \\ &\psi_j(x^f, t^f) = 0 \quad ; \quad j = 1, m \end{aligned} \tag{2.3.138}$$

is rather general in form. Occasionally, however, problems arise whose formal statement differs slightly from that just given. In this section, four variations of this problem are considered and the modified necessary conditions are developed.

(A) The Functional to be Minimized is an Integral of the Form

$$\int_{t^o}^{t^f} g(x,u)\,dt$$

In this case, it is a simplified matter to reduce the integral performance index to a terminal index of the form of equation (2.3.136). Let

$$x_{n+1} = \int_{t^o}^{t} g(x,u)\,dt$$

then

$$\dot{x}_{n+1} = g(x,u) \tag{2.3.139}$$

and the problem of minimizing the integral reduces to minimizing

$$J = \phi(x^f, t^f) = x^f_{n+1}$$

subject to equations (2.3.137) to (2.3.139) and the additional boundary condition.

$$x_{n+1} = 0 \quad ; \quad t = t^o \tag{2.3.140}$$

(B) The Differential Constraints Contain Time Explicitly with

$$\dot{x}_i = f_i(x, u, t)$$

The explicit time dependence in the differential constraints can be removed by a transformation. Let

$$x_{n+1} = t$$

with

$$\dot{x}_{n+1} = 1 \tag{2.3.141}$$

and

$$x_{n+1} = t^o \quad ; \quad t = t^o$$
$$x_{n+1} = t^f \quad ; \quad t = t^f \qquad (2.3.142)$$

The differential constraints

$$\dot{x}_i = f_i \; (x_1, x_2, \ldots x_{n+1}, u) = f_i \; (x, u)$$

are now no longer explicitly dependent on time. The optimization proceeds by applying the Maximum Principle to the n+1 dimensional system satisfying the combined boundary conditions of equations (2.3.138) and (2.3.142).

(C) The Initial Conditions are Not Completely Specified

Suppose that the initial conditions are not completely specified but rather are to satisfy the s constraints

$$\theta_i \, (x^\circ, t^\circ) = 0 \quad ; \quad i = 1, s \leq n+1 \qquad (2.3.143)$$

In this case the optimizing conditions are developed by working with the modified functional $\bar{\bar{J}}$ where

$$\bar{\bar{J}} = \phi + \mu_j \psi_j + \nu_k \theta_k + \int_{t^o}^{t^f} p_i \, (\dot{x}_i - f_i) \, dt$$

The functional $\bar{\bar{J}}$ is to be minimized with respect to variations in the control and in the initial and terminal points. For $\bar{\bar{J}}$ to be optimal with respect to variations that go through the optimal initial point in the set $\theta_i(x^\circ, t^\circ) = 0$, it is necessary that all the conditions stemming from the Maximum Principle be satisfied; that is,

$$\dot{p}_i = - p_j \frac{\partial f_j}{\partial x_i}$$
$$H(p, x, u_o) \geq H(p, x, \hat{u}) \qquad (2.3.144)$$
$$\left. \begin{aligned} & p_i + \frac{\partial \phi}{\partial x_i} + \mu_j \frac{\partial \psi_j}{\partial x_i} = 0 \\ & H = \frac{\partial \phi}{\partial t} + \mu_j \frac{\partial \psi_j}{\partial t} \end{aligned} \right\} \quad t = t^f$$

For the first variation of $\bar{\bar{J}}$ to vanish with respect to variations in the initial point, it follows that the additional boundary conditions

$$\left.\begin{aligned} & \nu_j \frac{\partial \theta_j}{\partial x_i} - p_i = 0 \qquad i = 1, n \\ & H + \nu_j \frac{\partial \theta_j}{\partial t} = 0 \end{aligned}\right\} \quad t = t^{\circ} \tag{2.3.145}$$

must hold at the start of the solution. Thus, the optimal solution, for the case in which the initial point lies in the set given by equation (2.3.143), must satisfy, in addition to equation (2.3.144), the boundary conditions of equation (2.3.145).

(D) The Admissible Control Set $\mathcal{U}$ Depends on the state x.

Up to this point, the set $\mathcal{U}$ has been a compact set in the r dimensional control space. In the variational development of Section 2.3.5, it was assumed that this set could be represented by an inequality of the form

$$u \in \mathcal{U} \Longleftrightarrow g(u) \leq 0 \tag{2.3.146}$$

Occasionally problems arise in which the set $\mathcal{U}$ depends on the state X, with the inequality in (2.3.146) replaced by

$$u \in \mathcal{U} \Longleftrightarrow g(x, u) \leq 0$$

However, the development of the appropriate necessary conditions in this case is straightforward if the variational approach of Section 2.3.5 is employed

As in Section 2.3.5, the functional $\bar{J}$ corresponding to equation (2.3.80a) is formed, but with the integrand F given by

$$F \equiv 0 = p_i(\dot{x}_i - f_i(x, \dot{z})) + \lambda(g(x, \dot{z}) + \dot{\eta}^2) \tag{2.3.147}$$

Requiring $\bar{J}$ to be optimal with respect to variations in the control and boundary conditions leads to the same necessary conditions as before; that is, equations (2.3.81) to (2.3.89). Substituting the F function of (2.3.147) into these equations provides

(1) Boundary Conditions

$$p_i + \frac{\partial \phi}{\partial x_i} + \mu_j \frac{\partial \psi_j}{\partial x_i} = 0$$

$$H = \frac{\partial \phi}{\partial t} + \mu_j \frac{\partial \psi_j}{\partial t} = 0 \qquad (2.3.148)$$

(2) Corner Conditions

$$p^{(+)} = p^{(-)}$$

$$H^{(+)} = H^{(-)} \qquad (2.3.149)$$

(3) Adjoint or Euler Conditions

$$\dot{p}_i = -p_j \frac{\partial f_j}{\partial x_i} + \lambda \frac{\partial g}{\partial x_i}$$

$$\lambda \dot{n} = 0 \qquad (2.3.150)$$

Note that the adjoint conditions are the same as before except for the additional term, $\lambda \partial g/\partial x_i$, in equation (2.3.150). The Weierstrass condition and the Euler condition of equation (2.3.82) combine to yield

$$H(x, p, u_o) \geq H(x, p, \hat{u}) \qquad (2.3.151)$$

where u_o denotes the optimal control and $\hat{u}$ denotes any other control both of which must lie in the set $\mathcal{V}$ given by

$$u \in \mathcal{V} \Longleftrightarrow g(x, u) \leq 0 \qquad (2.3.151)$$

The multiplier λ is zero if a definite inequality holds in the above expression. If an equality holds, then λ (and the optimal control) is determined from

$$\frac{\partial}{\partial u_k}(H + \lambda g) = 0 \quad ; k = 1, r$$

$$g(x, u) = 0 \qquad (2.3.152)$$

which must be satisfied for H to be maximized.

2.4 BOUNDED STATE SPACE PROBLEM

A problem of current interest in optimization theory is the bounded state space problem. This problem is characterized by an inequality constraint on the state variables of the form

$$g_1(x) \leq 0 \; ; \qquad [t^\circ \leq t \leq t^f] \tag{2.4.1}$$

that is, the functional J

$$J = \phi\,(x^f, t^f) = \text{MIN}$$

is to be minimized subject to the requirements that the control u lie in some specified set $\mathcal{V}$, that the equations

$$\dot{x}_i = f_i(x, u) \quad ; \quad i = 1, n$$

$$x = x^\circ \quad ; \quad t = t^\circ$$

$$\psi_j(x^f, t^f) = 0 \quad ; \quad j = 1, m$$

hold and that the optimal state history satisfies the inequality in (2.4.1). The feature which distinguishes this problem from those treated in Section 2.3 is the explicit independence of the function g_1 in (2.4.1) on the control action u.

Problems of this type arise frequently in flight and space mechanics. For example, in the minimum time to climb problem, the vehicle may be required to stay within a specified region on an altitude-velocity plot to prevent the onset of flutter, or stall, or excessive heating rates. This region of operation, called the flight envelope, can be defined by a state inequality of the form of (2.4.1).

The formulation and the development of the appropriate necessary condition for bounded state problems is straightforward and can be accomplished using the variational methods of Section 2.2. In fact, several different formulations are possible, depending on just how the inequality in (2.4.1) is adjoined to the proglem. All of these methods lead to slightly different but equivalent necessary conditions [see References (19) to (23) and chapter 6 of Reference (4)]. The difficult problem is the interpretation of the necessary conditions in regard to their use in a computational procedure.

There are, in general, four different types of inequality constraints that can occur in optimal control problems:

(1) State inequality

$$g_1(x) \leq 0 \tag{2.4.2}$$

(2) Control inequality

$$g_2(u) \leq 0 \tag{2.4.3}$$

(3) Coupled state and control inequality

$$g_3(x, u) \leq 0 \tag{2.4.4}$$

(4) Integral inequality

$$\int_{t^o}^{t^f} g_4(x, u)\, dt \leq 0 \tag{2.4.5}$$

The second and third type were considered in Section 2.3 and are frequently used as analytical representations for the admissible control set $\mathcal{U}$. The fourth type was treated briefly in Section 2.2.5.

Numerical computation of the optimal control when any or all of the last three types of inequalities are present, always leads to a two-point boundary value problem (that is, a set of differential equations with some of the boundary data given at the initial point and some at the terminal point). However, this is not the case if the optimization problem contains a state inequality of the form (2.4.2). When the state is bounded by the inequality (2.4.2), the numerical generation of the optimal control law requires the solution of at least a three-point boundary value problem; that is, a set of differential equations in which boundary data is given not only at the initial and terminal points, but also at one or more intermediate points. For example, in certain cases (to be discussed later), the junction of an interior segment $g_1(x) < 0$) with a boundary segment $g_1(x) = 0$ must be accomplished so that the optimal trajectory is tangent to the surface $g_1(x) = 0$. Thus, the optimal solution, in addition to meeting certain initial and terminal constraints, must also satisfy a tangency condition at the junction point. The occurence of these intermediate boundary conditions makes the development of numerical solutions more difficult than otherwise.

The intermediate boundary conditions arise from the application of the standard necessary conditions of the calculus of variations. However, the necessary conditions themselves require interpretation before the computation can be performed and there has been some difference of opinion in the literature as to just what constitutes the correct interpretation.

To circumvent difficulties of this type, the approach used here to develop the necessary conditions will be more intuitive and less mathematical than the preceding sections. It is felt that the loss in mathematical rigor by this approach is more than compensated for by a clearer understanding of what the necessary conditions are, and how they are to be used in constructing the solution.

2.4.1 Problem Statement and Composition of the Extremal Arc

The problem under consideration is the determination of the control u from the set $\mathcal{U}$ so that a function of the terminal state is minimized

$$J = \phi(x^f, t^f) = \mathrm{MIN} \tag{2.4.6}$$

subject to the differential constraints

$$\dot{x}_i = f_i(x, u) \quad ; \quad i = 1, n \tag{2.4.7}$$

and boundary conditions

$$\begin{aligned} x &= x^0 \quad ; \quad t = t^0 \\ \psi_j(x^f, t^f) &= 0 \quad ; \quad j = 1, M \end{aligned} \tag{2.4.8}$$

In addition, the state x(t) is required to satisfy the inequality

$$g_1(x) \le 0 \quad ; [t^0 \le t \le t^f] \tag{2.4.9}$$

The optimal trajectory is composed of two types of segments, interior segments $[g_1(x) < 0]$ and boundary segments $[g_1(x) = 0]$. In some problems, it may happen that there are no boundary segments [Figure (1a) and (1b)] or no interior segments [Figure (1c)] but in the general case, both types will be present [Figure (1d)].

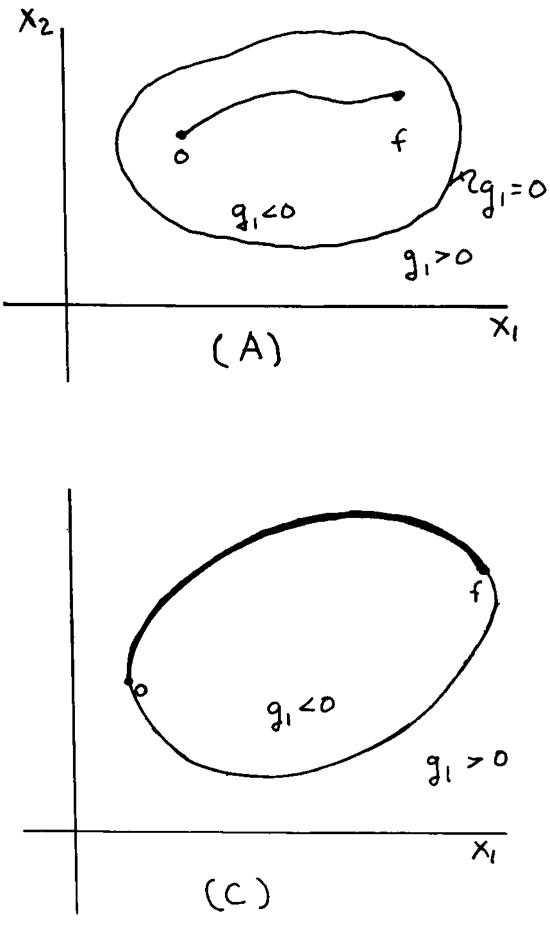


Figure I

Composition of the Extremal Arc in The Two Dimensional Problem

For convenience in developing the appropriate necessary conditions, it will be assumed that

(1) The control set $\mathcal{U}$ can be expressed analytically by the inequality

$$u \in \mathcal{U} \Longleftrightarrow g_2(u) \leq 0 \tag{2.4.10}$$

where the boundary curve $g_2(u) = 0$ is piecewise smooth.

(2) The state boundary curve $g_1(x) = 0$ has continuous second derivatives with respect to all its arguments. In certain cases, higher derivatives than the second will be needed; thus, when these cases arise, the existence of such derivatives will be tacitly assumed.

(3) The initial point X° and the target set $\psi_j\ (x^f, t^f)$ lie in the interior $g_1(x) < 0$.

These assumptions are made only to facilitate the development of the governing equations. The extension of the analysis to cases in which one or more of the above assumptions is relaxed, is again straightforward.

2.4.2 Necessary Conditions

A. Interior Segment

An interior segment $[g_1(x) < 0]$ must satisfy the same necessary conditions as an optimal trajectory for an unconstrained problem; that is, an optimization problem in which inequality (2.4.9) is absent. The reason for this is obvious. If the interior segment connecting the points 2 and f in Figure (1d) did not satisfy these conditions, the performance index ϕ could be decreased by varying this segment. Hence, it follows that along an interior segment

$$\dot{P}_i = -\frac{\partial f_j}{\partial x_i} P_j \tag{2.4.11}$$

$$P_i + \mu_j \frac{\partial \psi_j}{\partial x_i} + \frac{\partial \Phi}{\partial x_i} = 0 \quad ; \qquad i = 1, n \; ; t = t^f$$

$$H = \frac{\partial \phi}{\partial t} + \mu_j \frac{\partial \psi_j}{\partial t} \; ; \qquad t = t^f \tag{2.4.12}$$

$$H(x, P, u_0) \geq H(x, P, \hat{u}) \quad ; u_0, \hat{u} \in \mathcal{U} \tag{2.4.13}$$

B. Boundary Segment

Along a boundary segment, the equation

$$g_1(x)=0 \tag{2.4.14}$$

must hold and, as a result, the additional equations

$$\begin{aligned} \frac{dg_1}{dt} &= 0 \\ \frac{d^2g_1}{dt^2} &= 0 \\ &\vdots \\ \frac{d^ng_1}{dt^n} &= 0 \\ &\vdots \end{aligned} \tag{2.4.15}$$

are also satisfied; that is, all derivatives of the boundary curve are zero by virtue of the fact that the trajectory lies in the boundary surface. Hence, either the constraint (2.4.14) can be directly adjoined to the problem or any one of the derivatives in equation (2.4.15). It turns out to be convenient computationally to work with the first derivative that explicitly contains the control action u. For example, the first derivative takes the form

$$\frac{dg_1}{dt} = 0 = \frac{\partial g_1}{\partial x_i} f_i(x,u) \tag{2.4.16}$$

If the right hand side of (2.4.16) contains u explicitly, the first derivative is adjoined to the problem. If not, the second derivative is computed with

$$\frac{d^2g_1}{dt^2} = \frac{\partial}{\partial X_j}\left(\frac{\partial g_1}{\partial X_i} f_i(x,u)\right) f_j(x,u) = 0 \tag{2.4.17}$$

since the partial $\frac{\partial}{\partial u_K}\left(\frac{\partial g_1}{\partial x_i} f_i\right)$ is, by assumption, zero.

Let K be the first time derivative of g_1 which contains u explicitly after the $\dot{X}_i$ are replaced by $f_i(x,u)$ as in equations (2.4.16) and (2.4.17); that is,

$$\frac{d^K g_1}{dt^K} = G(x,u) = 0 \tag{2.4.18}$$

To find the optimal boundary segment, the modified functional $\overline{J}$

$$\overline{J} = \phi(x^f, t^f) + \mu_j \psi_j(x^f, t^f) + \int_{t^o}^{t^f} F\,dt \tag{2.4.19}$$

is formed where

$$F = \rho_i\left\{\dot{x}_i - f_i(x,u)\right\} + \lambda_1 G(x,u) + \lambda_2\left\{g_2(u) + \dot{\eta}^2\right\} \tag{2.4.20}$$

and where

$$\begin{aligned} \lambda_1 &= 0 \qquad &&; \text{ Along Interior Segment} \\ \lambda_1 &\neq 0 \qquad &&; \text{ Along Boundary Segment} \end{aligned} \tag{2.4.21}$$

By applying the standard necessary conditions to the functional $\overline{J}$, it follows that equations (2.4.11) to (2.4.13) hold along an interior segment ($\lambda_1 = 0$) awhile along a boundary segment ($\lambda_1 \neq 0$)

$$\dot{\rho}_i = -\frac{\partial f_j}{\partial x_i} \rho_j + \lambda_1 \frac{\partial}{\partial x_i}\left\{G(x,u)\right\} \tag{2.4.22}$$

$$\lambda_2 \dot{\eta} = 0 \tag{2.4.23}$$

$$\frac{\partial H}{\partial u_k} - \lambda_1 \frac{\partial}{\partial u_k}\left\{G(x,u)\right\} - \lambda_2 \frac{\partial g_2}{\partial u_k} = 0 \qquad ; k = 1, r \tag{2.4.24}$$

$$H(x,p,u_o) \geq H(x,p,\hat{u}) \tag{2.4.25}$$

In this last inequality u_o is the optimal control and $\hat{u}$ is any other control, both of which must satisfy

$$g_2(u) + \dot{\eta}^2 = 0 \left\{ \text{i.e., } u_o \text{ and } \hat{u} \in U \right\} \tag{2.4.26}$$

$$\frac{d^k g_1(x)}{dt^k} = G(x, u) = 0 \tag{2.4.27}$$

The first three necessary conditions result from the Euler equations while the last is the Weierstrass condition. Note that equation (2.4.24) is redundant since it must hold if equations (2.4.25) to (2.4.27) are satisfied. However, this equation does indicate how the multipliers λ_1 and λ_2 are to be computed. Also, the corner conditions [see equation (2.3.95)] required that the Hamiltonian and the P vector be continuous across an unconstrained corner (i.e., a discontinuity in the derivative $\dot{X}$ resulting from a discontinuity in the control u).

C. Junction Conditions

Since the equations governing the P vector and the control u along, both interior and boundary segments have been developed; it remains to determine how these two segments should be joined together to form the optimal solution. The optimal junction conditions are computed by requiring the first variation of the functional $\bar{J}$ to vanish with respect to variations in the place and time at which the junctions occur.

Let t_ℓ ($\ell = 1,s$) denote the junction times. Since the derivative and control may be discontinuous at t_ℓ, rewrite the functional J in equation (2.4.19) as

$$\bar{J} = \phi(x^f, t^f) + \mu_j \psi_j (x^f, t^f) + \int_{t^o}^{t_1^{(-)}} F\,dt + \int_{t_1^{(+)}}^{t_2^{(-)}} F\,dt + \ldots + \int_{t_s^{(+)}}^{t^f} F\,dt$$

Now, requiring the first variation of J to be zero provides

$$\delta\bar{J} = \left(\frac{\partial\phi}{\partial X_i} + \mu_j \frac{\partial\psi_j}{\partial X_i}\right) dX_i + \left(\frac{\partial\phi}{\partial t} + \mu_j \frac{\partial\psi_j}{\partial t}\right) dt \Big]^{t^f} + \int_{t^o}^{t_1^{(-)}} F_1\,dt + \int_{t_1^{(+)}}^{t_2^{(-)}} F_1\,dt + \ldots + \int_{t_s^{(+)}}^{t^f} F_1\,dt$$

where

$$F_1 = \frac{\partial F}{\partial X_i}\delta X_i + \frac{\partial F}{\partial \dot{X}_i}\delta \dot{X}_i + \frac{\partial F}{\partial u_k}\delta u_k + \frac{\partial F}{\partial \dot{\eta}}\delta\dot{\eta}$$

But, the quantity $\partial F \delta x_i / \partial \dot{x}_i$ can be integrated by parts and the identity

$$\delta x_i = dx_i - \dot{x}_i \, dt$$

can be used to reduce $\bar{J}$ to

$$\delta \bar{J} = \left(\frac{\partial \phi}{\partial x_i} + \mu_j \frac{\partial \psi_j}{\partial x_i} + p_i\right) dx_i + \left(\frac{\partial \phi}{\partial t} + \mu_j \frac{\partial \psi_j}{\partial t} - H\right) dt \Big]^{t^f}$$

$$+ \int_{t^o}^{t_1^{(-)}} F_z \, dt + \int_{t_1^{(+)}}^{t_2^{(-)}} F_z \, dt + \ldots + \int_{t_s^{(+)}}^{t^f} F_z \, dt$$

$$+ \sum_{\ell=1}^{s} \left\{ \left(\frac{\partial F^{(-)}}{\partial \dot{x}_i} - \frac{\partial F^{(+)}}{\partial \dot{x}_i} \right) dx_i - \left(\frac{\partial F^{(-)}}{\partial \dot{x}_i} \dot{x}_i^{(-)} - \frac{\partial F^{(+)}}{\partial \dot{x}_i} \dot{x}_i^{(+)} \right) dt \right\}^{t = t_\ell}$$

where

$$F_z = \left(\frac{\partial F}{\partial x_i} - \frac{d}{dt} \frac{\partial F}{\partial \dot{x}_i} \right) \delta x_i + \frac{\partial F}{\partial u_k} \delta u_k + \frac{\partial F}{\partial \dot{\eta}} \delta \dot{\eta}$$

In view of the optimizing conditions of equations (2.4.11), (2.4.12), and (2.4.22) to (2.4.25), this expression becomes

$$\delta \bar{J} = 0 = \sum_{\ell=1}^{s} \left\{ \left(p_i^{(-)} - p_i^{(+)} \right) dx_i - \left(H(x, p^{(-)}, u^{(-)}) - H(x, p^{(+)}, u^{(+)}) \right) dt \right\}^{t = t_\ell} = 0$$

Furthermore, since the junction variations, dx_i and dt, are unrelated at the different junction times t_ℓ , it follows that for $\delta \bar{J}$ to vanish

$$\left(p_i^{(-)} - p_i^{(+)} \right) dx_i - \left(H(x, p^{(-)}, u^{(-)}) - H(x, p^{(+)}, u^{(+)}) \right) dt = 0 \qquad (2.4.28)$$

$$\text{at } t = t_\ell \quad ; \qquad \ell = 1, s$$

Equation (2.4.28) determines how an interior segment is to be joined to a boundary segment so that the total solution is optimal. Note that if the dx_i and dt were unrelated, the usual corner conditions (unconstrained corner) would result with

$$P_i^{(-)} = P_i^{(+)}$$
$$H^{(-)} = H^{(+)}$$

that is, continuity on the P vector and the Hamiltonian across the corner. In this case, however, the dX_i and dt are related at a junction corner with the exact relationship depending on the form of the boundary surface $g_1(x)=0$.

To begin with, the junction must occur on the surface $g_1\ (x)=0$. Hence, the dX_i and dt satisfy the constraint

$$0\,dt + \frac{\partial g_1}{\partial X_i} dX_i = 0 \qquad at\ t = t_\ell\ ;\ \ell = 1, s \tag{2.4.29}$$

If the first time derivative $dg_1/dt = \partial g_1 f_i / \partial x_i$ contains u explicitly [K in equation (2.4.18) is unity] then equation (2.4.29) is the only additional relationship that must be satisfied at a junction point and the two equations, (2.4.28) and (2.4.29) combine to yield the optimal junction conditions

$$P_i^{(-)} = P_i^{(+)} + \nu_1 \frac{\partial g_1}{\partial X_i} \quad ;\ i = 1, n \tag{2.4.30}$$

$$H^{(-)} = H^{(+)} \tag{2.4.31}$$

where ν_1 is a constant to be determined. If the second derivative for K=2 is the first derivative to contain the control u explicitly, the junction must satisfy the two conditions

$$g_1(X) = 0$$
$$\frac{dg_1}{dt} = \frac{\partial g_1}{\partial X_i} f_i = 0 \qquad t = t_\ell\ ;\ \ell = 1, s \tag{2.4.32}$$

In this case, if the derivative $\frac{dg_1}{dt}$ were not zero, an incoming trajectory on striking the boundary surface $g_1(x)=0$ would go through the surface and violate the condition. Since the first derivative $\frac{dg_1}{dt}$ does not contain u explicitly, the only way that the trajectory will stay on the boundary surface at a junction is if the first derivative is zero. Hence, from (2.4.32) the dX_i and dt must satisfy the equations

$$
\begin{aligned}
&\frac{\partial g_1}{\partial x_i}\, dx_i = 0 \\
&\frac{\partial}{\partial x_i}\left(\frac{\partial g_1}{\partial x_i} f_i(x,u)\right) dx_i = 0
\end{aligned}
\qquad (2.4.33)
$$

Combining these equations with equation (2.4.28) provides the optimal junction conditions

$$
\left.
\begin{aligned}
&p_i^{(-)} = p_i^{(+)} + \nu_1 \frac{\partial g_1}{\partial x_i} + \nu_2 \frac{\partial}{\partial x_i}\left(\frac{\partial g_1}{\partial x_j} f_j\right) \\
&K=2 \\
&H^{(-)} = H^{(+)}
\end{aligned}
\right\} \text{ for } K=2 \qquad (2.4.34)
$$

The quantities ν_1 and ν_2 are constants to be determined. The general case in which K is some arbitrary integer leads to the conditions

$$
\begin{aligned}
&p_i^{(-)} = p_i^{(+)} + \nu_1 \frac{\partial g_1}{\partial x_i} + \nu_2 \frac{\partial}{\partial x_i}\left(\frac{dg_1}{dt}\right) + \dots \nu_K \frac{\partial}{\partial x_i}\left(\frac{d^{K-1} g_1}{dt}\right) \\
&H^{(-)} = H^{(+)}
\end{aligned}
\qquad (2.4.35)
$$

2.4.3 Interpretation of the Necessary Conditions for the Case K = 1

The development of the appropriate necessary conditions for the bounded state space problem is rather straightforward and involves essentially the same mathematical techniques used on other variational problems. The question to be considered next, and one that is not trivial, is the interpretation and use of these conditions in the computing of a solution.

A comparison of equations (2.4.11) and (2.4.12) with (2.4.22) to (2.4.27) indicates that the P vector must satisfy the differential equation

$$
\dot{p}_i = -p_j \frac{\partial f_j}{\partial x_i} + \lambda_1 \frac{\partial}{\partial x_i} G(x,u) \qquad (2.4.36)
$$

and the boundary conditions

$$
\left.
\begin{aligned}
&p_i + \frac{\partial \phi}{\partial x_i} + \mu_j \frac{\partial \psi_j}{\partial x_i} = 0 \\
&H = \frac{\partial \phi}{\partial t} + \mu_j \frac{\partial \psi_j}{\partial t}
\end{aligned}
\right\} t = t^f \qquad (2.4.37)
$$

where λ_1 is zero along an interior segment and non-zero along a boundary segment. The optimal control is computed from

$$
H(x,p,u_o) \geq H(x,p,\hat{u})
$$

where u is constrained to lie in the set $\mathcal{V}$ given by

$$u \in \mathcal{V} \iff g_2(u) \leq 0$$

for an interior segment, and where u must satisfy the two conditions

$$g_2 \leq 0$$

$$\frac{d^K g_1(x)}{dt^K} = G(x,u) = 0$$

on a boundary segment. In addition, the multipliers λ_1 and λ_2 are determined from

$$\frac{\partial H}{\partial u_k} - \lambda_1 \frac{\partial G}{\partial u_k}(x,u) - \lambda_2 \frac{\partial g_2}{\partial u_k} = 0 \quad ; k = 1, r$$

$$\lambda_1 g_1(x) = 0 \qquad (2.4.38)$$

$$\lambda_2 g_2(u) = 0$$

that is, λ_1 and λ_2 are both zero when g_1 and g_2 are both less than zero.

When the optimizing solution enters or leaves the boundary curve $g_1(x)=0$, the junction conditions of equation (2.4.35) must be satisfied. These same junction conditions result regardless of how the bounded state space problem is formulated. For example, if the functional J is formed [see equations (2.4.19) and (2.4.20)] by adjoining the constraint $g_1(x) \leq 0$ directly to the problem rather than the k^{th} derivative $d^k g_1/dt^k$, the same set of junction conditions result. However, while the form is the same, the meaning of the junction conditions [in particular, the value of the multipliers V_i in equation (2.4.35)] varies with the formulation being used. [see Reference (19)].

Another point of difficulty in regards to the junction conditions is that their interpretation depends on whether the junction is an entry corner (solution is entering the boundary surface) or an exit corner (solution is leaving the boundary surface). This point has been the cause of some disagreements in the literature.

To avoid any ambiguity in regards to the interpretation of the junction conditions, attention will be focused on one special case; the case in which K=1 and the first derivative of the boundary surface contains the control u explicitly with

$$G(x,u) = \frac{dg_1}{dt} = \frac{\partial g_1}{\partial x_i} f_i(x,u) = 0 \qquad (2.4.39)$$

Much of the analysis for this case carries over when $K \geq 2$. The interested reader should consult the literature [References (19) to (22)].

A. Strong Weierstrass Condition

From equation (2.4.35), the junction conditions for K=1 reduce to

$$
\begin{aligned}
&g_1(x) = 0 \\
&H(x, p^{(-)}, u^{(-)}) = H(x, p^{(+)}, u^{(+)}) \\
&p_i^{(-)} = p_i^{(+)} + \nu_1 \frac{\partial g_1}{\partial x_i} \quad ; \quad i = 1, n
\end{aligned}
\tag{2.4.40}
$$

with the first of these equations stating that the junction must occur at the boundary surface. These conditions have a simple geometrical interpretation when the strong version of the Weierstrass condition holds.

Recall that the Weierstrass condition requires that

$$H(x, p, u_o) \geq H(x, p, \hat{u}) \tag{2.4.41}$$

where u_o denotes the optimal control and $\hat{u}$ denotes any other admissible control. The strong Weierstrass condition is said to be satisfied if

$$H(x, p, u_o) > H(x, p, \hat{u}) \quad ; \quad \hat{u} \neq u_o \tag{2.4.42}$$

that is, the optimal control provides a larger value for the Hamiltonian than any other control, with an equality condition holding in (2.4.41) only if $\hat{u} = u^o$.

By employing the definition of the Hamiltonian

$$H = p_i f_i$$

Equations (2.4.40) can now be combined to yield

$$H(x, p^{(-)}, u^{(-)}) = H(x, p^{(+)}, u^{(+)})$$

$$p_i^{(-)} f_i^{(-)} = p_i^{(+)} f_i^{(+)} = \left(p_i^{(-)} - \nu_1 \frac{\partial g}{\partial x_i} \right) f_i^{(+)} = p_i^{(-)} f_i^{(+)} - \nu_1 \frac{dg}{dt}^{+}$$

But since $\frac{dg+}{dt}$ must be zero (assuming the + sign denotes the boundary side of the corner) it follows that

$$p_i^{(-)} f_i^{(-)} = p_i^{(-)} f_i^{(+)} \qquad (2.4.43)$$

or

$$H(x, p^{(-)}, u^{(-)}) = H(x, p^{(-)}, u^{(+)}) \qquad (2.4.44)$$

If the system is such that the strong version of the Weierstrass condition holds [inequality (2.4.42)], equation (2.4.44) reduces to the condition

$$u^{(-)} = u^{(+)} \qquad (2.4.45)$$

This condition requires that the solution be tangent to the boundary surface at each junction with the control continuous across the junction. However, if the strong Weierstrass condition does not hold, the tangency condition and continuity in the control vector are not necessary.

B. Computation of the Multiplier ν_1

If a solution in a bounded state space is to be computed, the values of the $P_i^{(-)}$ could be determined at the first entry corner simply by integrating the governing equations from the initial point. The junction conditions of equations (2.4.40) would then appear to determine the $P_i^{(+)}$ (that is, the P vector on the boundary side of the junction) so that the last two of equation (2.4.40) could be considered as a system of n + 1 equations in n + 1 unknown, the $P_i^{(+)}$ and ν_1. However, it can be shown that the coefficient determinant of this system vanishes and that the corner conditions are insufficient to determine the $P_i^{(+)}$ and ν_1. The reason for this is quite simple.

Consider the case in which the strong Weierstrass condition holds and the optimal solution is required to be tangent to the boundary. If the tangency condition is not met then the junction is not optimal and there is no choice of the multiplier ν_1, which will allow the junction conditions of (2.4.40) to be satisfied. On the other hand, if the solution is tangent, any choice of ν_1, will suffice; that is, the junction conditions of (2.4.40) will hold regardless of the value of ν_1. The point is, that an optimal junction cannot be made merely by selecting a certain value for ν_1.

If the junction is correctly made, the value of ν_1 is arbitrary. A rather interesting point is that this arbitrariness is valid only at one end of a boundary segment and not at both ends; that is, if ν_1 is selected to have some value at the entry corner, its value at that exit corner is fixed [see Reference (21)]. This can be shown as follows.

On the interior side of the corner (denoted by the superscript$^{(-)}$) the Hamiltonian must be maximized subject to the constraint

$$u \in U \Longleftrightarrow g_2(u) \leq 0$$

Hence

$$\frac{\partial}{\partial u_k}(p_i f_i - \lambda_2 g_2(u)) = 0 \ ; k = 1, r \tag{2.4.46}$$

From equation (2.4.43)

$$p_i^{(-)} f_i^{(-)} = p_i^{(-)} f_i^{(+)}$$

and therefore equation (2.4.46) must have at least two solutions, $[\lambda_2^{(-)}\ u^{(-)}]$ and $[\lambda_2^{(+)} u^{(+)}]$, both of which maximize the Hamiltonian. On the boundary side, the additional constraint

$$G(x, u^{(+)}) = \frac{\partial g_1}{\partial x_i} f_i(x, u^{(+)}) = 0$$

is imposed and the optimal control must satisfy

$$\frac{\partial}{\partial u_k}\left(p_i^{(+)} f_i^{(+)} - \lambda_2 g_2(u) - \lambda_1 \left[\frac{\partial g_1}{\partial x_i} f_i^{(+)}\right]\right) = 0 \qquad ; k = 1, r \tag{2.4.47}$$

Now, one of the solutions satisfying equation (2.4.46), namely $[\lambda_2^{(+)}, u^{(+)}]$ must also satisfy equation (2.4.47). Substituting

$$p_i^{(-)} = p_i^{(+)} + \nu_1 \frac{\partial g}{\partial x_i} \tag{2.4.48}$$

into (2.4.47) it follows that

$$\lambda_1 = \nu_1 \tag{2.4.49}$$

at a junction corner.

In getting onto the boundary $g_1(x)=0$, any value of ν_1 will do provided equations (2.4.40) are satisfied to begin with. It is then a matter of selecting λ_1 to satisfy equation (2.4.49); that is, the arbitrariness in ν_1 is removed by the appropriate choice of λ_1 at the entry corner. Once the value of λ_1 is specified at the entry corner, its time history along the boundary is determined from equations (2.4.38). Thus, ν_1 is not arbitrary at the exit corner, but must be selected to satisfy (2.4.49).

To summarize, the value for the multiplier ν_1 is arbitrary at one end of the boundary and fixed at the other end. There is no one correct value that this multiplier should take. However, the relationship between λ_1 and ν_1 given in (2.4.49) must be satisfied. For the junction to be optimal there must exist two values, $u^{(-)}$ and $u^{(+)}$, which maximize the Hamiltonian at the entry corner and with $u^{(+)}$ satisfying the additional constraint

$$G(x,u^+) = \frac{dg_1}{dt} = \frac{\partial g_1}{\partial x_i} f_i(x,u^+) = 0$$

If the strong Weierstrass condition holds, then $u^{(-)}=u^{(+)}$ and the solution is tangent to the boundary surface.

C. Getting Off the Boundary

In the process of computing a solution, it is possible that the interior segment will not satisfy the optimal junction conditions of equation (2.4.40); that is, the incorrect initial P vector may have been selected so that upon integrating to the boundary surface, the junction conditions are not met. In this case, it would be concluded that the initial P vector was wrong, and that the resulting interior segment was not a part of the optimal solution. In contrast, it is always possible to go from a boundary segment to an interior segment in an optimal fashion. This situation is analogous to the flying of an airplane in which the pilot finds the take-off (the transition from the two to the three-dimensional space) relatively easy to execute with the landing (the transition from the three to the two-dimensional space) much more difficult.

Since an optimal return to the interior can be made at any point along the boundary segment, the question arises as to when the return should be made. The answer to this question is relatively simple. The point at which the boundary segment should return to the interior is that point for which the remaining segments of the solution will be optimal and also satisfy the required boundary conditions of the problem (i.e., the state and transversality conditions of equations (2.4.8) and (2.4.12)). In other words, the point at which the boundary segment should be left is guessed; the governing state and adjoint equations are integrated to the terminal point, and the terminal conditions are tested to see if they have been satisfied. If they have not, the solution is not optimal.

In the case under consideration in which K=1 and in which the first derivative of $g_1(x)$ contains the control explicitly, the necessity of guessing the time to leave the boundary is easy to demonstrate. The state and adjoint system of differential equations is of order 2n; therefore, 2n+2 boundary conditions are required to generate a solution. The initial state constitutes n+1 conditions and the terminal state and transversability conditions

$$\begin{aligned}
&\psi_j(x^f, t^f) = 0 \quad ; \; j = 1, m \\
&p_i + \frac{\partial \phi}{\partial x_i} + \mu_j \frac{\partial \psi_j}{\partial x_i} = 0 \quad ; \; i = 1, n \\
&H = \frac{\partial \phi}{\partial t} + \mu_j \frac{\partial \psi_j}{\partial t}
\end{aligned} \tag{2.4.50}$$

constitutes another n+1 conditions. To compute a solution, n+1 quantities must be guessed at the initial point, say the initial P vector and the final time t^f, an integration performed, and the solution tested to see if the n+1 terminal conditions of equation (2.4.50) are satisfied. However, while an exit junction can always be made in an optimal manner, an entry junction can not. Therefore, one component of the initial P vector must be taken so that the entry junction conditions of (2.4.40) are satisfied. This leaves only n quantities to be guessed at the initial point, to satisfy n+1 terminal conditions -- for all practical purposes, an impossible situation. Hence, another degree-of-freedom (another quantity to be guessed) must be introduced, and this new quantity is the time at which the boundary surface is to be left and the return to the interior made. Thus, the degree-of-freedom lost in entering the boundary surface is restored at the exit due to the choice of the time at which the exit is to be made.

2.4.4 Discussion

The bounded state space problem differs from other optimization problems in that a multiple point (rather than a two point) boundary value problem must be solved. This makes the generation of numerical solutions extremely difficult. However, the numerical methods developed in Section 2.3.6 can be extended to handle these problems. Such an extension of the gradient method is developed in References (20) and (22).

2.5 LINEAR OPTIMIZATION PROBLEMS WITH CONTROL SEPARABLE

In recent years, a great deal of effort has been expended on the treatment of linear optimization problems, and many significant theoretical and computational advanced have been made. The linear problem differs from other optimization problems in that in many cases solutions can be effected either directly or through iterative techniques which are guaranteed to converge. Hence, it is not uncommon in engineering practice to attempt to replace the nonlinear optimization problem by some linearized approximation to which numerical solutions can be readily developed.

The linear problem with control separable is described by dynamical equations of the form

$$\begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \vdots \\ \dot{X}_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} + \begin{pmatrix} g_1(u) \\ g_2(u) \\ \vdots \\ g_n(u) \end{pmatrix} \tag{2.5.1}$$

or in the vector notation

$$\dot{X} = A\,X + g\,(u)$$

where A is an n x n matrix and g is an n dimensional vector function of the r dimensional control vector u. The boundary conditions take the form

$$X = X^\circ; \quad t = t^\circ \tag{2.5.2}$$

$$d_{ij}\,X_j + e_i = 0 \Longleftrightarrow DX + e = 0; \quad t = t^f \tag{2.5.3}$$

where D is an n x m constant matrix and e is a constant m vector. For convenience, it will also be assumed that the final time t^f is specified to facilitate the presentation; although this assumption is not necessary. The performance index [the quantity $\phi\ (X^f, t^f)$ which is to be minimized] and the admissible control set $\mathcal{U}$ vary from problem to problem along with the explicit dependence of the vector function g(u), appearing in equation (2.5.1), on the control u; thus, the ease with which the optimization problem can be solved depends on the particular form which these quantities take. In the following paragraphs two different linear problems are treated, one of which can be solved directly while for the second, a computational algorithm exists which insures convergence of the iterative process.

2.5.1 Linear Problems with Quadratic Integral Performance Index

In this problem, the control action u is to be determined so as to minimize the integral performance index

$$^{*}J = \int_{t^o}^{t^f} \left\{ \frac{X^T Q X}{2} + \frac{u^T R u}{2} \right\} dt \tag{2.5.4a}$$

or in the scalar notation

$$J = \int_{t^o}^{t^f} \left\{ \frac{x_i q_{ij} x_j}{2} + \frac{u_k r_{ik} u_i}{2} \right\} dt \tag{2.5.4b}$$

where Q is an n x n symmetric positive semidefinite matrix and R is an r x r symmetric positive definite matrix (that is, the quantity $u^T R u/2$ is always greater than zero for $u \neq 0$). The state equations for this system are

$$\dot{X} = A(t)x + B(t)u \tag{2.5.5}$$

where A is an n x n matrix and B is an n x r matrix. The initial state is specified by

$$X = X^\circ; \quad t = t^\circ$$

while the terminal state is required to satisfy the m constraints

$$Dx + e = 0 \Longleftrightarrow \delta_{ij} X_i + e_j = 0 \quad ; t = t^f \; ; j = 1, m \tag{2.5.6}$$

where D is an m x n matrix. The final time, t^f, is assumed to be specified. In addition, the admissible control set $\mathcal{U}$ is the entire r-dimensional control space; that is the control u is unconstrained.

To reduce this problem to the standard form treated in Section 2.3, the variable X_{n+1} is introduced with

$$\begin{aligned} \dot{X}_{n+1} &= X^T \frac{QX}{2} + \frac{u^T R u}{2} \\ X_{n+1} &= 0 \quad ; t = t^\circ \\ J &= X_{n+1}(t_f) = MIN \end{aligned} \tag{2.5.7}$$

*Superscript T denotes transpose

The corresponding adjoint variable P_{n+1} satisfies

$$\dot{P}_{n+1} = -\frac{\partial H}{\partial X_{n+1}} = 0$$
$$P_{n+1}(t^f) = -1 \tag{2.5.8}$$

with the remaining n components of the P vector governed by

$$\dot{P} = -A^T P \tag{2.5.9}$$

$$P + D^T \mu = 0 \,; \quad t = t^f \tag{2.5.10}$$

where μ is an m dimensional multiplier vector

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_m \end{pmatrix} \tag{2.5.11}$$

Note that the n transversality conditions of (2.5.10) in the m + n variables, μ_j and P_i, can be rewritten as a system of n - m equations in the n variables P_i (i=1, n) provided the matrix D has maximum rank (i.e., the terminal constraints are not redundant). If this is done, equation (2.5.10) becomes

$$\hat{D}\, P = 0 \Longleftrightarrow \begin{pmatrix} \hat{d}_{11} & \hat{d}_{12} & \cdots & \hat{d}_{1n} \\ \hat{d}_{21} & \hat{d}_{22} & \cdots & \hat{d}_{2n} \\ \hat{d}_{n-m,1} & \hat{d}_{n-m,2} & & \hat{d}_{n-m,n} \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{pmatrix} = 0 \,; \quad t = t^f \tag{2.5.12}$$

where $\hat{D}$ is an (n - m) x n matrix and is obtained from (2.5.10) by eliminating μ.

From equations (2.5.8) to (2.5.10), the Hamiltonian is given by

$$H = \sum_{i=1}^{n+1} P_i \dot{X}_i = \sum_{i=1}^{n} P_i \dot{X}_i - \dot{X}_{n+1} = \left(P^T \dot{X}\right)_{n\ \text{Demensional}} - \dot{X}_{n+1}$$
$$= P^T(Ax + Bu) - \frac{X^T Q X}{2} - \frac{u^T R u}{2} \tag{2.5.13}$$

and since no constraint is placed on the control (i.e., the set $\mathcal{V}$ is the entire r-dimensional control space) it follows that for H to be maximized

$$\frac{\partial H}{\partial u_k} = 0 \qquad ; \; k = 1, r \tag{2.5.14}$$

so that the optimal control is given by

$$u = R^{-1} B^T p \tag{2.5.15}$$

Combining equations (2.5.5), (2.5.9), and (2.5.15), the trajectory which optimizes the performance index given in (2.5.4a), satisfies the 2n system

$$\begin{aligned} \dot{x} &= Ax + BR^{-1}B^T p \\ \dot{p} &= -A^T p + Qx \end{aligned} \tag{2.5.16}$$

and the boundary conditions

$$x = x^o \quad ; \quad t = t^o \tag{2.5.17}$$

$$\left.\begin{aligned} Dx + e &= 0 \\ \hat{D} p &= 0 \end{aligned}\right\} \; t = t^f \tag{2.5.18}$$

Note that this system constitutes a linear two-point boundary value problem and can, therefore, be solved directly. Let Λ denote the 2n by 2n matrix satisfying the equations

$$\dot{\Lambda} = \begin{pmatrix} A & BR^{-1}B^T \\ Q & -A^T \end{pmatrix} \Lambda \; , \quad \Lambda(t = t^o) = I$$

where **I** is the 2n x 2n unit matrix. Then it is easy to verify that (2.5.16) has the solution

$$\begin{pmatrix} x^f \\ p^f \end{pmatrix} = \Lambda(t^f) \begin{pmatrix} x^o \\ p^o \end{pmatrix} \tag{2.5.19}$$

The initial P vector, P^o, is easily determined by substituting equation (2.5.19) into the n terminal conditions of equation (2.5.18).

From equation (2.5.19), it follows that the X and P vectors are linearly related. Hence, the optimal control of equation (2.5.15) is a linear function of the state of the system. This relationship proves useful in optimal guidance theory where knowledge of the control as a function of state allows for a rather simple feedback mechanization. This point is discussed in References (24) and (25) where it is also shown that optimal guidance problems are usually of the linear dynamics-quadratic performance type.

2.5.2 Linear Problems with Linear Cost

Again, the dynamical equations take the form

$$\dot{\hat{X}} = \hat{A}(t)\hat{X} + \hat{g}(u) \tag{2.5.20}$$

but in this case, the control u is to be selected from the set U so that the linear function of the state

$$\hat{\phi} = \hat{c}_i \hat{X}_i = \hat{c}^T \hat{X} = min. \tag{2.5.21}$$

is minimized subject to the boundary conditions

$$\hat{X} = \hat{X}^o \quad , \quad t = t^o \tag{2.5.22}$$

$$\hat{D}\hat{X} + \hat{e} = 0 \Longleftrightarrow \hat{d}_{ji}\hat{X}_i + \hat{e}_j = 0 \qquad j = 1, m \tag{2.5.23}$$

with the final time t^f specified. Unlike the previous problem, the set can be any compact region in the r-dimensional control space.

Problems of this type have been extensively analyzed in the literature. Because of the form of the performance index and the control set V, the combined state and adjoint equations are seldom linear and therefore solutions cannot be effected directly. However, L. Neustadt [References (26) and (27)] has developed an iterative process for solving such problems which will converge regardless of the starting condition. What's more, this iterative process has been shown to be highly effective on several difficult problems [References (28) and (7)]. The following paragraphs contain an outline of Neustadt's method for the linear problem given in equations (2.5.20) to (2.5.23).

To simplify the presentation, it is convenient to put the boundary conditions of equation (2.5.23) in a slightly different form. Let D be the n x n matrix

$$D = \begin{bmatrix} \hat{D} \\ 0, I \end{bmatrix} \tag{2.5.24}$$

where $\hat{D}$ is the m x n matrix in equation (2.5.23) and I is an n-m unit matrix, and consider the transformation

$$x = D\hat{x} \tag{2.5.25}$$

which is nonsingular provided $\hat{D}$ has maximum rank. With this transformation, the optimization problem in equations (2.5.20) to (2.5.23) reduces to minimizing

$$\phi = c_i \, x_i^f \tag{2.5.26}$$

subject to the equations

$$\dot{x} = Ax + g(u) \tag{2.5.27}$$

and boundary conditions

$$x = x^o \quad ; \quad t = t^o \tag{2.5.28}$$

$$\begin{aligned} x_j &= x_j^f \text{ (specified)} \; ; \; j = 1, m \; ; \; t = t^f \\ &= -e_i \end{aligned} \tag{2.5.29}$$

The hatted vectors and matrices $\hat{A}$, $\hat{X}$, $\hat{C}$, $\hat{g}$, $\hat{e}$ are related to the unhatted quantities A, X, C, g, e through the linear transformation in equation (2.5.25). At the terminal point, the first m components of the new state vector X, $\{x_1^f, x_2^f \ldots x_m^f\}$, are specified. Also, the quantity ϕ in equation (2.5.26) can be written as

$$\phi = \phi_1 + \phi_2 = \sum_{i=1}^{m} c_i \, x_i(t^f) + \sum_{i=m+1}^{n} c_i \, x_i(t^f)$$

where ϕ_1 is specified since the X_i^f (i = 1, m) are specified. Hence, minimizing ϕ in equation (2.5.26) is equivalent to minimizing the reduced quantity

$$\phi_2 = \sum_{i=m+1}^{n} c_i \, x_i(t^f) \tag{2.5.30}$$

Neustadt's method will now be used to determine the control u from the set $\mathcal{V}$ which minimizes $\emptyset_2$ in equation (2.5.30) subject to the constraints of equations (2.5.27) to (2.5.29).

Using the variation of parameters technique, equations (2.5.27) can be shown to have the quadrature

$$x(t^f) = \mathcal{X}(t^f)\left\{x^o + \int_{t^o}^{t^f} \mathcal{X}^{-1}(\tau)\, g(u(\tau))\, d\tau\right\} \tag{2.5.31}$$

where $\mathcal{X}$ is the n x n fundamental matrix solution satisfying the equation

$$\dot{\mathcal{X}} = A\mathcal{X} ; \qquad \mathcal{X}(t = t^o) = I, \quad \text{The Unit Matrix}$$

Let X_i^f , i = 1, m denote the specified terminal components of equation (2.5.29) and let $X_i(t^f)$, i = 1, n denote the terminal state resulting from the solution of (2.5.31) using some particular control u(t) (not necessarily optimal) in the set $\mathcal{V}$. Define the vector Y by

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} X_1^o \\ X_2^o \\ \vdots \\ X_n^o \end{pmatrix} - \mathcal{X}^{-1}(t^f) \begin{pmatrix} X_1^f \\ X_2^f \\ \vdots \\ X_m^f \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \tag{2.5.32}$$

and the vector Z by

$$Z = \begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{pmatrix} = \begin{pmatrix} X_1^o \\ X_2^o \\ \vdots \\ X_n^o \end{pmatrix} - \mathcal{X}^{-1}(t^f) \begin{pmatrix} X_1(t^f) \\ X_2(t^f) \\ \vdots \\ X_n(t^f) \end{pmatrix} \tag{2.5.33}$$

Now, note that Y is a fixed quantity (independent of the control u) whose value is determined from the boundary conditions of equations (2.5.20) and (2.5.29), while Z varies with and is dependent on the control action u. Further, it follows from the definition of Z and equation (2.5.31) that

$$Z = -\int_{t^o}^{t^f} \mathcal{X}^{-1}(\tau)\, g(u(\tau))\, d\tau \tag{2.5.34}$$

Turning to the Maximum Principle, the function H is formed where

$$H = p^T \dot{x} = p^T A x + p^T g(u) \tag{2.5.35}$$

with

$$\dot{p} = -A^T p \tag{2.5.36}$$

and

$$p_i(t^f) = -C_i \quad ; \; i = m+1, n \tag{2.5.37}$$

so that minimization of ϕ_2 requires that the control action be selected from the admissible control set such that

$$H_1 = p^T g(u) = MAX \tag{2.5.38}$$

is maximized.

To determine a solution, the initial value of the P vector must be known. Denote this value P° where P° must be chosen to satisfy the terminal conditions of equation (2.5.37). Therefore, only m components of P° can be selected independently. Let η be an m dimensional vector

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_m \end{pmatrix}$$

with $P^\circ = P^c(\eta)$; that is, the n-dimensional vector P° is determined by the m-dimensional vector η and the n-m boundary conditions of equation (2.5.37). Furthermore, it follows that an optimal control, that is, one satisfying the minimum condition of equation (2.5.38) depends only on η . Such a control will be denoted by $u=u(t,\eta)$. Further, to emphasize the dependence on η , the P vector is denoted by

$$P = P(\eta)$$

and the Z vector by

$$z = z(\eta) = -\int_{t^o}^{t^f} X^{-1}(\tau)\, g(u(\tau,\eta))\, d\tau \tag{2.5.39}$$

Consider the dot product $P^\circ(\eta)\cdot Z(\eta)$. From equation (2.5.36), it follows that

$$p^{T}_{o}(\eta) \cdot X^{-1}(t) = p^T(t,\eta)$$

Hence

$$p_o^T(\eta)\cdot z(\eta) = \int_{t^o}^{t^f} -p(\tau,\eta)\, g(u(\tau,\eta))\, d\tau \qquad (2.5.40)$$

Thus

$$p_o^T(\eta)\cdot z(\eta) \leq p_o^T(\eta)\cdot z(\eta^*) \ ; \qquad \eta^* = \eta \qquad (2.5.41)$$

since if $P_o^T(\eta).Z(\eta)$ were not less than, or equal to $P_o^T(\eta).Z(\eta^*)$, the integrand $\{-p^T(\tau,\eta)\cdot g(u(\tau,\eta))\}$ would have to be larger than $\{-p^T(\tau,\eta)\cdot g(u(\tau,\eta^*))\}$ for some τ between t^o and t^f which contradicts the maximum condition of equation (2.5.38). It is this inequality, (2.5.41), which is the basis of Newstadt's computational scheme.

Let $\hat{\eta}$ be the value of η which solves the problem. Then from (2.5.41)

$$p_o^T(\eta)\cdot\left\{z(\hat{\eta}) - z(\eta)\right\} \geq 0$$

But from the definition of Y [equation (2.5.32)], it follows that

$$p^o(\eta)\cdot z(\hat{\eta}) = p^o(\eta)\cdot y - \phi_{2\,MIN}$$

where $\phi_{2_{Min}}$ is the desired minimum value of ϕ_2. Hence

$$p_o^T(\eta)\cdot(y - z(\eta)) \geq \phi_{2MIN}.$$

At this point, if the function $F(\eta)$ is defined by

$$F(\eta) = p_o^T(\eta)\cdot\left\{y - z(\eta)\right\} \qquad (2.5.42)$$

then

$$F(\eta) \geq \phi_{2\,MIN}. \qquad (2.5.43)$$

and

$$F(\hat{\eta}) = \phi_{z\,MIN.}$$

Thus η is to be chosen so that $F(\eta)$ is a minimum; that is, the problem of determining the correct initial P° vector has been reduced to that of determining the minimum value of the function F.

To determine η for which F is a minimum, Neustadt suggests the iteration scheme

$$[\eta]_{j+1} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_m \end{pmatrix}_{j+1} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_m \end{pmatrix}_j - K \nabla F(\eta) \tag{2.5.44}$$

where $[\eta]_{j+1}$ is the j+1 iteration for η and K is a small constant equal to the magnitude of the correction vector. Usually, the selection of the quantity K proves difficult and frequently an alternate scheme is used. One such scheme, developed by M. Powell, is discussed in Reference (29) with the combined Neustadt-Powell procedure for solving linear optimization problems treated in Reference (28).

Like the gradient approach of equation (2.5.44), Powell's method also requires the computation of the gradient, ∇F . One of the advantages of Newstadt's approach is that this quantity can be determined analytically, with the value of the gradient easily computed at any point where the value of the function F is known.

From equation (2.5.41)

$$\nabla F = \nabla\left\{P^\circ(\eta)\cdot(y - z(\eta))\right\} = \nabla\left\{P^\circ(\eta)\cdot y\right\} - \nabla\left\{P^\circ(\eta)\cdot z(\eta)\right\} \tag{2.5.45}$$

The quantity $\nabla(P^\circ . Y)$ can be calculated once the dependence of P° on η is specified. It is given in component form by the expression

$$\nabla\left\{P^\circ\cdot y\right\} = \sum_{j=1}^{n} \frac{\partial P_j^\circ}{\partial \eta_i} y_j \quad ; \quad i = 1, m$$

The second quantity on the right hand side of (2.5.45) is given by

$$\nabla\left\{P^\circ\cdot z\right\} = \nabla_{P^\circ}\left\{P^\circ\cdot z\right\} + \nabla_z\left\{P^\circ\cdot z\right\} \tag{2.5.46}$$

where the symbol ∇p° indicates the gradient operator operating only on the P° vector in the product $P^\circ \cdot Z$. The second term $\nabla_Z(P^\circ \cdot Z)$ can be shown to vanish.

Let

$$\theta(\eta) = p^\circ(\eta) \cdot z(\eta)$$

and

$$\theta(\eta + \Delta\eta_i) = p^\circ(\eta) \cdot z(\eta + \Delta\eta_i)$$

then for $\nabla_Z(P^\circ \cdot Z)$ to vanish, it must be shown that

$$\lim_{\Delta\eta_i \to 0} \left| \frac{\theta(\eta + \Delta\eta_i) - \theta(\eta)}{\Delta\eta_i} \right|$$

goes to zero. Now

$$\theta(\eta + \Delta\eta_i) - \theta(\eta) = p^\circ(\eta) \cdot z(\eta + \Delta\eta_i) - p^\circ(\eta) \cdot z(\eta)$$

and from (2.5.41)

$$\left| \theta(\eta + \Delta\eta_i) - \theta(\eta) \right| = p^\circ(\eta) \cdot z(\eta + \Delta\eta_i) - p^\circ(\eta) \cdot z(\eta)$$

Hence,

$$\left| \frac{\theta(\eta + \Delta\eta_i) - \theta(\eta)}{\Delta\eta_i} \right| = \frac{1}{|\Delta\eta_i|} \left\{ p^\circ(\eta) \cdot z(\eta + \Delta\eta_i) - p^\circ(\eta) \cdot z(\eta) \right\} \quad (2.5.47)$$

Adding and subtracting the quantity $P^\circ(\eta + \Delta\eta_i) \cdot Z(\eta)$ from the right hand side of (2.5.47) provides

$$\left| \frac{\phi(\eta + \Delta\eta_i) - \phi(\eta)}{\Delta\eta_i} \right| = \frac{1}{|\Delta\eta_i|} \Big\{ p^\circ(\eta) \cdot z(\eta + \Delta\eta_i) - p^\circ(\eta + \Delta\eta_i) \cdot z(\eta)$$

$$- (p^\circ(\eta) - p^\circ(\eta + \Delta\eta_i)) \cdot z(\eta) \Big\}$$

and since

$$p^\circ(\eta + \Delta\eta_i) \cdot z(\eta) \geq p^\circ(\eta + \Delta\eta_i) \cdot z(\eta + \Delta\eta_i)$$

it follows that

$$\left|\frac{\phi(\eta+\Delta\eta_i)-\phi(\eta)}{\Delta\eta_i}\right| \le \frac{1}{|\Delta\eta_i|}\left\{\left[p^\circ(\eta)-p^\circ(\eta+\Delta\eta_i)\right]\cdot\left[z(\eta+\Delta\eta_i)-z(\eta)\right]\right\}$$

$$\le \frac{1}{|\Delta\eta_i|}\left|p^\circ(\eta)-p^\circ(\eta+\Delta\eta_i)\right|\left|z(\eta+\Delta\eta_i)-z(\eta)\right|$$

Taking the limit, provides the desired result; i.e.,

$$\nabla_z (p^\circ \cdot z) = 0$$

Combining this result with equations (2.5.45) and (2.5.46) gives the gradient in component form

$$\nabla F(\eta) = \sum_{j=1}^{N} \frac{\partial p_j}{\partial \eta_i}\,(y_j - z_j)\ , \quad i = 1, m \tag{2.5.48}$$

To summarize, the determination of the optimal control action requires the determination of the correct value of the vector P°. This vector, P°, in turn, is an n dimensional vector whose components are determined from the m dimensional vector η and the n-m boundary conditions of equation (2.5.37). Thus, it is the correct value of η which is sought. It follows that the particular η which solves the problem also minimizes the function $F(\eta)$ given by equation (2.5.42). Therefore, starting with a first guess of η, corrections in η are made in the direction in which $F(\eta)$ is decreasing. Such a correction process requires a knowledge of the gradient, ∇F, which is given analytically by equation (2.5.48).

3.0 RECOMMENDED PROCEDURES

Both the Maximum Principle and the Calculus of Variations represent simple but rather general techniques for formulating optimization problems. Thus, little remains to be accomplished as far as broadening or extending these formulations. There are, however, many special problems and applications that warrant addition investigation.

Among these special problems is the singular arc problem discussed in connection with the lunar soft landing (Section 2.3.2) and with the orbital transfer maneuver (Section 2.3.4). In this case, the Maximum Principle becomes degenerate and higher order terms in the series expansion about the optimal solution must be examined to determine if these arcs are minimizing. Such a procedure has been carried out in the literature [References (12) and (13)] and additional necessary conditions developed which the singular arc must satisfy. However, little is known at this writing as to when such arcs will occur.

Another problem area is the optimization of stochastic systems. If the system under consideration contains parameters or elements which have a statistical rather than a deterministic description, then the usual optimization methods are not directly applicable. Though some stochastic problems have been analyzed using the Maximum Principle and the Calculus of Variations in modified form, the techniques have not been standardized to the extent that they have in the deterministic case. Such a standardization will evolve only through the extensive analysis of a variety of stochastic optimization problems.

In regards to applications, there are many problems in both the trajectory and control areas that require additional work. For example, most optimal control studies have been conducted on an open-loop basis despite the fact that almost all controllers operate closed-loop. While a complete synthesis of an optimal controller may require faster computers and better computational techniques, some studies should be initiated at this time which are concerned with closed-loop operation.

Applications in trajectory analysis have been limited for the most part to vehicles which move in an airless inverse square gravitational field. However, there are many interesting optimization problems which arise in connection with the atmospheric section of the flight and where the inverse square field assumption is a poor approximation of the total force applied. Among these problems are horizontal take-off systems, planetary entry maneuvers for high L/D vehicles and lunar flyby and return missions. All of these problems can be analyzed and solved using existing techniques.

One of the major difficulties in optimization theory is the generation of numerical solutions. With few exceptions, the variational formulation leads to a nonlinear two-point boundary value problem, the solution of which must be developed iteratively. While considerable progress has been made during the past few years in this area, and many new computational methods have been developed, there still exists no one method which will consistently work for a relatively large class of problems. Thus, while the trajectory analyst may find the numerical problems severe, with patience he can develop optimal trajectories using existing numerical techniques. In control theory, on the other hand, the numerical difficulties rule out the use of optimal controllers as part of a feedback mechanization. In this case, the numerical methods and the available computing equipment are not sufficiently fast of sufficiently reliable for a closed-loop operation and some technical breakthrough is needed before optimal controllers will become feasible. A more wide spread use of the techniques of optimization theory, therefore impatiently awaits the development of better computational methods.

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