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GUIDANCE, FLIGHT MECHANICS AND TRAJECTORY OPTIMIZATION

Volume III - The Two-Body Problem

by G. E. Townsend and M. B. Tamburro

Prepared by
NORTH AMERICAN AVIATION, INC.
Downey, Calif.
for George C. Marshall Space Flight Center

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FOREWORD

This report was prepared under contract NAS 8-11495 and is one of a series intended to illustrate analytical methods used in the fields of Guidance, Flight Mechanics, and Trajectory Optimization. Derivations, mechanizations and recommended procedures are given. Below is a complete list of the reports in the series.

Volume I	Coordinate Systems and Time Measure
Volume II	Observation Theory and Sensors
Volume III	The Two Body Problem
Volume IV	The Calculus of Variations and Modern Applications
Volume V	State Determination and/or Estimation
Volume VI	The N-Body Problem and Special Perturbation Techniques
Volume VII	The Pontryagin Maximum Principle
Volume VIII	Boost Guidance Equations
Volume IX	General Perturbations Theory
Volume X	Dynamic Programming
Volume XI	Guidance Equations for Orbital Operations
Volume XII	Relative Motion, Guidance Equations for Terminal Rendezvous
Volume XIII	Numerical Optimization Methods
Volume XIV	Entry Guidance Equations
Volume XV	Application of Optimization Techniques
Volume XVI	Mission Constraints and Trajectory Interfaces
Volume XVII	Guidance System Performance Analysis

The work was conducted under the direction of C. D. Baker, J. W. Winch, and D. P. Chandler, Aero-Astro Dynamics Laboratory, George C. Marshall Space Flight Center. The North American program was conducted under the direction of H. A. McCarty and G. E. Townsend.

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LIST OF SYMBOLS

A	Area
a	Semimajor axis
b	Semiminor axis
E	Eccentric anomaly
e	Eccentricity
F	Force vector magnitude; or hyperbolic anomaly
f	Force per unit mass; coefficient employed to generate $\vec{r}$ (t) from $\vec{r}$ (o), $\vec{v}$ (o)
G	Newton's universal gravitation constant
g	Coefficient employed to generate $\vec{r}$ (t) from $\vec{r}$ (o), $\vec{v}$ (o)
H	Total angular momentum
h	Angular momentum per unit mass
i	Orbital inclination to the reference plane (generally equatorial reference)
$J_n(ne)$	Bessel function of the first kind
l	A vector directed toward the node $[\vec{l} = (\hat{h}\cdot\hat{y})\hat{x} - (\hat{h}\cdot\hat{x})\hat{y}]$
M	Mean anomaly
m	Mass
N	The nodal unit vector
$P_n(\mu)$	Legendre polynomial
p	Semilatus rectum
r	Radial distance (from the center of force unless otherwise specified)
T_o	Time of periapse passage
t	Time of the instant
U.T.	Universal time
v	Velocity (or speed)
X,Y,Z	Three components of the force vector
x,y,z	Components of position (cartesian)
γ	Flight path angle relative to local horizontal
ϵ	Total energy unit per mass
θ	True anomaly (angle from periapse to the instantaneous position); Euler angle equivalent to right ascension
μ	The gravitational constant G_m; argument of Legengre polynomial
τ	Orbital period
φ	Central angle from the node to the instantaneous ; or position $(\theta + \omega)$; Eugler angle equivalent to latitude

Ω	Longitude of the ascending node
ω	Argument of perigee

Subscripts

a	Apoapse
o	Initial
p	Periapse

Superscript Notations

$\hat{}$	Unit vector
$\vec{}$	Vector
$\dot{}$	$\frac{d}{dt}(\ )$
$\ddot{}$	$\frac{d^2}{dt^2}(\ )$

1.0 STATEMENT OF THE PROBLEM

The approximate analysis of the motion of near-earth (planet) satellites, and/or the generation of precise trajectories via derivatives of an Encke formulation (employing a reference trajectory) or via an osculating conic formulation, require that the nature of the motion be well known and expressible in a simple well-determined form. Thus, the fundamental objective of this Monograph is the presentation of information adequate to satisfy these requirements and sufficient to introduce material to be prepared in other Monographs of the series. This objective will be achieved by developing the classical solution and modifying its form to assure that a completely deterministic solution is available.

The two-body problem (the analysis of the motion of two bodies acted upon only by their own mutual attraction) was one of the earliest problems in dynamics to be solved. Thus, the material to be presented does not represent the current status of a rapidly changing field on analysis as do some of the presentations in other Monographs. Rather, the material is intended to express the results of these previous analyses, to express the observations regarding indeterminacies in the most commonly used form of the solution, and to provide alternate formulations of the motion to avoid the computational problems. In addition, this presentation is intended to function as a reference volume providing detailed tabulations of equations relating the most basic parameters of the motion and the dynamics.

2.0 STATE OF THE ART

2.1 THE LAW OF GRAVITATION

The discussion of particle dynamics has its mathematical origin with Sir Isaac Newton and his statement of three principles of mechanics. These principles are:

1) Every particle continues in a state of rest or uniform motion unless compelled by an external force to change that state.

2) The product of the mass and the acceleration of a particle is proportional to the force applied to the particle, and the acceleration is in the same direction as the force.

3) When two particles exert forces on each other, the forces have the same magnitudes and act in opposite directions along the line joining the two particles.

The first two laws, which are related, can be written in the following mathematical form:

$$\bar{F} = K m \bar{a}$$

or,if the units of time, mass, length and force are selected properly

$$\bar{F} = m \bar{a}$$

This law of motion is general in form since no restrictions have been placed on the form of the force or its origin. Thus, the general solution to this sixth order set of equations (i.e., second order in each of three coordinates) involves the description of the force as a function of position or time and the analytical or numerical solution for the resultant motion. The special case of two-body orbits requires the description of the particular force which is the result of the mutual attraction of the masses. (Forces of other origin and their effects are presented in other monographs of this series.) Since the description of such a force is most easily accomplished by referring to the deductions of astronomers in the 16th and 17th centuries, a brief resume of the steps leading to a law of gravitation will be made.

Copernicus (1543) expounded the theory that the motions of the planets were sun centered (heliocentric) rather than earth centered as assumed in the Ptolemaic system. This theory laid the ground work for Johannes Kepler, who in 1609 deduced three laws of planetary motion from the observations of Tycho Brahe, and in 1619 deduced a fourth. These laws are:

1) The heliocentric motions of the planets take place in fixed planes passing through the sun.

2) The area of the sector traced by the radius vector from the sun, between any two points in the orbit, is proportional to the time spent in the arc.

3) The planetary orbit is an ellipse with the sun at one focus.

4) The square of the period is proportional to the cube of the semi-major axis.

Items 2, 3, and 4 are commonly referred to as Kepler's first, second, and third laws, respectively.

With these laws of motion as a guide, it is possible to derive the law of gravitation.

Consider Kepler's observations. Since the observed motions are planer, the force vector can be concluded to lie in the plane of motion. Further, since the areal velocity

$$\dot{A} = \tfrac{1}{2} \left| \vec{r} \times \dot{\vec{r}} \right|$$

$$= \tfrac{1}{2} r^2 \dot{\theta} = h/2 \qquad \text{(a constant)} \qquad (1)$$

was observed to be constant, the coordinate system which suggests itself is polar. Thus, the first step in the derivation of the force law is the derivation of the acceleration vector in such a system. Consider an inertial frame with the fundamental plane lying in the plane of the observed motion, with the origin of coordinates at the center of **mass** and with principal direction selected in such a manner that it locates the minimum radius (periapse) for the ellipse (or some definable spatial feature for the case of a circle) describing the motion of some **body.**

Now, since $\vec{r} = r \hat{r}$

the velocity and acceleration vectors are

$$\dot{\vec{r}} = \dot{r} \hat{r} + r \dot{\hat{r}}$$

$$= \dot{r} \hat{r} + r \dot{\theta} \hat{\theta}$$

and

$$\ddot{\vec{r}} = \ddot{r} \hat{r} + 2 \dot{r} \dot{\hat{r}} + r \ddot{\hat{r}}$$

$$= (\ddot{r} - r \dot{\theta}^2) \hat{r} + (2 \dot{r} \dot{\theta} + r \ddot{\theta}) \hat{\theta}$$

But the coefficient of $\hat{\theta}$ is observed to be $\frac{1}{r} \frac{d}{dt}(h)$, which is zero. Thus, by equating components of the force and acceleration vectors, it is observed that f is directed along the radius and satisfies the equation

$$\ddot{r} - r \dot{\theta}^2 = -f \qquad (2)$$

Further, $\dot{\theta}$ can be eliminated by employing equation (1).

Thus, if a variable u is defined to be $1/r$ and if the following derivatives are evaluated

$$\dot{r} = \frac{-1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} = -h\frac{du}{d\theta}$$

$$\ddot{r} = -h\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -h\frac{d^2u}{d\theta^2}\frac{d\theta}{dt} = -h^2u^2\frac{d^2u}{d\theta^2}$$

the equation of the force is obtained as

$$f = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) \tag{4}$$

This is a general force law for radial acceleration in the planar problem as it appears in many texts (e.g., Reference 1). But Kepler's second law (Item 3) states that the orbit is an ellipse with the central force at the focus; thus the motion satisfies the equation

$$r = \frac{p}{1 + e\cos\theta}$$

where p and e are constants which describe the geometry of the ellipse and θ is the angle between $\bar{r}$ and the half of the major axis containing the central force (the focus). Differentiating the polar form of the ellipse and forming the function $\frac{f}{h^2u^2}$ now reveals that

$$u + \frac{d^2u}{d\theta^2} = \frac{1}{p}$$

and

$$f = \frac{h^2u^2}{p} = \frac{K}{r^2} \tag{5}$$

From these equations and the observational data available to him, Newton was able to deduce that the constant K must be of the form

$$K = GM$$

where G is an absolute constant which is independent of the masses and of their distance (the best experimental value of this constant is currently

$$G = 6.670 \times 10^{-8}\ cm^3/gr.\ sec^2)$$

and where the quantity M is **a function** of the masses, i.e.,

$$M = m_1 m_2$$

Thus, the force exerted on the orbiting mass assumes the form

$$\bar{F} = m_2\bar{a} = -m_2 f\hat{r}$$

$$= -G m_1 m_2 \frac{\hat{r}}{r^2}$$

$$\equiv \frac{-\mu_1 m_2}{r^3}\bar{r}$$

The universality of this law can be seen from the proofs of Bertrand (Comptes Rendus LXXVII, page 849) which state that the only two laws of attraction which can lead to elliptic motion are:

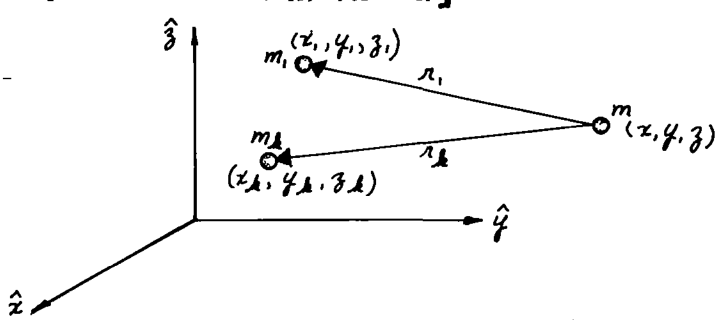
$$f = K/r^2 \text{ and } f = Kr$$

However, to date no cases in which the latter law applies have been observed; thus, it must be assumed that the former law is always applicable.

Before passing into the discussion of the two-body problem, it is noted that this law was obtained for motions in which the dimensions of both bodies were small, relative to the distance between them. Thus, modifications to the form must be expected for smaller distances where mass asymmetry relative to the plane of motion becomes more significant (see Appendix C). However, to preclude the restriction of the law to the case of particle motion, it will be shown that the form of the law is identical to that obtained for the case where the bodies have finite dimensions, but are constructed in homogeneous concentric layers. In this manner, the law can be extended to the earth (or planets) and close satellites to the first order.

The process of proving that finite bodies constructed in homogeneous concentric shells produce force fields of the same form as do particles will be accomplished by introducing the gravitational potential and comparing the potential function for such bodies with that for a collection of discrete particles.

Consider a mass particle m with inertial coordinates (x, y, z) and a system of n other mass particles m_1, m_2, . . . m_n [where the coordinates of the kth particle are (x_k, y_k, z_k)].



The attraction exerted on m by m_1 is Gmm_1/r_1^2, in the direction from m to m_1 (r_1 is the distance from m to m_1). Let X, Y, Z, represent the components of the force exerted on m by m_1. Then

$$X_1 = \frac{G m m_1}{r_1^2} \frac{x_1 - x}{r_1}$$

But

$$r_1^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2$$

Thus

$$r_1 \frac{\partial r_1}{\partial x} = x - x_1$$

and

$$\frac{G m m_1}{r_1^2}\left(\frac{\partial r_1}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{G m m_1}{r_1}\right)$$

The sum of the attractions on m in the x direction due to the n masses is therefore

$$X = \frac{\partial}{\partial x} \sum_{i=1}^{n} \frac{G m m_i}{r_i}$$

The summation of terms in the last expression is defined as the work function due to mass particles $m_1, m_2, \ldots m_n$ and its negative equivalent as the potential at (x, y, z) due to mass particles $m_1, m_2, \ldots m_n$, i.e.,

$$U = -\sum_{i=1}^{n} \frac{G m m_i}{r_i} \tag{6}$$

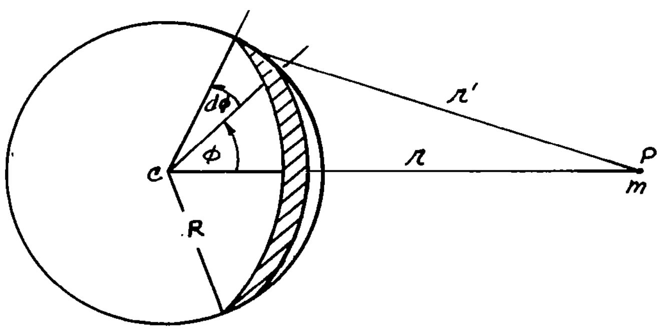
and the force is observed to be expressible as the negative gradient of the potential function, e.g.,

$$X = -\frac{\partial U}{\partial x}, \quad Y = -\frac{\partial U}{\partial y}, \quad Z = -\frac{\partial U}{\partial z} \tag{7}$$

Now consider the potential of a mass m at point P due to an annulus of a thin spherical shell cut by two planes normal to the line between the center C and the point P, is given by

$$dU = -\frac{G m \, dm_e}{r'}$$

where dm_e is the mass of the annulus and r' is the distance from any part of the annulus to the point P.



The mass of the annulus can be expressed in terms of the mass density ρ, the shell thickness t, and the radius "a" as

$$dm_e = 2\pi R^2 \rho t \sin\phi \, d\phi$$

Thus, the potential relative to the entire shell is

$$U = -2\pi m R^2 G\rho t \int_0^{\pi} \frac{\sin\phi \, d\phi}{r'}$$

but

$$r'^2 = R^2 + r^2 - 2 R \cos\phi$$

where r is the distance between the center C of the spherical shell and the point P .

Hence

$$r' dr' = a r \sin\phi \, d\phi$$

Substituting this expression into the expression for the potential and integrating yields

$$\frac{2\pi m a G\rho t}{r} \int_{r-R^2}^{r+R} dr' = -\frac{4\pi m R^2 G\rho t}{r}$$

But the mass m_e of the shell is $4\pi R^2\rho t$; hence, the potential becomes $U = -G m_e m / r$, or the potential of a spherical shell is the same as a mass particle having the same mass of the shell and situated at its center. Now a solid sphere whose mass distribution is radially symmetric can be thought of as consisting of an infinite number of constant density layers. So the total potential of an infinite number of infinitesimally thick layers is

$$U_T = \sum_{i=1}^{\infty} U_i$$

$$= \frac{Gm}{r} \sum_{i=1}^{\infty} m_{e_i}$$

$$= \frac{Gm}{r} M_e \tag{8}$$

where M_e is the total mass.

2.2 REDUCTION OF THE TWO-BODY PROBLEM

Since the force vector was derived from the observed motion, there would seem, on the surface, to be little merit in proving that the motion which results from this law is elliptical. However, this conclusion is not well founded for several reasons:

1) The previous discussions did not explore the nature of the motion to determine if other than elliptic trajectories could be produced.

2) No means of describing the motion in three dimensional space as a function of time was provided (i.e., the parameters of the ellipse were not related to the dynamics).

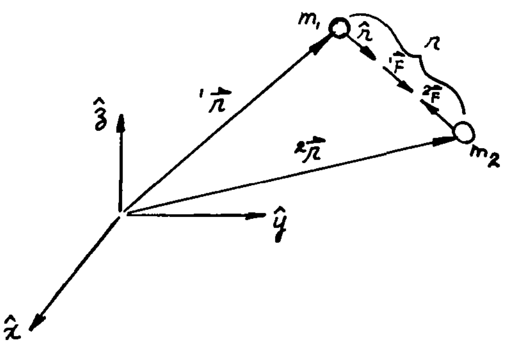
3) No attempt was made to determine if the form of the equations employed ever produced indeterminacies.

It is to these ends that the remaining sections of the monograph have been prepared.

Consider two masses in an inertial coordinate system (having an arbitrary origin) described by three rectangular cartesian unit vectors ($\hat{x}$, $\hat{y}$, $\hat{z}$). In this system, the position vectors* of mass particles m_1 and m_2 are ${}^1\vec{r}$ and ${}^2\vec{r}$, respectively. The forces exerted on m_1 and m_2 due to mutual gravitational attractions are ${}^1\vec{F}$ and ${}^2\vec{F}$ where

$$ {}^1\vec{F} = -{}^2\vec{F} $$

(Newton's third law of action and reaction).



*Note - Vector quantities will use the superscript-subscript notation established in Reference (2) where the subscripts, when given, indicate the coordinate system the vector utilized to express; and the superscripts, the body considered.

Newton's law of gravitation then states that

$$ {}^2\vec{F} = -\frac{G m_1 m_2}{r^2}\hat{r} \quad , {}^1\vec{F} = \frac{G m_1 m_2}{r^2}\hat{r} $$

where r is the distance between m_1 and m_2, and $\hat{r}$ is a unit vector from m_1 to m_2, i.e.,

$$ r = |{}^2\vec{r} - {}^1\vec{r}| \quad , \quad \hat{r} = \frac{{}^2\vec{r} - {}^1\vec{r}}{|{}^2\vec{r} - {}^1\vec{r}|} $$

According to the second law of motion, the time rate of change of linear momentum ($m\vec{v}$) of m_1 and m_2 is respectively equal to the forces ${}^1\vec{F}$ and ${}^2\vec{F}$, i.e.,

$$ m_1 {}^1\ddot{\vec{r}} = \frac{G m_1 m_2}{r^2}\hat{r} \tag{9a} $$

$$ m_2 {}^2\ddot{\vec{r}} = \frac{-G m_1 m_2}{r^2}\hat{r} \tag{9b} $$

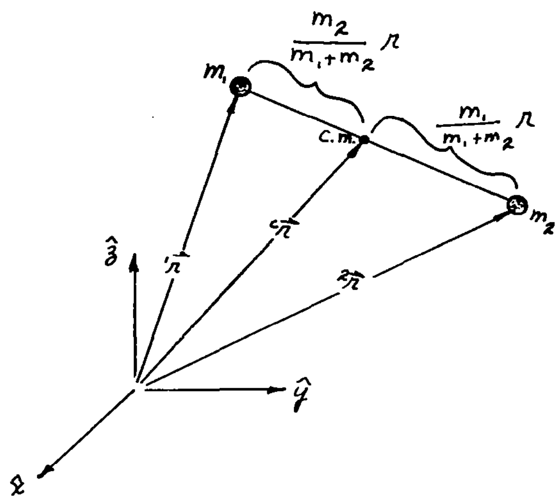
Equations (9a) and (9b) describe a second order system with 12 constants of integration. The first 6 of these constants can be evaluated by adding (9a) and (9b) and integrating the result twice.

$$ m_1 {}^1\vec{r} + m_2 {}^2\vec{r} = \vec{\alpha} t + \vec{\beta} \tag{10} $$

where $\vec{\alpha}$ and $\vec{\beta}$ are vector constants of integration.

These constants can be interpreted by defining a vector ${}^c\vec{r}$ which locates the center of mass of the system

$$ {}^c\vec{r} = \frac{m_1 {}^1\vec{r} + m_2 {}^2\vec{r}}{m_1 + m_2} $$



This definition allows equation (10) to be rewritten as

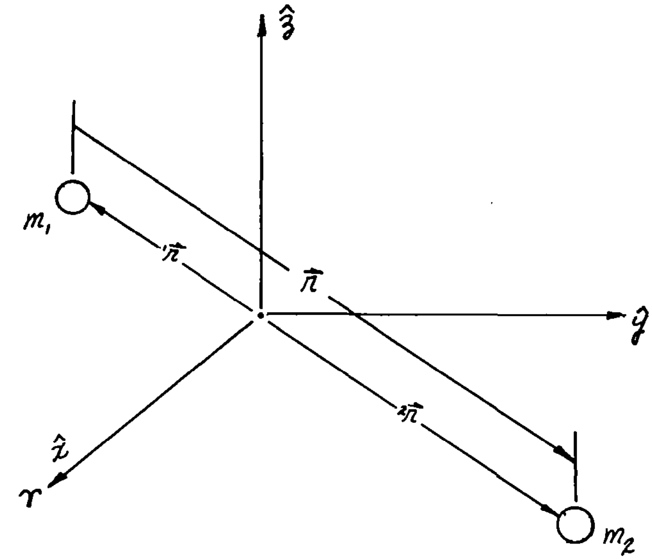
$$(m_1 + m_2)\, {}^c\vec{r} = \vec{\alpha}\, t + \vec{\beta}$$

or

$$ {}^c\vec{r} = \frac{\vec{\alpha}}{m_1 + m_2}\, t + \frac{\vec{\beta}}{m_1 + m_2} \tag{11}$$

Equation (11) states that the center of mass moves in the plane of ${}^c\vec{r}\,(0)$, ${}^c\vec{v}\,(0)$ with a constant velocity ($\vec{\alpha}/(m_1 + m_2)$. This result redefines 6 of the required 12 constants.

The remaining constants can be evaluated more easily if, at this point, a transformation is made to reduce the solution to the equivalent problem of motion with respect to the center of one of the two masses. In the process, motion will be referenced to the center of mass of the system. (The center of mass is not accelerating and may, thus, be utilized as the origin of coordinates without loss of generality.) Henceforth, in the discussion, the vectors ${}^1\vec{r}$ and ${}^2\vec{r}$ will be considered to be defined in the following sketch.



where

$$m_1\,{}^1\bar{r} + m_2\,{}^2\bar{r} = 0 \tag{12a}$$

The final step in the transformation is taken by defining a vector $\bar{r}$ to be

$$\bar{r} = {}^2\bar{r} - {}^1\bar{r} \tag{12b}$$

This definition allows the equations of motion to be reduced to

$$\ddot{\bar{r}} = {}^2\ddot{\bar{r}} - {}^1\ddot{\bar{r}}$$

$$= -\frac{G(m_1 + m_2)}{r^2}\,\hat{r}$$

$$\equiv -\frac{\mu}{r^2}\,\hat{r} \tag{13}$$

Equation (13) represents the motion of both bodies since equations (12) allow both ${}^1\bar{r}$ and ${}^2\bar{r}$ to be recovered

$${}^1\bar{r} = \frac{-m_2}{m_1 + m_2}\,\bar{r} \qquad ; \quad {}^2\bar{r} = \frac{m_1}{m_1 + m_2}\,\bar{r}$$

The solution of equation (13) is generally referred to as the Kepler problem. This solution is the subject of the discussions which follow.

2.3 FIRST INTEGRAL OF THE REDUCED PROBLEM

The general solution of a second order vector differential equation such as (13), requires six constants of integration. The first three of these constants can, however, be obtained by recognizing that the vector product of the position and force vectors of each mass particle equals the time rate of change of the angular momentum of the particle. That is, for particle 1

$$ {}^1\bar{r} \times (m_1\, {}^1\ddot{\bar{r}}) = {}^1\bar{r} \times ({}^1\bar{F}) $$

$$ {}^1\bar{r} \times m_1\, {}^1\ddot{\bar{r}} + {}^1\dot{\bar{r}} \times m_1\, {}^1\dot{\bar{r}} = {}^1\bar{r} \times {}^1\bar{F} $$

$$ \frac{d}{dt}\left({}^1\bar{r} \times m_1\, {}^1\dot{\bar{r}}\right) = {}^1\bar{r} \times {}^1\bar{F} $$

where $({}^1\bar{r} \times m_1\, {}^1\dot{\bar{r}})$ is the angular momentum of particle 1. This principle can be employed to advantage in the Kepler problem since

$$ {}^1\bar{F} = |{}^1\bar{F}|\, {}^1\hat{r} $$

or

$$ \frac{d}{dt}\left({}^1\bar{r} \times m_1\, {}^1\dot{\bar{r}}\right) = {}^1\bar{r} \times |{}^1\bar{F}|\, {}^1\hat{r} = 0 $$

Thus by integration

$$ {}^1\bar{r} \times m_1\, {}^1\dot{\bar{r}} = {}^1\bar{H} \quad \text{, a constant} \qquad (14a) $$

Similarly, $ {}^2\bar{r} \times m_2\, {}^2\dot{\bar{r}} = {}^2\bar{H} $, a constant (14b)

Now, since the total angular momentum is

$$ \bar{H} = {}^1\bar{H} + {}^2\bar{H} \quad , \qquad (15) $$

it is also constant and can be evaluated from equation (15) or its equivalent

$$ \bar{H} = \bar{r} \times \left(\frac{m_1 m_2}{m_1 + m_2}\right) \dot{\bar{r}} \qquad (16) $$

Equation (16) is a rigorous proof of the first two of Kepler's laws, since $\vec{H} \cdot \vec{r}(t) \equiv 0$ for all t and since the area of an infinitesimal sector of the ellipse swept by the radius vector is

$$dA = \tfrac{1}{2}|\vec{r} \times d\vec{r}|$$

and
$$\dot{A} = \tfrac{1}{2}|\vec{r} \times \dot{\vec{r}}|$$

or
$$dA = \tfrac{1}{2}|\vec{H}|\left(\frac{m_1+m_2}{m_1 m_2}\right)dt$$

Equation (16) is recognized as the angular momentum of a particle of mass $m_1m_2/(m_1+m_2)$ whose position vector is $\vec{r}$. The solution to the two-body problem is, therefore, equivalent to that of one body whose position vector satisfies equation (13) and whose mass is $m_1m_2/(m_1+m_2)$. Thus, if the angular momentum per unit mass is defined as

$$\vec{h} = \frac{\vec{H}}{\dfrac{m_1 m_2}{m_1+m_2}} ,$$

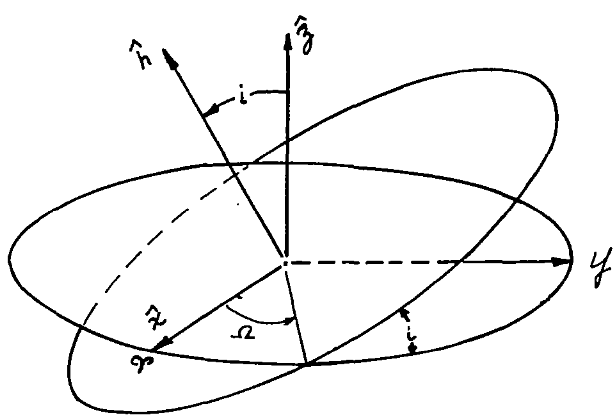
then,

$$\vec{h} = \vec{r} \times \dot{\vec{r}} \tag{17}$$

is a vector constant of integration of (13). Further, since this vector is constant and normal to the instantaneous plane of motion (the plane of $\vec{r}$ and $\dot{\vec{r}}$), the vectors $\vec{r}(t)$ and $\dot{\vec{r}}(t)$ must lie in the same plane. This plane of motion is determined by giving the components of the unit normal vector

$$\hat{h} = \vec{h}/|\vec{h}|$$

or, equivalently, a set of orientation angles such as those illustrated in the following sketch. This latter representation is common practice since the orientation angles are more readily visualized.



The angle "i", called the inclination angle, is the angle between the orbit and reference planes measured in a plane perpendicular to their line of intersection. The cosine of this angle is

$$\cos i = \hat{h} \cdot \hat{z} \qquad 0 \leq i \leq 180°$$

The angle " Ω " is the longitude of the ascending node, and is measured in the reference plane from the principal direction to the ascending node. Since the equation of the line of nodes is given by

$$y = -\frac{\hat{h} \cdot \hat{x}}{\hat{h} \cdot \hat{y}}\, x \quad ,$$

the angle " Ω " is defined by the equation

$$\tan \Omega = \frac{\hat{h} \cdot \hat{x}}{\hat{h} \cdot \hat{y}}$$

The quadrant of Ω is fixed by checking the signs of the numerator and denominator. This form of representation of the plane of motion has a major failing, however, since for inclination angles of 0° or 180°, the longitude of the ascending node is indeterminate (both $\hat{h} \cdot \hat{x}$ and $\hat{h} \cdot \hat{y}$ are zero). This problem leads to considerable computational difficulty in some analyses and requires that either the reference plane be altered or that the components of $\hat{h}$ itself be employed to describe the plane. This problem (along with others which will be discussed subsequently) serves as motivation for the development of more uniformly deterministic elements presented later in the monograph.

For the special case in which $\vec{h}$ is zero, an alternate logic is required since the motion is rectilinear (i.e., $\vec{r} \cdot \vec{v} = rv$) for all times. In this case, a line, rather than a plane, contains all possible trajectories and the orientation of the line is defined by $\vec{r}(0)$ or $\vec{v}(0)$.

2.4 CONIC MOTION INTEGRAL

So far, three of the six constants of integration required for the reduced problem have been determined. However, the three remaining constants can be obtained by forming the vector product of $\bar{h}$ and equation (13)

$$(\ddot{\vec{r}}) \times \vec{h} = (-\frac{\mu}{r^2} \hat{r}) \times \vec{h}$$

$$= \frac{\mu}{r^2} \hat{r} \times (\vec{r} \times \dot{\vec{r}}) \tag{18}$$

but

$$\hat{r} \times (\vec{r} \times \dot{\vec{r}}) = (\hat{r} \cdot \dot{\vec{r}})\vec{r} - (\hat{r} \cdot \vec{r})\dot{\vec{r}}$$

$$= \dot{r}\vec{r} - r\dot{\vec{r}} \tag{19}$$

by vector identity. Thus, substitution of (19) into (18) yields

$$\ddot{\vec{r}} \times \vec{h} = \mu \frac{r\dot{\vec{r}} - \dot{r}\vec{r}}{r^2}$$

$$= \mu \frac{d}{dt}\left(\frac{\vec{r}}{r}\right)$$

$$= \mu \frac{d}{dt}(\hat{r}) \tag{20}$$

However, the vector $\vec{h}$ is a constant so the left side of (20) can be written as the time derivative of ($\dot{\vec{r}} \times \vec{h}$), i.e.,

$$d/dt\, (\dot{\vec{r}} \times \vec{h}) = \mu\, d/dt\, (\hat{r}) \tag{21}$$

Finally, integration of (21) yields

$$\dot{\vec{r}} \times \vec{h} = \mu(\hat{r} + \vec{e}) \tag{22}$$

where $\vec{e}$ is a vector whose components are the remaining constants of integration (two components are independent ; the third is not, since $\vec{e} \cdot \vec{h} = 0$).

The vector constant $\vec{e}$ can be related to the more conventional form of the solution by forming the scalar product of $\vec{r}$ and equation (22) and employing the identity

$$\vec{r} \cdot (\dot{\vec{r}} \times \vec{h}) = (\vec{r} \times \dot{\vec{r}}) \cdot \vec{h}$$

$$\equiv h^2$$

so

$$h^2 = \mu(\vec{r} \cdot \hat{r} + \vec{r} \cdot \vec{e})$$

$$= \mu\left[r + |\vec{r}||\vec{e}| \cos(\vec{r}, \vec{e})\right] \tag{23}$$

or

$$r = \frac{h^2/\mu}{1 + |\vec{e}| \cos(\vec{r}, \vec{e})} \tag{24}$$

This equation proves Kepler's third observation, since it is recognized as the polar form of the equation of a conic section (see Appendix A). For this reason, equation (24) is sometimes referred to as the conic motion integral. The conventional parameters of the conic sections are defined by comparing (24) with the classical form; i.e.,

$$r = \frac{p}{1 + e \cos\theta} \tag{25}$$

where r is the radial distance of a point on the conic section from the focus, p is the semi-latus rectum, e is the eccentricity, and θ is the angle between the radius vector and the vector directed toward periapse from the focus.

Thus $$p = \frac{h^2}{\mu}, \quad e = |\vec{e}|, \quad \theta = \cos^{-1}(\vec{r}, \vec{e})$$

and the vector constant $\vec{e}$ can now be interpreted as a vector whose magnitude is equal to the eccentricity, and whose direction is defined by the position of the periapse (i.e., $\theta = 0$).

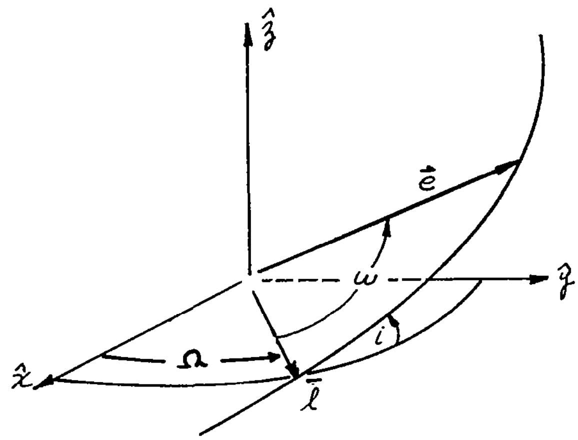
Since the vector $\vec{e}$ lies in the orbital plane and since a geometrical interpretation of the constants of integration is frequently desirable, the direction of $\vec{e}$ is generally defined by specifying the argument of periapse (ω). This angle is measured in the orbital plane from the ascending node to the line from the focus to periapse. If $\vec{l}$ denotes a vector directed toward the ascending node

$$\vec{l} = (\hat{h} \cdot \hat{y})\hat{x} - (\hat{h} \cdot \hat{x})\hat{y}$$

then the argument of periapse is calculated from

$$\cos\omega = \frac{\vec{l} \cdot \vec{e}}{l e} \qquad \begin{matrix} e \neq 0 \\ l \neq 0 \end{matrix} \tag{26}$$

These quantities are illustrated in the following sketch.

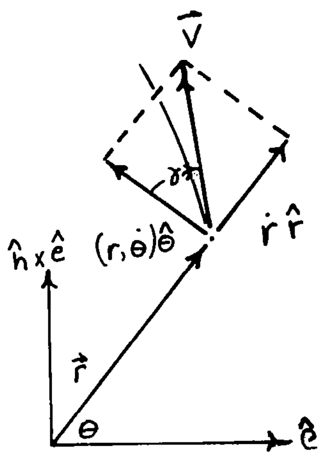


Thus, a geometrical explanation of the fact that only five of the six constants of integration are independent can be given. Two are required to define the orientation of the orbital plane (i, Ω) one is required to orient the line of apsides in the plane of motion (ω), and two define the angular momentum and eccentricity of trajectory (p, e). The sixth constant defines the position on the trajectory at the specified epoch (discussions relating the position-time relationship will be presented in subsequent sections of this monograph).

2.5 DESCRIPTION OF THE VELOCITY VECTOR AND THE ENERGY OF THE SYSTEM

The first integral of the motion and the conic motion integral have so far been used to generate five independent constants of integration for the Kepler problem. These constants and equation (16) determine a solution path function φ of the radius $\vec{r}$ such that $\varphi(\vec{r}) = 0$. However, no explicit relations have been derived which exhibit information directly pertaining to the velocity vector. Thus motivated, consider the equation for the velocity vector in rotating coordinates, and the following sketch drawn in the plane of motion.

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\,\hat{\theta} \tag{27}$$



Now differentiating equation (24), yields the radial velocity component

$$\dot{r} = \frac{pe\sin\theta}{(1+e\cos\theta)^2}\,\dot{\theta}$$

But $\quad h = r^2\dot{\theta}$

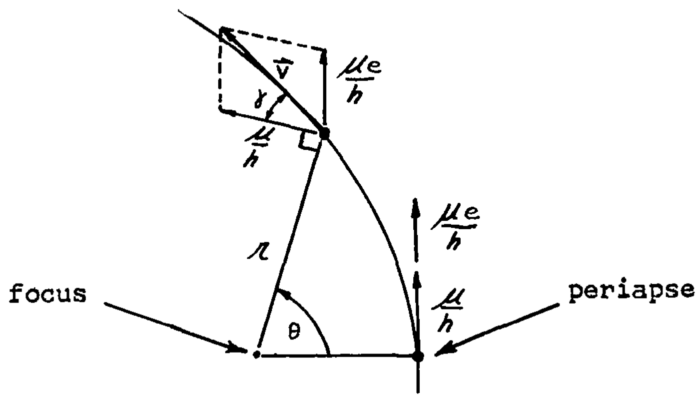
Thus,

$$\dot{r} = \frac{\mu}{h}\,e\sin\theta \tag{28}$$

and $\quad r\dot{\theta} = \frac{h}{r}$

$$= \frac{h}{p}(1 + e\cos\theta) \tag{29}$$

Equations (28) and (29) suggest a geometrical construction of the velocity vector in terms of two vectors of fixed magnitudes. One vector has the magnitude (μ/h) and lies along the instantaneous $\hat{\theta}$. This vector provides the constant term in equation (29). The other vector of magnitude $\mu e/h$ is aligned with the velocity vector at periapse. The velocity vector at any point in the orbit is then given by the vector sum as illustrated in the following sketch



A second representation of the velocity vector involving the magnitude of $\vec{V}$ and the flight path angle γ can be obtained by considering the following scalar product

$$\dot{\vec{r}} \cdot \ddot{\vec{r}} = \dot{\vec{r}} \cdot \left(\frac{-\mu}{r^2}\right)\hat{r}$$

$$d/dt\left(\frac{\dot{\vec{r}} \cdot \dot{\vec{r}}}{2}\right) = -\frac{\mu}{r^2}\dot{r}$$

But $\quad \dot{\vec{r}} \cdot \dot{\vec{r}} = V^2$

and $\quad \frac{\dot{r}}{r^2} = \frac{d}{dt}(1/r)$

Thus $\quad \frac{d}{dt}\left(\frac{V^2}{2} - \frac{\mu}{r}\right) = 0$

or

$$\frac{V^2}{2} - \frac{\mu}{r} = \epsilon \tag{30}$$

where ϵ is a scalar constant of integration recognized to be the total energy per unit mass of the equivalent one body $(\frac{m_1 m_2}{m_1+m_2})$. Thus, given a set of initial conditions (or equivalently ϵ) and a value of θ (or r)

$$v = +\sqrt{2\left(\frac{\mu}{r} + \epsilon\right)} \tag{31}$$

and

$$\begin{aligned} \tan\gamma &= \frac{\dot{r}}{r\dot{\theta}} \\ &= \frac{e \sin\theta}{1 + e\cos\theta} \end{aligned} \tag{32}$$

Use of the set of equations (28, 29) or the corresponding set (31, 32) is primarily a matter of preference, since they are of the same order of complexity.

Since six constants of integration have already been obtained for the Kepler problem, ϵ must be a dependent constant. This dependency is shown by forming the dot product of $\vec{e}$ with itself, i.e.,

$$\vec{e}\cdot\vec{e} = \left[\hat{r} - \tfrac{1}{\mu}(\dot{\vec{r}} \times \vec{h})\right]\cdot\left[\hat{r} - \tfrac{1}{\mu}(\dot{\vec{r}} \times \vec{h})\right]$$

$$= 1 - \frac{2}{\mu}\hat{r}\cdot(\dot{\vec{r}} \times \vec{h}) + \frac{V^2 h^2}{\mu^2}$$

but it has previously been shown that $\vec{r}\cdot(\dot{\vec{r}} \times \vec{h}) = h^2$

Thus, $$\begin{aligned} e^2 &= 1 - 2\frac{h^2}{\mu r} + \frac{V^2 h^2}{\mu^2} \\ &= 1 + 2\frac{h^2}{\mu^2}\left(\frac{V^2}{2} - \frac{\mu}{r}\right) \end{aligned} \tag{33}$$

Therefore

$$e^2 = 1 + 2\epsilon\frac{h^2}{\mu^2} \tag{34a}$$

and the dependency of ϵ on the previously obtained integration constants is demonstrated. This dependency can also take another form by referring to Appendix B and noting that since

$$a = \frac{p}{1-e^2}$$

substitution of (34a) will produce

$$a = -\frac{\mu}{2\epsilon}$$

or

$$\epsilon = -\frac{\mu}{2a} \tag{34b}$$

At this point, one additional formula will be developed to show the relationship between the various energy components. Consider the moment of inertia (a scalar) of a system of n particles

$$I = \sum_{i=1}^{n} m_i r_i^2 = \sum_{i=1}^{n} m_i \vec{r}_i \cdot \vec{r}_i$$

Differentiating with respect to time yields

$$\dot{I} = 2 \sum_{i=1}^{n} m_i \vec{r}_i \cdot \dot{\vec{r}}_i$$

$$\ddot{I} = 2 \sum_{i=1}^{n} m_i \vec{r}_i \cdot \ddot{\vec{r}}_i + 2 \sum_{i=1}^{n} m_i \dot{r}_i^2$$

Now if the force field is conservative and if U and T are employed to denote the **total potential and kinetic energies (i.e., for the system),**

$$\ddot{I} = -2 \sum_{i=1}^{n} \vec{r}_i \cdot \vec{\nabla}_i U + 4T$$

$$= -2 \sum_{i=1}^{n} \left(x_i \frac{\partial U}{\partial x_i} + y_i \frac{\partial U}{\partial y_i} + z_i \frac{\partial U}{\partial z_i} \right) + 4T$$

But U is homogeneous in the coordinates and of the order -1. That is, U has the property that the substitution $x = \lambda x$, $y = \lambda y$ and $z = \lambda z$ merely reproduces the original U multipled by λ^n where n is the order of the function (in this case -1). Thus, the form of $\ddot{I}$ can be simplified as follows:

$$U(\lambda x, \lambda y, \lambda z) = \lambda^n U(x, y, z)$$

Now differentiating with respect to λ yields

$$n\lambda^{n-1} U(x, y, z) = \frac{\partial U}{\partial(\lambda x)} \frac{\partial(\lambda x)}{\partial \lambda} + \frac{\partial U}{\partial(\lambda y)} \frac{\partial(\lambda y)}{\partial \lambda} + \frac{\partial U}{\partial(\lambda z)} \frac{\partial(\lambda z)}{\partial \lambda}$$

$$= x \frac{\partial U}{\partial(\lambda x)} + y \frac{\partial U}{\partial(\lambda y)} + z \frac{\partial U}{\partial(\lambda z)}$$

But λ is an arbitrary constant so it can be selected to be 1. Substitution of this value yields Euler's theorem for homogeneous functions.

$$nU = \vec{r} \cdot \vec{\nabla} U$$

Therefore

$$\ddot{I} = 4T - 2U$$

Finally, if the system is to be stable (i.e., none of the $\vec{r}_i \to \infty$) and does not collapse (i.e., $\vec{r}_i \to 0$), then the time average of $\ddot{I}$ must be zero.

Thus

$$T' = -\tfrac{1}{2} U' \tag{35}$$

where the prime denotes a time average. With this equality, the total energy of the system reduces to

$$\begin{aligned} \epsilon &= U + T \\ &= \tfrac{1}{2} U' \text{ or } -T' \end{aligned} \tag{36}$$

This is the Virial Theorem.

2.6 TIME DEPENDENT NATURE OF THE MOTION

To this point the components of the position and velocity vectors have been related to one another or to the true anomaly, θ. However, no attempt has been made to express the solution in terms of the independent variable time. This final step in the development of the basic two-body formulae will require the consideration of each of the three distinct conic sections. Thus, consider the case of elliptic, hyperbolic and parabolic motion, respectively.

2.6.1 Elliptic Motion

The scalar magnitude of the angular momentum vector (all conic motions) is

$$h = r^2 \dot{\theta}$$

Thus, the time dependence of the motion can be obtained from

$$\int_{t_0}^{t} dt = \int_{0}^{\theta} \frac{r^2 d\theta}{h}$$

$$= \frac{h^3}{\mu^2} \int_{0}^{\theta} \frac{d\theta}{(1 + e \cos\theta)^2} \tag{37}$$

The solution to equation (37) is presented in most tables of integrals; however, it is informative from the standpoint of the introduction of variables as yet undefined to perform the integration.

The eccentric anomaly illustrated in Appendix B is defined as follows:

$$r = a(1 - e\cos E) = p/(1 + e\cos\theta) \tag{38a}$$

where

$$\sin E = \frac{r \sin\theta}{a\sqrt{1 - e^2}} \tag{38b}$$

solving equation (38a) for $E = E(\theta)$, yields

$$\cos E = \frac{e + \cos\theta}{1 + e\cos\theta} \tag{38c}$$

Thus differentiating (38c) yields

$$-\sin \frac{dE}{d\theta} = \frac{(1+e\cos\theta)(-\sin\theta)-(e+\cos\theta)(-e\sin\theta)}{(1+e\cos\theta)^2}$$

$$= -\frac{\sin\theta\,(1-e^2)}{(1+2\cos\theta)^2}$$

and substituting (38b) reduces this equation to

$$\frac{dE}{d\theta} = \frac{\sqrt{1-e^2}}{1+e\cos\theta} \qquad (39)$$

Finally, substituting (38a) and (39) into (37) yields

$$t - t_0 = \frac{h^3}{\mu^2}\int_0^E \frac{1}{(1+e\cos\theta)^2} \quad \frac{(1+e\cos\theta)}{\sqrt{1-e^2}}\, dE$$

$$= \frac{h^3}{\mu^2}\int_0^E \frac{(1-e\cos E)}{(1-e^2)^{3/2}}\, dE$$

$$= \sqrt{\frac{a^3}{\mu}}\,\left[E - e\sin E\right] \qquad (40)$$

Equation (40), which is commonly referred to as Kepler's equation, defines the position variable E as an implicit function of the time from the periapse (E = 0 when $\theta = 0$; thus, t_0 is the time of periapse passage, the 6th independent constant) and completes the basic development of elliptic motion.

Implicit in equation (40) is the relationship for the orbital period since values of the true anomaly (θ) and the eccentric anomaly (E) are the same for $\theta = N\pi$. Therefore one anomalistic period (corresponds to the time required for θ to increase from zero to 2π) is

$$\Upsilon = 2\pi\sqrt{\frac{a^3}{\mu}}$$

or

$$\tau = \frac{2\pi}{n} \tag{41}$$

where n is the mean motion (i.e. the mean angular rate in radians per unit of time).

The solution of Kepler's equation for the mean anomaly as a function of the eccentric anomaly is direct. However, the reverse determination for the eccentric anomaly as a function of time (or mean anomaly) involves the solution of a transcendental equation. This reverse solution (while at times burdensome) is not unmanageable and is always unique since M is a monotonically increasing function of E. That is

$$\frac{dM}{dE} = 1 - e \cos E$$

is positive definite for all E($e < 1$). Once E is found, the true anomaly can be evaluated explicitly as shown in Appendix B, i.e.,

$$\sin \theta = \frac{a}{r} \sqrt{1-e^2} \sin E \tag{42a}$$

$$\cos \theta = \frac{a}{r} (\cos E - e) \tag{42b}$$

$$\tan \frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2} \tag{42c}$$

Since the inverse solution of Kepler's equation is more generally attempted than the direct solution (most trajectory problems employ time as independent variable), many techniques have been devised to resolve the problem. Two of these techniques (series expansion and Newton's method of numerical iteration) will be discussed since they are easily adapted for manual or digital computer solutions.

Consider the quantity E-M = e sin E. This is a periodic function of either of the anomalies, E or M. Let this quantity be considered as a function of M. Then, since it is periodic (with period P = 2π), it can be written as a Fourier series in M as

$$E - M = a_0 + \sum_{n=1}^{\infty} (a_n \cos_n M + b_n \sin_n M) \tag{43}$$

where the coefficients are given by

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} e \sin E \, dM \quad , \quad a_n = \frac{1}{\pi}\int_0^{2\pi} e \sin E \cos nM \, dM \, ,$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} e \sin E \sin nM \, dM$$

From equation (42) dM = (1 - e cos E) d E and the coefficient α_0 becomes

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} (e \sin E - e^2 \sin E \cos E) \, dE = 0$$

Also, since (e sin E) is an odd periodic function, then a_n = 0. The expression for b_n is integrated by parts to give

$$b_n = -\frac{1}{n\pi} e \sin E \cos nM \Big|_0^{2\pi} + \frac{1}{n\pi}\int_0^{2\pi} \cos nM \, d(e \sin E) \tag{44}$$

The first term on the right side of (44) is zero. Now the second term can be evaluated by substituting (E - M) for (e sin E).

$$b_n = \frac{1}{n\pi}\int_0^{2\pi} \cos nM \, dE - \frac{1}{n\pi}\int_0^{2\pi} \cos nM \, dM$$

The second integral of this expression is also zero. Finally, since the cosine function is even, the limits on the single remaining integral can be changed to give

$$b_n = \frac{2}{n\pi}\int_0^{\pi} \cos(nE - ne \sin E) \, dE \tag{45}$$

The form of equation (45) is recognized as that of the Bessel function $J_n(ne)$ of the first kind of order n where

$$J_n(ne) = \frac{1}{\pi}\int_0^{\pi} \cos(nE - ne \sin E) \, dE$$

so that $b_n = \frac{2}{n} J_n(ne)$. Thus, the Fourier series expansion of $E = E(M)$ becomes

$$E = M + \sum_{n=1}^{\infty} \frac{2}{n} J_n(ne) \sin nM \tag{46}$$

Where,for purposes of calculation, the Bessel function of the first kind of order n is

$$J_n(ne) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{ne}{2}\right)^{2k+n}}{k! \; (k+n)!}$$

This series is convergent for all $M (e < .66274)$.

Once an estimate of E has been obtained from this series from a moderately accurate graphical solution, or from a numerical search, the estimate can be refined to the desired precision by employing Newton's method. This method is a simple restatement of the Taylor expansion in which all but first-order terms have been neglected (Thus, the neighborhood of the position variable must have been located.). To be specific

$$M = M_i + \left. \frac{\partial M}{\partial E} \right|_{E=E_i} \Delta E$$

$$\Delta E = \left(M - E_i + e \sin E_i\right) \Big/ \left(\left. \frac{\partial M}{\partial E} \right|_{E=E_i} \right)$$

and

$$E_{i+1} = \frac{M - E_i + e \sin E_i}{1 - e \cos E_i} + E_i \tag{47}$$

This iteration converges quite rapidly (generally at the rate of about 2 digits per iteration) given an estimate which is accurate through the second digit.

Before leaving the discussion of elliptic motion, it is noted that the arguments advanced when expanding Kepler's equation could be applied to any of the position or velocity dependent variables. Thus, for those cases where the eccentricity is sufficiently small to assure convergence of the series, a series representation can be utilized to replace the process of solving for eccentric anomaly, the true anomaly, etc. References 3 and 4 present the general forms of the Fourier-Bessel expansions for most of these parameters. The more useful of these equations have been expanded and are presented below through terms of order e^6 .

$$E = M + 2\sum_{n=1}^{\infty} \frac{1}{n} J_n(ne) \sin nM$$

$$= M + \left(e - \frac{e^3}{8} + \frac{e^5}{192} - \cdots\right) \sin M$$

$$+ \left(\frac{e^2}{2} - \frac{e^4}{6} + \frac{e^6}{48} - \cdots\right) \sin 2M$$

$$+ \left(\frac{3e^3}{8} - \frac{27e^5}{128} + \cdots\right) \sin 3M$$

$$+ \left(\frac{e^4}{3} - \frac{4e^6}{15} + \cdots\right) \sin 4M$$

$$+ \left(\frac{125e^5}{384} - \cdots\right) \sin 5M$$

$$+ \left(\frac{27e^6}{80} - \cdots\right) \sin 6M \qquad (48)$$

$$\sin E = \frac{2}{e}\sum_{n=1}^{\infty} \frac{1}{n} J_n(ne) \sin nM$$

$$= \left(1 + \frac{e^2}{8} + \frac{e^4}{192} - \frac{e^6}{9216} + \cdots\right) \sin M$$

$$+ \left(\frac{e}{2} - \frac{e^3}{6} + \frac{e^5}{48} - \cdots\right) \sin 2M$$

$$+ \left(\frac{3e^2}{8} - \frac{27e^4}{128} + \frac{243e^6}{5120} - \cdots\right) \sin 3M$$

$$+ \left(\frac{e^3}{3} - \frac{4e^5}{15} + \cdots\right) \sin 4M$$

$$+ \left(\frac{125e^4}{384} - \frac{3125e^6}{9216} + \cdots\right) \sin 5M$$

$$+ \left(\frac{27e^5}{80} - \cdots\right) \sin 6M$$

$$+ \left(\frac{16807e^6}{46{,}080} - \cdots\right) \sin 7M \qquad (49)$$

$$\cos E = \frac{1}{2} e + \sum_{n=1}^{\infty} \frac{2}{n^2} \frac{d}{de} \left\{ J_n (ne) \right\} \cos nM$$

$$= -\frac{e}{2} + \left(1 - \frac{3e^2}{8} + \frac{5e^4}{192} - \frac{7e^6}{9216} + \ldots\right) \cos M$$

$$+\left(\frac{e}{2} - \frac{e^3}{3} + \frac{e^5}{16} - \ldots\right) \cos 2M$$

$$+\left(\frac{3e^2}{8} - \frac{45e^4}{128} + \frac{567e^6}{5120} - \ldots\right) \cos 3M$$

$$+\left(\frac{e^3}{3} - \frac{2e^5}{5} + \ldots\right) \cos 4M$$

$$+\left(\frac{125e^4}{384} - \frac{4375e^6}{9216} + \ldots\right) \cos 5M$$

$$+\left(\frac{81e^5}{240} - \ldots\right) \cos 6M$$

$$+\left(\frac{16{,}807e^6}{46{,}080} - \ldots\right) \cos 7M \qquad (50)$$

$$\theta = M + \sum_{n=1}^{\infty} \frac{2}{n} \sin nM \sum_{k=-\infty}^{+\infty} f^{|n|} J_{n+k}(ne)$$

$$\theta = M + \left(2e - \frac{e^3}{4} + \frac{5e^5}{96} + \ldots\right) \sin M$$

$$+\left(\frac{5e^2}{4} - \frac{11e^4}{24} + \frac{17e^6}{192} - \ldots\right) \sin 2M$$

$$+\left(\frac{13e^3}{12} - \frac{43e^5}{192} + \ldots\right) \sin 3M$$

$$+\left(\frac{103e^4}{96} - \frac{451e^6}{480} + \ldots\right) \sin 4M$$

$$+\left(\frac{1097e^5}{960} - \ldots\right) \sin 5M$$

$$+\left(\frac{1223e^6}{960} - \ldots\right) \sin 6M \qquad (51)$$

$$f \equiv 1 - \frac{\sqrt{1-e^2}}{e} = \frac{e}{2} + \frac{e^3}{8} + \frac{e^5}{16} + \ldots$$

$$\sin\theta = 2\sqrt{1-e^2}\sum_{n=1}^{\infty}\frac{1}{n}\frac{d}{de}\{J_n(ne)\}\sin nM$$

$$= \left(1 - \frac{7e^2}{8} + \frac{17e^4}{192} + \frac{317e^6}{9216} + \ldots\right)\sin M$$

$$+\left(e - \frac{7e^3}{6} + \frac{e^5}{3} - \ldots\right)\sin 2M$$

$$+\left(\frac{9e^2}{8} - \frac{207e^4}{128} + \frac{3681e^6}{5120} + \ldots\right)\sin 3M$$

$$+\left(\frac{4e^3}{3} - \frac{34e^5}{15} + \ldots\right)\sin 4M$$

$$+\left(\frac{625e^4}{384} - \frac{29363}{9216} + \ldots\right)\sin 5M$$

$$+\left(\frac{81e^5}{40} - \ldots\right)\sin 6M$$

$$+\left(\frac{117{,}649e^6}{46{,}080} - \ldots\right)\sin 7M \tag{52}$$

$$\cos\theta = -e + \frac{2(1-e^2)}{e}\sum_{n=1}^{\infty} J_n(ne)\cos nM$$

$$= -e + \left(1 - \frac{9e^2}{8} + \frac{25e^4}{192} - \frac{49e^6}{9216} + \ldots\right)\cos M$$

$$+\left(e - \frac{4e^3}{3} + \frac{3e^5}{8} - \ldots\right)\cos 2M$$

$$+\left(\frac{9e^2}{8} - \frac{225e^4}{128} + \frac{3969e^6}{5120} - \ldots\right)\cos 3M$$

$$+\left(\frac{4e^3}{3} - \frac{12e^5}{5} + \ldots\right)\cos 4M$$

$$+\left(\frac{625e^4}{384} - \frac{30{,}625e^6}{9216} + \ldots\right)\cos 5M$$

$$+\left(\frac{81e^5}{40} - \ldots\right)\cos 6M \quad +\left(\frac{117{,}649e^6}{46{,}080} - \ldots\right)\cos 7M \tag{53}$$

$$\frac{r}{a} = 1 + \frac{e^2}{2} - 2e \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{d}{de}\left\{ J_n(ne) \right\} \cos nM$$

$$= 1 + \frac{e^2}{2} - \left(e - \frac{3e^3}{8} + \frac{5e^5}{192} - \dots\right) \cos M$$

$$-\left(\frac{e^2}{2} - \frac{e^4}{3} + \frac{e^6}{16} - \dots\right) \cos 2M$$

$$-\left(\frac{3e^3}{8} - \frac{45e^5}{128} + \dots\right) \cos 3M$$

$$-\left(\frac{e^4}{3} - \frac{2e^6}{5} + \dots\right) \cos 4M$$

$$-\left(\frac{125}{384} e^5 - \dots\right) \cos 5M$$

$$-\left(\frac{81e^6}{240} - \dots\right) \cos 6M \qquad (54)$$

$$\frac{r \cos \theta}{a} = -\frac{3e}{2} + 2 \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{d}{de} \left\{ J_n(ne) \right\} \cos nM$$

$$= -\frac{3e}{2} + \left(1 - \frac{3e^2}{8} + \frac{5e^4}{192} - \frac{7e^6}{9216} + \dots\right) \cos M$$

$$+\left(\frac{e}{2} - \frac{e^3}{3} + \frac{e^5}{16} - \dots\right) \cos 2M$$

$$+\left(\frac{3e^2}{8} - \frac{45e^4}{128} + \frac{567e^6}{5120} - \dots\right) \cos 3M$$

$$+\left(\frac{e^2}{3} - \frac{2e^5}{5} + \dots\right) \cos 4M$$

$$+\left(\frac{125e^4}{384} - \frac{4375}{9216} e^6 + \dots\right) \cos 5M$$

$$+\left(\frac{81e^5}{240} - \dots\right) \cos 6M$$

$$+\left(\frac{16{,}807e^6}{46{,}080} - \dots\right) \cos 7M \qquad (55)$$

$$\frac{r\sin\theta}{a} = \frac{2}{e}\sqrt{1-e^2}\sum_{n=1}^{\infty}\frac{1}{n}J_n(ne)\sin nM$$

$$= \left(1 - \frac{5e^2}{8} - \frac{11e^4}{192} - \frac{457e^6}{9216} - \ldots\right)\sin M$$

$$+ \left(\frac{e}{2} - \frac{5e^3}{12} + \frac{e^5}{24} - \ldots\right)\sin 2M$$

$$+ \left(\frac{3e^2}{8} - \frac{51e^4}{128} + \frac{543e^6}{5120} - \ldots\right)\sin 3M$$

$$+ \left(\frac{e^3}{3} - \frac{13e^5}{30} + \ldots\right)\sin 4M$$

$$+ \left(\frac{125e^4}{384} - \frac{4625e^6}{9216} + \ldots\right)\sin 5M$$

$$+ \left(\frac{27e^5}{80} - \ldots\right)\sin 6M$$

$$+ \left(\frac{16{,}807e^6}{46{,}080} - \ldots\right)\sin 7M \qquad (56)$$

2.6.2 Hyperbolic Motion

The anomaly of Kepler's equation for the case of hyperbolic motion (i.e., $e > 1$, $h \neq 0$) can also be obtained from equation (37). In contrast to the previous approach, however, the solution will be obtained from a table of integrals as

$$t - t_o = \sqrt{\frac{-a^3}{\mu}} \left[-F + e \sinh F \right] \qquad (a<0) \tag{57}$$

where F is referred to as the hyperbolic anomaly and is defined by the equation

$$r = a(1 - e \cosh F) \tag{58}$$

comparison of equations (57) and (40) and/or equations (58) and (38) reveals that this solution is identical to that for elliptic motion under the following substitution

$$E = -iF \tag{59}$$

Thus, all of the basic results derived previously for the specific case of elliptic motion can be extended directly to the case of hyperbolic motion, i.e.

$$\sin \theta = \frac{a}{r}\sqrt{e^2-1} \sinh F \tag{60a}$$

$$\cos \theta = \frac{a}{r}(\cosh F - e) \tag{60b}$$

$$\tan \frac{\theta}{2} \sqrt{\frac{e+1}{e-1}} \tanh \frac{E}{2} \tag{60c}$$

The inverse solution of (57) is accomplished in exactly the same fashion as was (40). However, the series expansion employed previously must be discarded for this case. This conclusion is due to the fact that the upper limit on the eccentricity was placed at .66274... if the series was to be uniformly convergent. In fact, no series approximations for initial estimates which are valid over large variations of time can be constructed because the functions M-F and F are not periodic.

2.6.3 Parabolic Motion

Return once again to equation (37) for the case of parabolic motion (i.e., $e = 1$, $h \neq 0$). For this type of motion

$$r = \frac{p}{1+\cos\theta}$$

But

$$1 + \cos\theta = 2\cos^2\frac{\theta}{2}$$

and

$$\sec^2\frac{\theta}{2} = 1 + \tan^2\frac{\theta}{2}$$

Thus (37) becomes

$$t - t_o = \frac{h^3}{\mu^2}\int_0^\theta \sec^2\frac{\theta}{2}\left(1+\tan^2\frac{\theta}{2}\right)d\theta$$

$$= \sqrt{\frac{p^3}{\mu}}\left[\tan\frac{\theta}{2} + \frac{1}{3}\tan^3\frac{\theta}{2}\right] \tag{61}$$

In contrast to the cases of elliptic and hyperbolic motion, no intermediate variable analogous to E or F is required. However, as before, it is necessary to solve (61) iteratively for $\theta = \theta(t)$ or for this case solve the cubic equation (in $\tan\theta/2$) and subsequently evaluate the true anomaly.

2.7 DEFINITIVE ORBITAL ELEMENTS

Throughout the text of the previous discussions, reference has been made to computational problems arising from certain combinations of initial conditions which tended to make the solution in terms of the geometrical set of orbital elements (a, e, i, Ω, ω, T_0) indeterminate. (Some of these conditions were $e = 0$, $i = 0°$, $i = 180°$.) This behavior causes no concern in some problems where the trajectories of interest are well defined and lie beyond the range over which numerical problems arising from the formulation can be expected to occur. However, there are many other problems (such as those arising in the construction of general conic reference trajectories for an Encke integration of an arbitrary satellite trajectory) in which a "fix" is required to avoid the ambiguity at the ill-conditioned points and the associated loss of significance in the neighborhood of these points. Thus, the following paragraphs will be directed to the task of modifying the computational procedure so as to effect the desired behavior.

Before embarking on this task, however, it is important to note that all of the problems which must be resolved arise from the attempt to employ a set of angles (i, Ω, ω) to define the orientation of the orbital plane and of the line of apsides in the orbital plane to be utilized as the reference direction. Thus, attention will be directed toward means of expressing the motion in the inertial coordinate system without the aid of these quantities. Several means exist to accomplish this objective.

1. Select combinations of variables which taken as a group are well defined.

2. Discard this set of orbital elements and derive a set of new elements (employ the initial position and velocity components directly as elements).

The set which will be recommended for use will then depend on the data available, the method of calculation, the degree of complexity involved in calculating with any given set, the numerical accuracy desired, and the sufficiency of a given set to define the orbit.

2.7.1 The Set $[a, e \cos E_0, e \sin E_0, \vec{h}\ T]$

As was shown earlier, the specification of the components of $\hat{h}$ resolved problems associated with inclinations of o and π. Thus, if $\hat{h}$ is given, attention need only be directed toward the problem of defining a principal direction other than $\vec{r}_p$ to be utilized for the purpose of defining time along the trajectory. This set of variables is established to utilize the initial position vector $\hat{r}_o$ (unit magnitude) as such a direction.

Consider Kepler's equation in the form

$$(t-T)-(t_o-T) = \frac{1}{n}\left[E - e \sin E - E_o + e \sin E_o \right]$$

or

$$M - M_o = E - E_o - e\left[\sin E - \sin E_o \right] \qquad (62)$$

Now if E is replaced by the quantity

$$E = E_o + (E - E_o) = E_o + \Delta E$$

and if this quantity is substituted into Equation (62) and the result expanded, the change in the mean anomaly can be expressed as

$$M - M_o = e \sin E_o + \Delta E - e \cos E_o \sin \Delta E$$

$$- e \sin E_o \cos \Delta E$$

$$\equiv S + \Delta E - C \sin \Delta E - S \cos \Delta E \qquad (63)$$

where

$$S = e \sin E_o$$
$$C = e \cos E_o$$

Now, if the constants S and C can be evaluated uniquely for the case of circular motion; and if ΔE can be related to the angle between $\vec{r}_o$ and $\vec{r}$; then a deterministic solution is possible for all elliptic motions. The first of these objectives is accomplished by referring to

$$r = a(1 - e\cos E)$$

or

$$e\cos E_o \equiv C = \frac{a - r_o}{a} = 1 - \frac{r_o}{a} \tag{64}$$

Now, since Equation (64) may be differentiated to yield

$$e\sin E_o \equiv S = \dot{r}_o / (a\dot{E}_o)$$

and since Kepler's equation may be differentiated to yield

$$\begin{aligned} \dot{M} &= \dot{E}(1 - e\cos E) \\ &= \dot{E}(r/a) \end{aligned}$$

$\dot{E}_o$ may be written as:

$$\begin{aligned} \dot{E}_o &= \dot{M}_o (a/r_o) \\ &= \sqrt{\frac{\mu}{a^3}}(a/r_o) = \sqrt{\frac{\mu}{a r_o^2}} \end{aligned}$$

and

$$\begin{aligned} S &= r_o \dot{r}_o / (\mu a)^{1/2} \\ &= \vec{r}_o \cdot \dot{\vec{r}}_o / (\mu a)^{1/2} \end{aligned} \tag{65}$$

The final step in the derivation is the relation of the change in the eccentric and true anomalies. This step is essential since the change in the true anaomaly $(\Delta\theta)$ will orient the vector $\vec{r}$ in the plane of motion (defined by $\hat{h}$) with respect to $\vec{r}_o$. Referring to Equations (B9) and (B10), it follows that:

$$\cos\Theta = \cos(\Theta - \Theta_o + \Theta_o) = \frac{a}{r}(\cos E - e) \tag{66a}$$

$$\sin\Theta = \sin(\Theta - \Theta_o + \Theta_o) = \frac{a}{r}\sqrt{1-e^2}\,\sin E \tag{66b}$$

Thus, expanding Equation (66) in terms of cos $(\Theta - \Theta_0)$ and sin $(\Theta - \Theta_0)$ yields the linear equations below, written in matrix form

$$\begin{bmatrix} \cos\Theta_o & -\sin\Theta_o \\ \sin\Theta_o & \cos\Theta_o \end{bmatrix} \cdot \begin{bmatrix} \cos(\Theta-\Theta_o) \\ \sin(\Theta-\Theta_o) \end{bmatrix} = \begin{bmatrix} \frac{a}{r}(\cos E - e) \\ \frac{a}{r}\sqrt{1-e^2}\,\sin E \end{bmatrix} \tag{67}$$

A determinant solution for cos $(\Theta - \Theta_0)$ and sin $(\Theta - \Theta_0)$ then yields

$$\cos(\Theta-\Theta_o) = \begin{vmatrix} \frac{a}{r}(\cos E - e) & -\frac{a}{r_o}\sqrt{1-e^2}\,\sin E_o \\ \frac{a}{r}\sqrt{1-e^2}\,\sin E & \frac{a}{r_o}(\cos E_o - e) \end{vmatrix} \tag{68}$$

$$\sin(\Theta-\Theta_o) = \begin{vmatrix} \frac{a}{r_o}(\cos E_o - e) & \frac{a}{r}(\cos E - e) \\ \frac{a}{r_o}\sqrt{1-e^2}\,\sin E_o & \frac{a}{r}\sqrt{1-e^2}\,\sin E \end{vmatrix} \tag{69}$$

Expanding the determinants (68) and (69) in terms of $(E - E_0)$ now yields

$$r\cos(\Theta-\Theta_o) = \frac{a^2}{r_o}\left[S^2 - C(1-C) + S(1-C)\sin\Delta E + (1 - C - S^2)\cos\Delta E \right] \tag{70}$$

$$r\sin(\Theta-\Theta_o) = \frac{a^2\sqrt{1-e^2}}{r_o}\left[S + (1-C)\sin\Delta E - S\cos\Delta E \right] \tag{71}$$

The expressions (70) and (71) are immediately recognized as the components of the position vector $\vec{r}$ in the $\hat{r}_o$ direction and in the direction of increasing anomaly ($\hat{h} \times \hat{r}_o$), respectively, i.e.

$$\vec{r} = r \cos(\theta - \theta_o)\, \hat{r}_o + r \sin(\theta - \theta_o)\, \hat{\theta}_o \tag{72}$$

where $\hat{\theta}_o = \hat{h} \times \hat{r}_o$

Thus, since S and C, and thus $r\cos(\theta-\theta_0)$ and $r\sin(\theta-\theta_0)$, are deterministic for all motions $e<1$, Equation (72) is a unique representation of the vector $\vec{r}$ and is valid for all eccentricities $e<1$.

The process of utilizing these variables to define the position vector as a function of time is as follows.

1. Equations (64) and (65) are utilized to define the variables C and S .
2. Kepler's equation in the form (62) is solved iteratively for ΔE .
3. Equations (70) and (71) are solved for $r\cos(\theta-\theta_0)$ and $r\sin(\theta-\theta_0)$.
4. Equation (72) is solved for $\vec{r}$.

The solution for the velocity vector may now be obtained by differentiating Equation (72) and substituting for the derivatives of the coefficients of $\hat{r}_o$ and $\hat{\theta}_o$ their equivalents. Or, by expressing the velocity vector in terms of its magnitude and flight path angle. This latter approach is more attractive since both of the required quantities are easily defined (the velocity magnitude from the energy equation and the flight path angle from the angular momentum). Thus,

$$\vec{V} = V \sin\gamma\, \hat{r} + V \cos\gamma\, \hat{\theta} \tag{73}$$

where

$$V = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)} \tag{74a}$$

$$\cos\gamma = \sqrt{\mu p}\, / \,(r V) \tag{74b}$$

and where $\sin \gamma'$ is fixed by expressing γ' in terms of $\theta - \theta_0$ or ΔE (to avoid sign ambiguities in the angle γ')

$$\tan \gamma = \frac{e}{\sqrt{1-e^2}} \sin E$$

$$= \frac{e}{\sqrt{1-e^2}} \sin (E_0 + \Delta E)$$

$$= \frac{1}{\sqrt{1-e^2}} [S \cos \Delta E + C \sin \Delta E] \tag{74c}$$

This set of variables, while valid for all $e < 1$, is also extendable to the case of hyperbolic motion. This fact is observed by noting that the substitution of $E = iF$ into (63) through (74) yields purely real terms.

2.7.2 The Set $[\vec{r}_o, \vec{v}_o, T]$

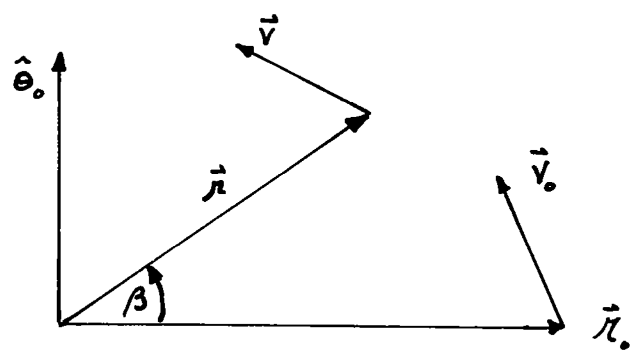
Since the angular momentum for the angular momentum for the central force problem is constant, any vector lying in the plane of motion (e.g., A) can be expressed as a linear combination of any two non-colinear vectors in that plane. Thus, the initial position and velocity vectors themselves can be utilized as "elements" for the purpose of constructing the radius vector at some time (t) as

$$\vec{r} = f\,\vec{r}_o + g\,\vec{V}_o \tag{75}$$

where f and g are expressions to be determined. Furthermore, since $\vec{r}_o$ and $\vec{v}_o$ are constants, then (75) can be differentiated to yield the velocity vector

$$\dot{\vec{r}} = \dot{f}\,\vec{r}_o + \dot{g}\,\vec{V}_o \tag{76}$$

Consider the following sketch illustrating motion in the orbit plane



where the normal to $\vec{r}_o$ in the orbit plane is given by $\hat{\theta}_o = \hat{h} \times \hat{r}_o$.

The position vector at time t is resolved as the sum of components in the direction of the given radius and its normal, i.e.

$$\begin{aligned} \vec{r} &= r\cos\beta\ \hat{r}_o + r\sin\beta\ \hat{\theta}_o \\ &= \frac{r}{r_o}\left[\cos\beta\ \vec{r}_o + \sin\beta\ \frac{(\vec{r}\times\vec{V})}{h}\times\vec{r}_o\right] \end{aligned} \tag{77}$$

But, the vector identity for the triple cross product is

$$(\vec{r}_o \times \vec{V}_o) \times \vec{r}_o = (\vec{r}_o \cdot \vec{r}_o)\, \vec{V}_o - (\vec{r}_o \cdot \vec{V}_o)\, \vec{r}_o \tag{78}$$

Substituting this identity into (77) yields

$$\vec{r} = \frac{r}{r_o}\left[\cos\beta\, \vec{r}_o + \frac{\sin\beta}{h}\left(r_o^2\, \vec{V}_o - \vec{r}_o \cdot \vec{V}_o\, \vec{r}_o \right)\right] \tag{79}$$

$$= \frac{r}{r_o}\left(\cos\beta - \frac{\vec{r}_o \cdot \vec{V}_o}{h} \sin\beta \right)\vec{r}_o + \frac{r\, r_o}{h} \sin\beta\, \vec{V}_o \tag{80}$$

Now, since equations (75) and (80) represent the same vector,

$$f = \frac{1}{r_o}\left(r \cos\beta \right) - \left(\frac{\vec{r}_o \cdot \vec{V}_o}{r_o h} \right) r \sin\beta \tag{81}$$

$$g = \left(\frac{r_o}{h} \right) r \sin\beta \tag{82}$$

At this point the geometrical element β must be tied into the dynamics of the problem. This step is accomplished by recognizing that the angle β is the difference in true anomalies for the vectors $\vec{r}$ and $\vec{r}_o$, i.e.,

$$r \cos\beta = r \cos(\theta - \theta_o)$$

$$r \sin\beta = r \sin(\theta - \theta_o)$$

But these relationships have already been evaluated [equations (70), (71)] in terms of the change in the eccentric anomaly. Thus, substituting, the function f is found to be

$$f = 1 - \frac{a}{r}\left(1 - \cos\Delta E \right) \tag{83}$$

and g is

$$g = t - t_o - \sqrt{\frac{a^3}{\mu}}\left(\Delta E - \sin\Delta E \right) \tag{84}$$

The coefficients $\dot{f}$ and $\dot{g}$ can now be obtained by differentiating Kepler's equation as follows

$$\frac{d}{dt}\left[\sqrt{\frac{\mu}{a^3}}\,(t - t_o) \right] = \frac{d}{dt}\left[E - e\sin E - E_o + e\cos E_o \right]$$

or

$$\sqrt{\frac{\mu}{a^3}} = \dot{E}\,(r/a)$$

Now noting that

$$\Delta\dot{E} \equiv \dot{E}$$

$$= \frac{1}{r}\sqrt{\frac{\mu}{a}}$$

The desired functions can be expressed as

$$\dot{f} = \frac{d}{dt}(f)$$

$$= -\frac{a}{r_0} \sin \Delta E \Delta\dot{E}$$

$$= -\frac{\sqrt{\mu a}}{r\, r_0} \sin \Delta E \tag{85}$$

and

$$\dot{g} = \frac{d}{dt}(g)$$

$$= 1 - \sqrt{\frac{a^3}{\mu}}\,(1 - \cos \Delta E)\Delta\dot{E}$$

$$= 1 - \frac{a}{r}(1 - \cos \Delta E) \tag{86}$$

Now, the solution in terms of these variables proceeds as follows

1. solve Kepler's equation Eq (63) $\Delta E = \Delta E(t)$
2. evaluate f, $\dot{f}$, g, $\dot{g}$ [Eqs. (83), (84), (85), (86)]
3. evaluate $\vec{r}$, $\vec{v}$ [Eqs. (75), (76)]

This representation of $\vec{r}$, $\vec{v}$ is completely deterministic for $e < 1$ and can be extended to the case $e > 1$ simply by substituting

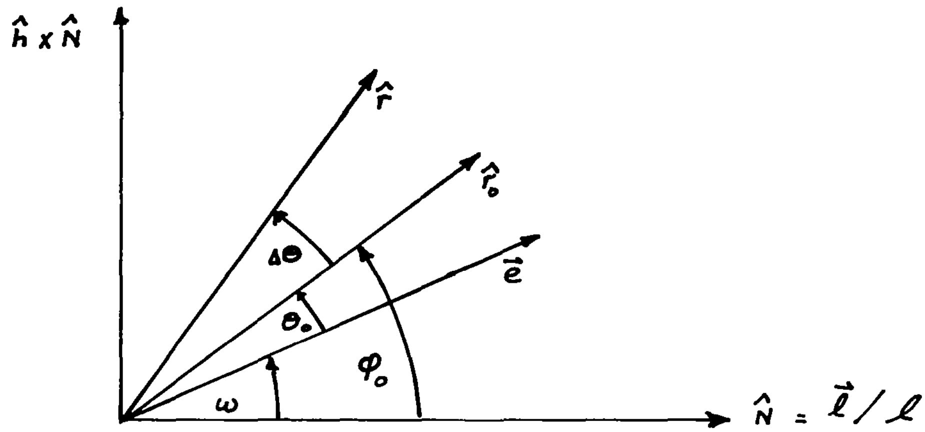
$$E = -i\,F$$

and noting that $[a < 0 \quad e > 1]$ for hyperbolic motion.

This solution is somewhat simpler than that employed in the case of the variables [a, $e \cos E_0$, $e \sin E_0$, $\hat{\tau}$]; thus, unless personal preference dictates the use of the former, the latter is recommended.

2.7.3 The Set $[\vec{h}, \vec{e}, T]$

The set of parameters $(\vec{h}, \vec{e})$ can be utilized for predicting the motion if the data which is given is $\vec{r}_o, \vec{V}_o$ and if the nature of the indeterminacies are examined. Consider the sketch



As the vector $\vec{e}$ shrinks to zero its direction (the angle ω) becomes undefined. The other vectors of the sketch, however, are still well defined. Thus, the most direct method of solution is constructed by employing either the vector $\hat{N}$ or the vector $\hat{r}_o$ for the reference direction.

As was mentioned in the text, the vector $\hat{N}$ is poorly defined in the vicinity of inclination of 0^o or 180^o. However, as was also mentioned this, indeterminacy is easily resolved by selecting any other reference plane (e.g., if $i = o$ for the equatorial reference, select the ecliptic as the fundamental plane). With $\hat{N}$ known, the procedure for determining $\vec{r}, \vec{V}$ as functions of time is:

solve for $\vec{e}$ and $\vec{h}$

solve Kepler's equation [Eq. (63)] for ΔE

solve Equations (70) and (71) for $\Delta\Theta$

solve for φ_o from

$$\varphi_o = \cos^{-1}(\hat{r}_o \cdot \hat{N}) \qquad \begin{array}{ll} 0 < \varphi_o < \pi, & r_{o_z} > 0 \\ \pi < \varphi_o < 2\pi, & r_{o_z} < 0 \end{array}$$

Compute $\vec{r}$ from

$$\hat{r} = \cos(\varphi_o + \Delta\theta)\,\hat{N} + \sin(\varphi_o + \Delta\theta)\,\hat{h} \times \hat{N}$$

$$\vec{r} = \frac{h^2/\mu}{1 + \vec{e}\cdot\hat{r}}\,\hat{r} \tag{87}$$

define $\vec{v}$ from

$$\vec{v} = \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{h} \times \hat{r}$$

$$= \left[\frac{\mu}{h}\,(\vec{e} \times \hat{r}) \cdot \hat{h}\right]\hat{r} + \frac{h}{r}\,\hat{h} \times \hat{r} \tag{88}$$

This approach can be simplified slightly if the decision function regarding the definition of the plane is removed. This objective can be accomplished simply by noting that $\hat{r}$ is also defined by

$$\hat{r} = \cos\Delta\theta\,\hat{r}_o + \sin\Delta\theta\,(\hat{h} \times \hat{r}_o)$$

Note the similarity of equations (87) and (88) with equations presented earlier for other approaches to deterministic elements. This fact graphically illustrates the relationships between the geometrical elements $\vec{e}$ and $\vec{h}$ and the variables employed in the previous discussions. Also note that as before the extension from the case $e < 1$ to the case $e > 1$ is direct.

2.7.4 The Set $[\vec{r}_1, \vec{r}_2, \Delta t]$

In many cases of orbit determination, two position vectors and their time difference are either known or given as fixed boundary conditions in place of traditional orbital elements. The set $(\vec{r}_1, \vec{r}_2, \Delta t)$, however, is not a definitive set unless the following conditions are specified.

1. $\vec{r}_1$, $\vec{r}_2$ noncolinear, i.e., $\vec{r}_1 \neq k\vec{r}_2$

2. the direction or sense of rotation is specified

This set of specifications and the equations

$$\left(\frac{\mu}{a^3}\right)^{\frac{1}{2}} \Delta t = E_2 - E_1 - e(\sin E_2 - \sin E_1) \tag{89}$$

$$\cos^{-1}(\hat{r}_1 \cdot \hat{r}_2) = \Delta\Theta = \cos^{-1}\left(\frac{p - r_2}{e\, r_2}\right) - \cos^{-1}\left(\frac{p - r_1}{e\, r_1}\right) \tag{90}$$

can be used to determine the parameters a, e of the orbit in the elliptical case through the use of two-dimensional numerical search techniques when the direction or sense of rotation is specified to resolve the ambiguity in $\Delta\Theta$. Since the simultaneous solution of the equation by slope methods requires a close first estimate of the parameters a and e, it is advisable to use a general method involving interaction of only a single variable developed by Euler (for the case of hyperbolic motion and extended for the elliptic and parabolic cases by Lambert and Lagrange) or (for the special case where the values of the time difference are small), a method based on series expansions.

2.7.4.1 Small values of Δt

Since the solution to the vector differential equation of motion $\ddot{\vec{r}} = -\frac{\mu}{r^2}\hat{r}$ can be expressed in a Taylor series convergent for an adequately small time increment, Δt then

$$\vec{r}_2 = \vec{r}_1 + \dot{\vec{r}}_1 \Delta t + \frac{\ddot{\vec{r}}_1}{2}(\Delta t)^2 + \frac{\dddot{\vec{r}}_1}{3!}(\Delta t)^3 + \cdots$$

Substituting for the derivatives of $\vec{r}$ higher than $\dot{\vec{r}}$ the expressions

$$\ddot{\vec{r}} = -\left(\frac{\mu}{r^3}\right)\vec{r}_1$$

$$\dddot{\vec{r}} = -\left(\frac{\mu}{r^4}\right)\dot{r}_1\,\vec{r}_1 - \left(\frac{\mu}{r^3}\right)\dot{\vec{r}}_1$$

$$\cdots\cdots$$

then gives

$$\vec{r}_2 = f\,\vec{r}_1 + g\,\dot{\vec{r}}_1 \tag{91}$$

where

$$f = 1 - \frac{\mu}{r^3}\frac{(\Delta t)^2}{2} + \left(\frac{\mu}{r^4}\right)\dot{r}_1\,\frac{(\Delta t)^3}{3!} + \cdots \tag{92}$$

$$g = \Delta t - \frac{\mu}{r^3}\,\frac{(\Delta t)^3}{3!} + \cdots \tag{93}$$

Thus, for a sufficiently small time increment Δt, the velocity vector at time t_1 ($\dot{\vec{r}}_1$) can be calculated from equations (61-93) from $\vec{r}_1$, $\vec{r}_2$, and Δt. This solution, in terms of the f and g series, does converge rapidly to a single value of $\dot{\vec{r}}_1$ for moderate values of Δt so that a solution for any of the other sets of elements (sections 2.7.1 and 2.7.3) is possible.

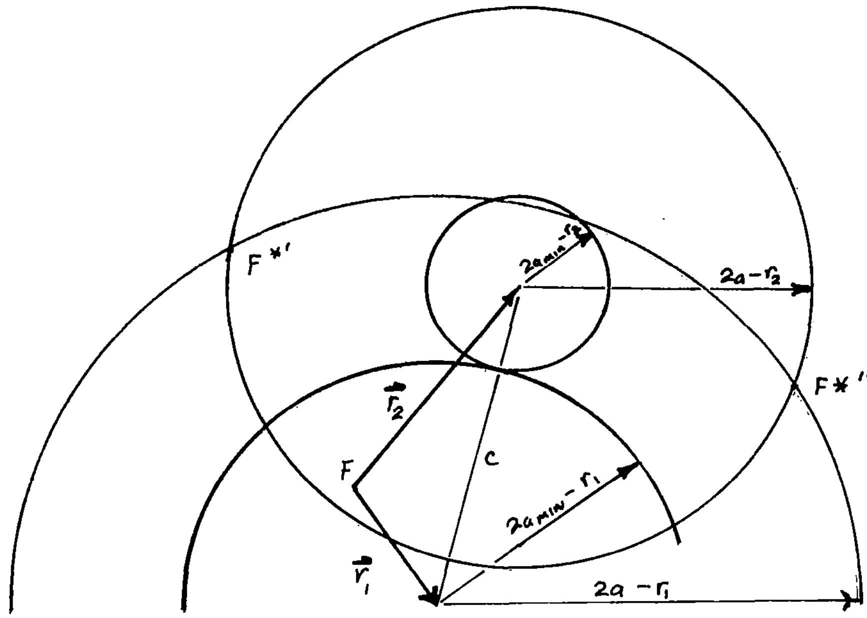
2.7.4.2 The Method of Lambert-Euler-Lagrange

The method of Lambert-Euler-Lagrange, valid for all time increments, considers a conic section having semimajor axis length "a" and containing the vectors $\vec{r}_1$, $\vec{r}_2$. Expressions are then developed relating Δt, $r_1 + r_2$, and $c = |\vec{r}_2 - \vec{r}_1|$ to "a." These expressions are simultaneously solved by numerical iteration of the single parameter "a."

Consider an elliptical path as the desired solution and let r_1', r_2' denote the distance from the end points of $\vec{r}_1$ and $\vec{r}_2$, respectively, to the non-central body focus F^*. Since the sums $(r_1 + r_1')$ and $(r_2 + r_2')$ must equal 2a, then

$$r_1' = 2a - r_1$$
$$r_2' = 2a - r_2$$

The intersection of two circles having radii r_1' and r_2' drawn about the end points of $\bar{r}_1$ and $\bar{r}_2$, respectively, locate two foci $F^{*\prime}$ and $F^{*\prime\prime}$.



Hence, two elliptical orbits exist for every value of the semimajor axis chosen. This fact is also true for the other conic sections as well. For elliptical orbits, however, there is some minimum semimajor axis for which the two circles of radius r_1' , r_2' just touch. For this case,

$$r_1' + r_2' = c$$

Substituting for r_1' and r_2' then yields

or $$(2a_{MIN} - r_1) + (2a_{MIN} - r_2) = c$$

$$a_{MIN} = \frac{r_1 + r_2 + c}{4}$$

All elliptical orbits have a semimajor axis length greater than or equal to this value since no elliptical orbits exist for $a < a$ min. It is observed in the diagrams that all elliptic orbits can be classified as being either of two types, depending on whether or not the line connecting the two foci intersects the chord c. For each of these orbit types in turn, there exists two paths: one for each sense of rotation, i.e., clockwise or counter-clockwise. Therefore, corresponding to every semimajor axis $a > a$ min, there are four possible elliptical paths containing $\vec{r}_1$ and $\vec{r}_2$ and four generally different times. Let E_2 and E_1 be the eccentric anomalies of $\vec{r}_1$ and $\vec{r}_2$ for one of these paths; then

$$r_1 = a(1 - e\cos E_1)$$

$$r_2 = a(1 - e\cos E_2)$$

Adding these equations yields

$$r_1 + r_2 = 2a\left[1 - \frac{e}{2}(\cos E_1 + \cos E_2)\right].$$

Now let

$$2G = E_2 + E_1 \quad , \quad 2g = E_2 - E_1$$

Then, in terms of G and g

$$r_1 + r_2 = 2a\left[1 - \frac{e}{2}\cos(G-g) - e\cos(G+g)\right]$$
$$= 2a\left[1 - e\cos G\cos g\right]$$

But the chord can be expressed from the equations for the x and y coordinates measured from the center of the ellipse as

$$c^2 = (a\cos E_2 - a\cos E_1)^2 + (a\sqrt{1-e^2}\sin E_2 - a\sqrt{1-e^2}\sin E_1)^2$$
$$= a^2(\cos E_2 - \cos E_1)^2 + a^2(1-e^2)(\sin E_2 - \sin E_1)^2$$

or in terms of G and g

$$c^2 = 4a^2\sin^2 G\sin^2 g + 4a^2(1-e^2)\sin^2 g\cos^2 G$$

This latter equation can in turn be simplified by letting

$$\cos h = e\cos G$$

Under this substitution the equation of the chord becomes

$$c^2 = 4a^2\sin^2 g\left[\sin^2 G + (1-e^2)\cos^2 G\right]$$
$$= 4a^2\sin^2 g\left[1 - e^2\cos^2 G\right]$$
$$= 4a^2\sin^2 g\left[1 - \cos^2 h\right]$$
$$= 4a^2\sin^2 g\sin^2 h$$

or

$$c = 2a \sin g \sin h$$

At this point, define

$$\alpha = h + g \quad , \quad \beta = h - g \tag{94}$$

and sum the expression for r_1, r_2 and c to obtain

$$\begin{aligned} r_1 + r_2 + c &= 2a\left[1 - \cos g \cos h + \sin g \sin h\right] \\ &= 2a\left[1 - \cos(h+g)\right] \\ &= 2a\left[1 - \cos\alpha\right] \\ r_1 + r_2 + c &= 4a \sin^2 \frac{\alpha}{2} \end{aligned} \tag{95}$$

In a similar manner, the relation

$$r_1 + r_2 - c = 4a \sin^2 \frac{\beta}{2} \tag{96}$$

is obtained.

Finally, the differenced form of Kepler's equation becomes

$$\begin{aligned} \left(\frac{\mu}{a^3}\right)^{\frac{1}{2}} \Delta t &= E_2 - E_1 - e(\sin E_2 - \sin E_1) \\ &= 2g - 2e \sin g \cos G \\ &= 2g - 2 \sin g \cos h \\ &= (\alpha - \beta) - 2 \sin\left(\frac{\alpha - \beta}{2}\right) \cos\left(\frac{\alpha + \beta}{2}\right) \end{aligned}$$

or

$$\Delta t = \left(\frac{a^3}{\mu}\right)^{\frac{1}{2}} \left[(\alpha - \sin\alpha) - (\beta - \sin\beta)\right] \tag{97}$$

When solved simultaneously, Eq. (95), (96), and (97) constitute Lambert's theorem for elliptic motion. Since four paths satisfy any single value of "a" chosen, it is apparent that the Δt given by Eq. (97) represents only one of the four time solutions possible for an ellipse. By means of a geometrical argument described in Reference 5, the solution (97) with positive values of α and β is shown to correspond to the path whose sector area contains neither of the two foci.

If only positive values of α and β are consistently used from Eq. (95) and (96), the solution for Δt for the case of the sector area containing the central body focus F is

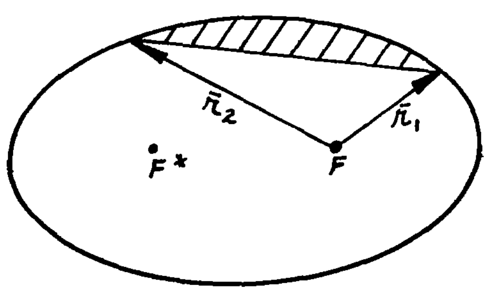
$$\Delta t_F = \left(\frac{a^3}{\mu}\right)^{\frac{1}{2}} \left[(\alpha - \sin\alpha) + (\beta - \sin\beta)\right] \tag{98}$$

The remaining two solutions for sectors containing both foci F , F*, and the empty foci F* are simply obtained by subtracting the times Eq. (97) and (98) from the period of the motion, i.e.,

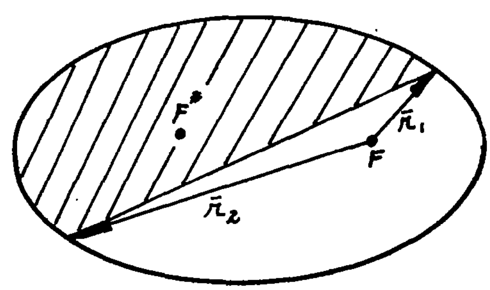
$$\Delta t_{FF*} = 2\pi\left(\frac{a^3}{\mu}\right)^{\frac{1}{2}} - \Delta t \tag{99}$$

$$\Delta t_{F*} = 2\pi\left(\frac{a^3}{\mu}\right)^{\frac{1}{2}} - \Delta t_F \tag{100}$$

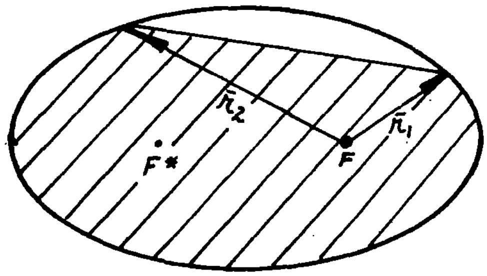
The sector geometry of the paths corresponding to these times is illustrated below:



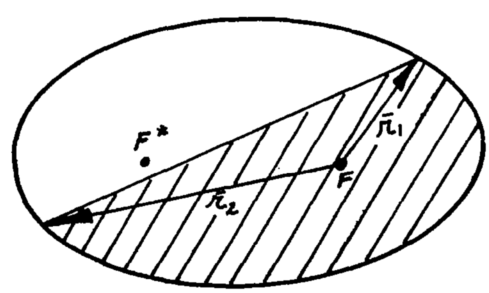
(a) Sector Area contains no foci
($\Delta\theta < 180°$)



(b) Sector Area contains empty focus
($\Delta\theta < 180°$)



(c) Sector Area contains both foci
($\Delta\theta > 180°$)



(d) Sector Area contains central body
($\Delta\theta > 180°$)

Expanding the functions $\sin\alpha$ and $\sin\beta$ in power series in Eq. (97) and (98), and taking the limit as "a" $\to \infty$, the time difference for parabolic orbits for both anomalistic angular differences less than and greater than 180° yields

$$\Delta t_{P^-} = \frac{1}{6}\left(\frac{1}{\mu^{\frac{1}{2}}}\right)\left[(r_1+r_2+c)^{3/2} - (r_1+r_2-c)^{3/2}\right] \qquad (\Delta\theta < \pi) \tag{101}$$

$$\Delta t_{P^+} = \frac{1}{6}\left(\frac{1}{\mu^{\frac{1}{2}}}\right)\left[(r_1+r_2+c)^{3/2} - (r_1+r_2-c)^{3/2}\right] \qquad (\Delta\theta > \pi) \tag{102}$$

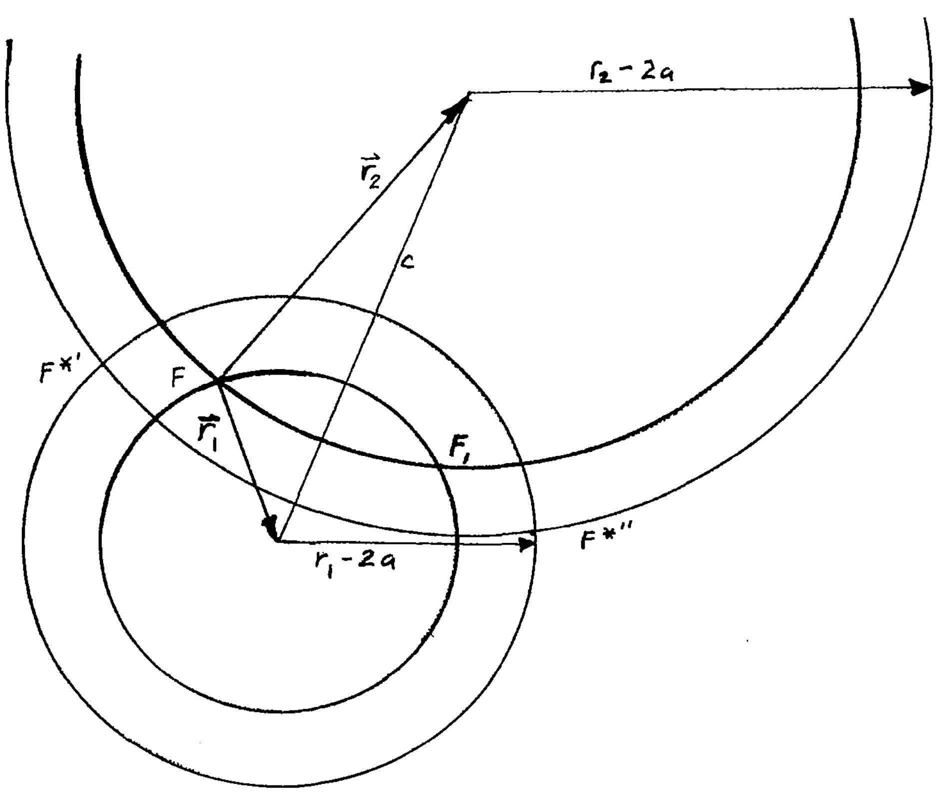
Next, consider a hyperbolic path as being the desired solution and, as before, let r_1', r_2' denote the distance from the end points of $\vec{r}_1$ and $\vec{r}_2$, respectively, to the non-central body focus F^*. Since the difference in distances of a point on a hyperbola from the foci is a positive constant equal to $-2a$,* then

$$r_1' - r_1 = -2a \qquad\qquad r_1' = r_1 - 2a$$

or

$$r_2' - r_2 = -2a \qquad\qquad r_2' = r_2 - 2a$$

The intersection of two circles having radii r_1' and r_2' drawn about the end points of $\vec{r}_1$ and $\vec{r}_2$, respectively, locates two foci $F^{*\prime}$ and $F^{*\prime\prime}$.



Hence, two hyperbolic paths exist for every value of the parameter (-2a). For one of these hyperbolas (having empty focus $F^{*''}$), the line between the foci intersects the chord c and the anomalistic angular difference is less than 180°. For the other (having empty focus $F^{*'}$) the line between the foci does not intersect the chord and the anomalistic angular difference is greater than 180°.

In the limit as $a \longrightarrow 0$, the empty focus is located at F_0 in the diagrams and the chord c becomes the only solution path. Since the total energy $\mathcal{E} = -\frac{\mu}{2a}$, the straight line solution corresponds to the infinite energy case.

By substituting for the eccentric anomaly, $F = i E$, in the elliptic formulation of Lambert-Euler's method and letting

$$\epsilon - \delta = F_2 - F_1$$

$$\cosh\left(\frac{\epsilon + \delta}{2}\right) = \epsilon \cosh\left(\frac{F_2 + F_1}{2}\right)$$

then Eq. (95) and (96) become

$$r_1 + r_2 + c = (-4a)\sinh^2\frac{\epsilon}{2} \tag{103}$$

$$r_1 + r_2 - c = (-4a)\sinh^2\frac{\delta}{2} \tag{104}$$

and (97) becomes

$$\Delta t = \left(\frac{-a^3}{\mu}\right)^{\frac{1}{2}}\left[(\sinh\epsilon - \epsilon) - (\sinh\delta - \delta)\right] \qquad \Delta\theta < 180^\circ \tag{105}$$

Eq. (103), (104), and (105) are the hyperbolic form of the Lambert-Euler method and can be solved simultaneously by an iteration procedure involving the single parameter (-a). By means of a geometrical argument in Reference 5, it is shown that the time increment given by Eq. (105) corresponds to the case where the anomalistic angular difference is less than 180°. Also, it is shown that for an anomalistic angular difference greater than 180°, the time increment is given as:

$$\Delta t_F = \left(\frac{-a^3}{\mu}\right)^{\frac{1}{2}}\left[(\sinh\epsilon - \epsilon) + (\sinh\delta - \delta)\right] \qquad \Delta\theta > 180^\circ \tag{106}$$

Since $[\vec{r}_1, \vec{r}_2, \Delta t]$ uniquely determines the path once the conditions 1 and 2 are satisfied, only one of the formulations of the Lambert-Euler method is satisfied. The decision as to which of the three formulations should be employed is aided by first evaluating the parabolic time increment from Eq. (101) for $\Delta\theta < 180°$ or Eq. (102) for $\Delta\theta > 180°$. If Δt is greater than the parabolic time increment, the elliptic formulation is applicable. If t is less than the parabolic time, the hyperbolic formulation is used.

3.0 RECOMMENDED PROCEDURES

Many classes of motion and types of analysis have been considered in regard to the application of the material presented in Section 2.0 of this Monograph. This review revealed many cases in which the classical set of elements $(a, e, i, \omega, \Omega, M_o)$ is to be preferred due to the graphic display of information. However, these studies also revealed that a more universal form of the solution should be adopted for those cases in which the prime concern was in generating values of $\vec{r}$ and $\vec{v}$ as functions of time. This conclusion was made based on the indeterminacies resulting in the solution for the cases $e = 0$, $i = 0$ and/or $i = \pi$ and on difficulties arising in the solution for values of $e > 1$.

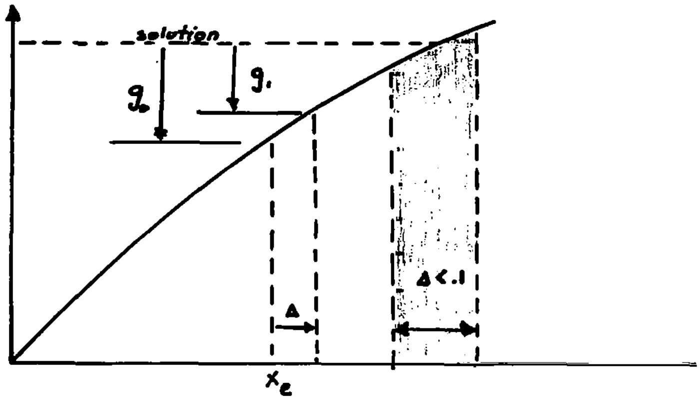
Based on the analysis of the material presented in this Monograph, it is recommended that the set of variables $\vec{r}_o$, $\vec{v}_o$ (and the associated parameters f, g, $\dot{f}$, and $\dot{g}$) be employed for this latter application. This set exhibits none of the problems associated with the classical set and is generally well behaved. Further, an extremely simple computational procedure can be constructed to mechanize these equations for elliptic or hyperbolic motion. One such mechanization is presented in diagramatic form on another page for the case in which $\vec{r}_o$, $\vec{v}_o$ are given and $\vec{r}$, $\vec{v}$ are desired at $t-t_o$.

Special note is made that the procedure employed for determining the initial estimate of X ($X = E-E_o$ or $F-F_o$) is a numerical search. This approach (rather than series expansion, for example) was taken to assure that the rationale would be valid for elliptic and hyperbolic motion. The significance of the two branches is

Branch 1 - (In this case, values of $\vec{r}$ and $\vec{v}$ are being generated either for the first time or at large time steps.) The initial value of X is selected as $M-M_o$ and a grid for the numerical search is set at $\frac{(X)}{10}_{\text{mod } 360°}$

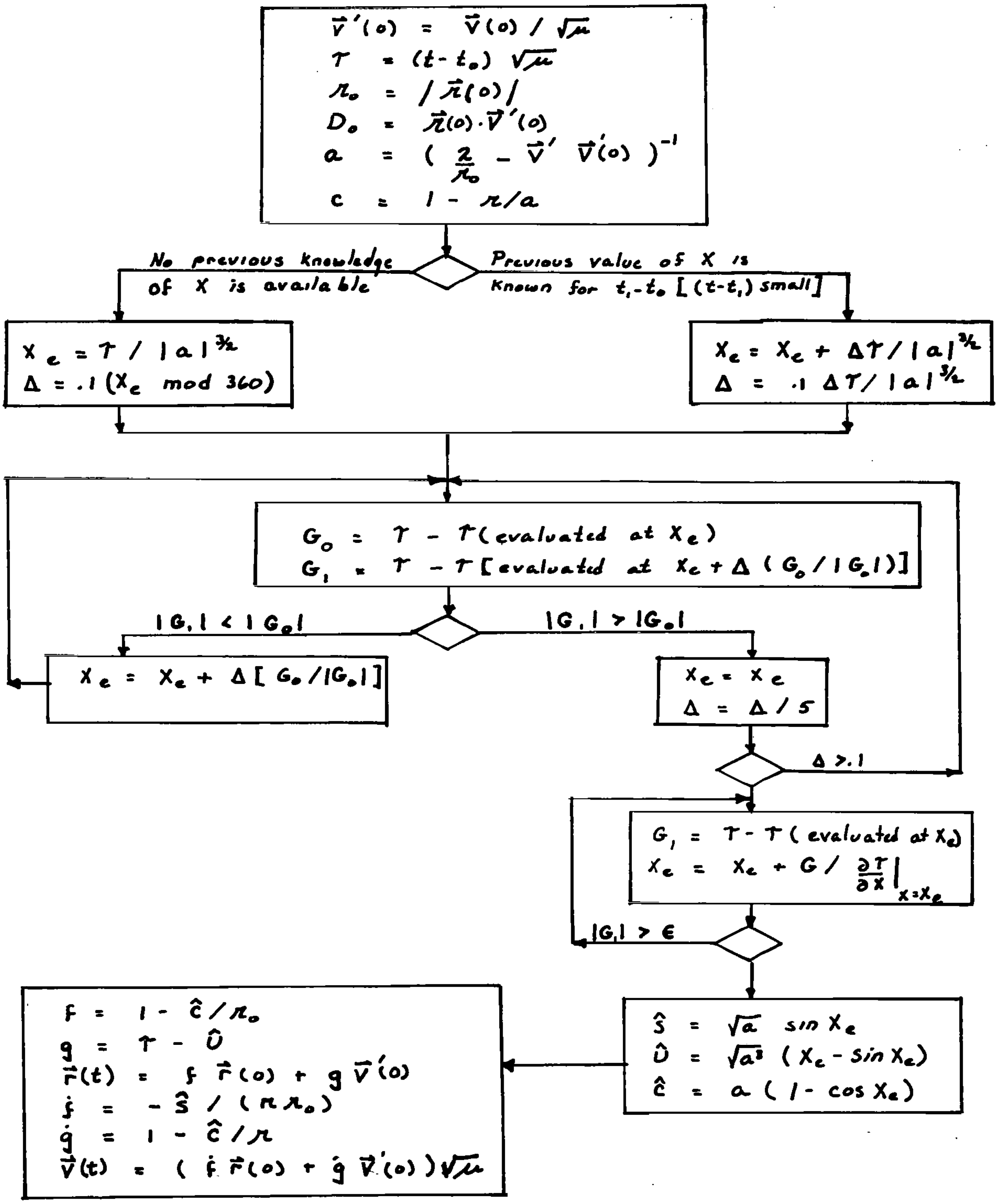
Branch 2 - (In this case, values of $\vec{r}$ and $\vec{v}$ are being generated at reasonably small time steps and the previous value of X is available.) The initial value of X is selected as the previous value plus the change in the mean anomaly (M_n-M_{n-1}) and the grid size is set at $\frac{M_n-M_{n-1}}{10}$. This case is extremely useful in the case of an Encke integration, for example.

Once this data is available, a search process is initiated as illustrated in the sketch



A fixed step (Δ) is employed in this process until $|q_0| > |q_1|$. At this time, Δ is reduced and the process continued. The numerical search phase terminates when $\Delta < .1$, and a Newton iteration is initiated. This process continues until $|q_1| < \epsilon$. At this point in the process, the position dependent variables $(\hat{c}, \hat{s}, \hat{u})$ are evaluated (only the case of elliptic motion is shown; however, only minor change is required for adding the hyperbolic capability) and the coefficients f, g, $\dot{f}$ and $\dot{g}$ defined. The solution is now complete.

Compute $\vec{r}(t)$, $\vec{V}(t)$



4.0 REFERENCES

The material presented in Section 2.0 is self-contained to a very high degree. Nonetheless, the discussion represents only one approach to the problem. For this reason, a list of references has been compiled to assist in isolating additional information should the need arise. These references, in conjunction with other monographs of this series, provide an extremely thorough presentation of basic and advanced theories of motion.

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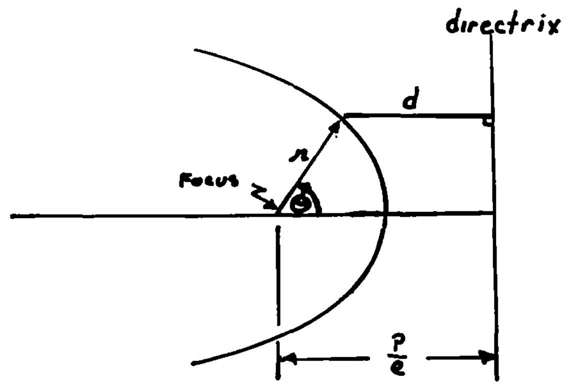
APPENDIX A
POLAR FORM OF THE EQUATIONS OF A CONIC SECTION AND THE CLASSIFICATION OF SOLUTION PATHS

It will be verified that equation (24) of the text does indeed represent the polar form of the equation of a conic section. To do so, the following definition of a conic section is used:

> A conic section is the locus generated by a point whose distance from a point called the focus is in a fixed ratio to its perpendicular distance to a straight line called the directrix.

The geometry of a general conic section is shown in the following diagram:



Let e be the fixed ratio of the distance to the focus to the distance to the directrix. The parameter e, called the eccentricity, will then be given by $e = r/d$. Let the fixed distance from the focus to the directrix be p/e , then

$$r \cos\theta + d = \frac{p}{e}.$$

where the angle θ is the true anomaly and is measured from the direction of least radial distance. Substituting $d = r/e$ into the above gives:

$$r\left(\cos\theta + \frac{1}{e}\right) = \frac{p}{e}$$

or

$$r = \frac{p}{1 + e\cos\theta}$$

which is identical to the integral of the motion (24). For e = 0 the path is one of constant radius, namely a circle. For $|e| < 1$ is always finite for all θ and the path is an ellipse. For $|e| > 1$, r becomes infinite as $\cos\theta \rightarrow (-1/e)$ and the path is a hyperbola. When e = 1, the solution path is a parabola. The parameter p, known as the semi-latus rectum, is half the width of the conic section at $\theta = \pi/2$. The solution path for p = 0 is rectilinear (a straight line) and is of little practical interest. For this reason, it will not be considered here.

Since $e^2 = 1 + 2\epsilon h^2/\mu^2$, the eccentricity will be greater than, equal to, or less than one depending on whether the total energy/unit mass (ϵ) is greater than, equal to, or less than zero (the quantity h^2/μ^2 always being positive). Thus, the solution paths can be classified in terms of the eccentricity e, the total energy/unit mass, or the relative values of $V^2/2$ and μ/r. A table giving the values of e, ϵ and the values of $V^2 r$ for the different conic sections is given below:

Conic Section	Eccentricity, e	Total Energy /unit mass, ϵ	Value of $V^2 r$
Circle	e = 0	$\epsilon = -\frac{\mu^2}{2h^2}$	$V^2 r = \mu$
Ellipse	e < 1	$\epsilon < 0$	$V^2 r < 2\mu$
Parabola	e = 1	$\epsilon = 0$	$V^2 r = 2\mu$
Hyperbola	e > 1	$\epsilon > 0$	$V^2 r > 2\mu$

APPENDIX B

SOME PROPERTIES OF THE CONIC SECTION

The Ellipse

An ellipse is the locus of points whose distance to the directrix is less than the distance to the focus, i.e., $e < 1$. The curve satisfying this condition is closed about the focus and is symmetrical about the line $\theta = 0$. The rectangular cartesian form of the equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{B1}$$

where x and y are measured along the major axis and minor axis, respectively, from an origin located at the center of the major axis. This curve is symmetrical about both the x and the y axes with two foci located on the x axis and has axis intercepts at $x = \pm a$ and $y = \pm b$.

The maximum and minimum distances of the ellipse from the focus are found from the polar form of the ellipse to be

$$r_{max} = \frac{p}{1-e} \qquad r_{min} = \frac{p}{1+e}$$

and the major axis length "2a" is the sum of these distances.

$$2a = \frac{p}{1-e} + \frac{p}{1+e}$$

Hence,

$$a = \frac{p}{1-e^2} \tag{B2}$$

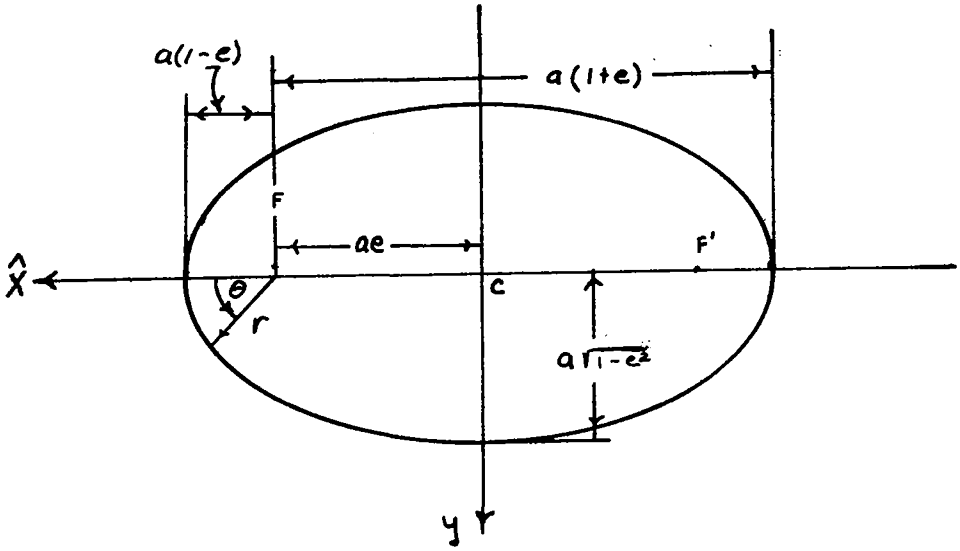
or

$$p = a(1-e^2) \tag{B3}$$

In terms of the semi-major axis length, the maximum and minimum distances are therefore

$$r_{max} = a(1+e) \qquad r_{min} = a(1-e)$$

These distances are indicated for a general ellipse in the following diagram:



The points F and F' donate the two foci of the ellipse, and point C the center. By symmetry, CF=CF' where

$$CF = a(1+e) - a$$

$$= ae$$

Therefore, the x coordinate of a point on the ellipse given by polar coordinates (r, θ) is

$$x = ae + r\cos\theta$$

Hence,

$$\cos\theta = \frac{x - ae}{r}$$

and the polar form of the equation for an ellipse becomes

$$r = \frac{a(1-e^2)}{1 + e\left(\frac{x-ae}{r}\right)}$$

or

$$r = a(1-e^2) - e(x-ae) \qquad \text{(B4)}$$

Since

$$r^2 = (x-ae)^2 + y^2 \qquad \text{(B5)}$$

then substituting (B1) into (B5) and rearranging terms gives

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1 \qquad \text{(B6)}$$

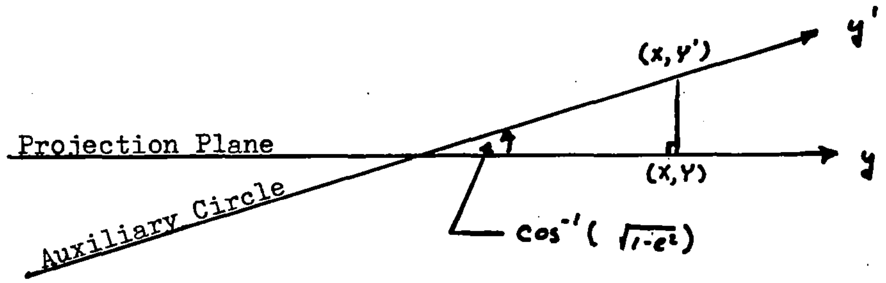
Comparing (B1) and (B6) now yields the semi-minor axis length in terms of the semi-major axis length and the eccentricity, i.e.,

$$b = a\sqrt{1-e^2} \qquad \text{(B7)}$$

Equation (B6) suggests a construction of an ellipse by projection of a circle of radius (a) on a plane inclined to that of the circle by an angle ($\cos^{-1}\sqrt{1-e^2}$) and intersecting at the center. Pursuing this development, the equation of a circle of radius (a) in rectangular cartesian coordinates is

$$\frac{x^2 + y'^2}{a^2} = 1 \qquad \text{(B8)}$$

where y' is the ordinate of the circle in a plane which has been rotated through an angle ($\cos^{-1}\sqrt{1-e^2}$) from the y axis.



The projection of a point (x, y') on the circle is given by (x, y) where

$$y = y' \cos\left(\cos^{-1}\sqrt{1-e^2}\right)$$
$$= y'\sqrt{1-e^2}$$

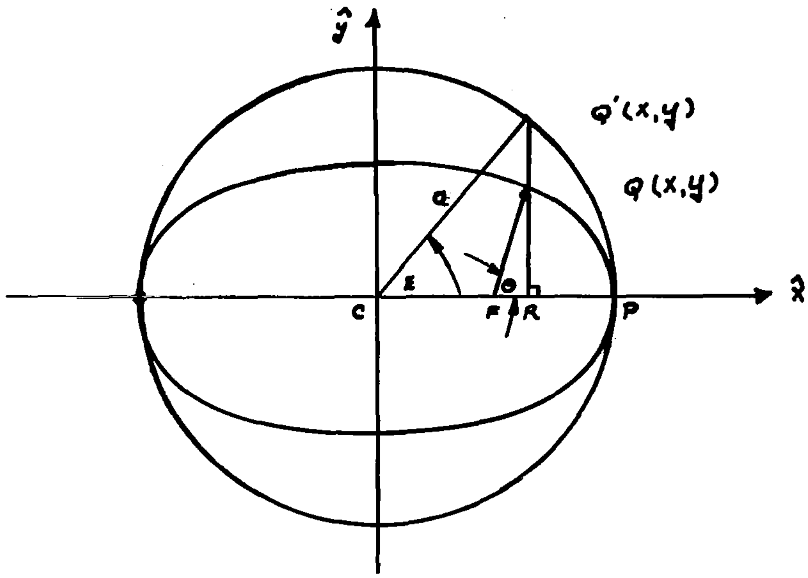
Hence, $y'^2 = y^2/(1-e^2)$. **Substitution of this expression** in the equation of the circle (B8) then gives the equation of the projection of the circle, namely the equation (B6), for an ellipse of eccentricity e. The area of an ellipse is, therefore, the area of the projection of its generating circle, i.e.,

$$A = \pi a^2 \cos\left(\cos^{-1}\sqrt{1-e^2}\right)$$
$$= \pi a^2 \sqrt{1-e^2}$$

or in terms of the semi-minor axis

$$A = \pi a b$$

The generation of an ellipse by an auxiliary circle also enables an angle called the eccentric anomaly to be defined. The eccentric anomaly is the central angle of the generating circle which intersects the circle at a point Q' whose projection Q onto the ellipse is determined by the true anomaly. A diagram showing the relationship between the eccentric and true anomalies, with both the generating circle (also called the auxiliary circle) and its ellipse drawn in the same plane, is given:



The coordinates of the point Q' on the circle are, in terms of the eccentric anomaly,

$$x = a \cos E \qquad\qquad y' = a \sin E$$

The coordinates of the projection of this point (i.e., Q) on the ellipse are

$$x = CF + FR \qquad\qquad y = QR$$
$$= ae + r\cos\theta \qquad\qquad = r\sin\theta$$

Equating the expressions for the x coordinate then gives

$$r\cos\theta = a(\cos E - e) \tag{B9}$$

and since $y = y'\sqrt{1-e^2}$ then

$$r\sin\theta = a\sqrt{1-e^2}\,\sin E. \tag{B10}$$

Squaring (B9) and (B10) and then adding gives

$$r^2 = a^2(\cos^2 E - 2e\cos E + e^2) + a^2(1-e^2)\sin^2 E$$
$$= a^2(1 - 2e\cos E + e^2\cos^2 E) \tag{B11}$$

Taking the square root of both sides of (B11) then gives the relation between the radial distance of the ellipse from the focus as a function of the eccentric anomaly of the auxiliary circle.

$$r = a(1 - e \cos E) \tag{B12}$$

The eccentric and true anomalies can be related to each other by rewriting (B9) in terms of the half angle $\frac{\theta}{2}$ giving

$$r\left(2 \cos^2 \frac{\theta}{2} - 1\right) = a \left(\cos E - e\right) \tag{B13}$$

$$r\left(1 - 2 \sin^2 \frac{\theta}{2}\right) = a \left(\cos E - e\right) \tag{B14}$$

then substituting (B12) in (B13) and (B14) gives

$$2 r \cos^2 \frac{\theta}{2} = a (1 - e)(1 + \cos E) \tag{B15}$$

$$2 r \sin^2 \frac{\theta}{2} = a (1 - e)(1 - \cos E) \tag{B16}$$

and finally dividing (B16) by (B15) and employing the half-angle substitutions for E gives the following relation.

$$\tan \frac{\theta}{2} = \left[\frac{1 + e}{1 - e}\right]^{\frac{1}{2}} \tan \frac{E}{2} \tag{B17}$$

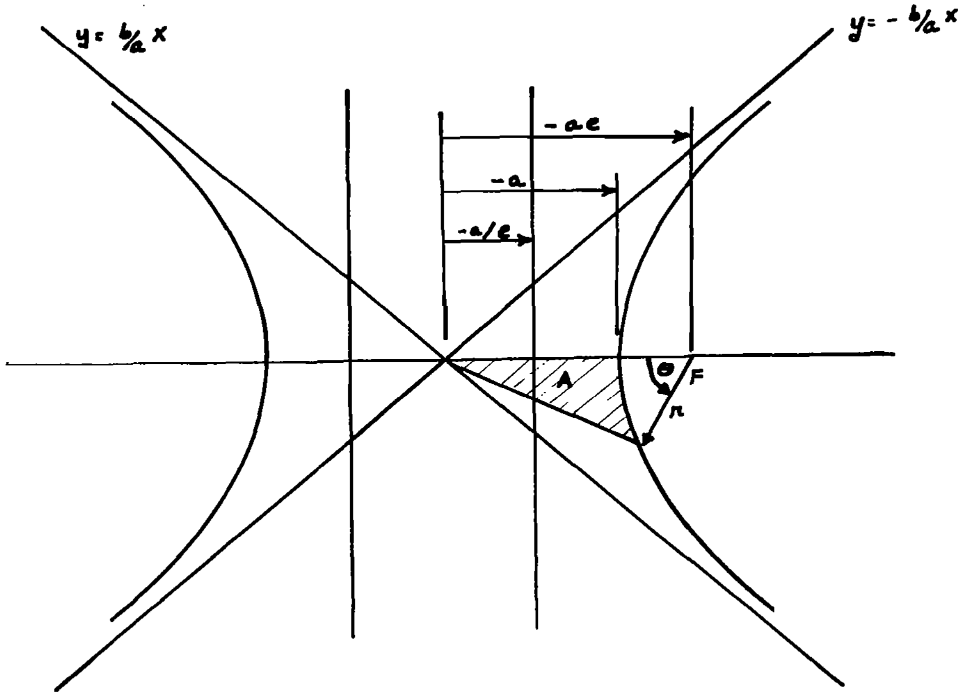
The Hyperbola

A conic section having eccentricity greater than one is a hyperbola. Since the radial distance of the hyperbola from its focus approaches infinity as the true anomaly approaches the value $\cos^{-1}(-1/e)$, the hyperbola is an open unbounded curve (see equation (24). This fact is also apparent from the cartesian form of the equation for a hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{B18}$$

Some clarification is, however, necessary in conjunction with the use of the parameter "a" in equation (B18). For the ellipse, it represented the semi-major axis length. For the hyperbola, its magnitude represents the distance of the periapse from the center or origin of coordinates. This conflict will be resolved by noting that the quantity "a" is positive for the ellipse and negative for the hyperbola (equations 27 and 33).

Since no real values of y exist which satisfy (B18) for the interval $|x| < -a$, the two branches of the hyperbola are symmetric to each other about the y axis. The only axes intercepts are those at x = ± a. Each branch has its own focus (located on the x axis at distances of ± ae from the center), and directrix (parallel to the y axes at distances ± a/e from the center).



For large values of x and y these branches lean toward the straight lines $y = \pm(b/a)\, x$ called asymptotes. Since this situation occurs as θ approaches $\cos^{-1}(-1/e)$, then

$$\tan^{-1}\left(\frac{b}{a}\right) = \cos^{-1}\left(-\frac{1}{e}\right)$$

Solving for b then gives

$$b = a \tan\left[\cos^{-1}\left(-\frac{1}{e}\right)\right]$$

or

$$b^2 = a^2 (e^2 - 1)$$

The hyperbolic anomaly defined in equation (58) , though imaginary, has a geometrical interpretation. This interpretation is obtained by realizing that F is

$$F = \frac{-A}{2\pi a b}$$

where A is the measured area shown on the previous sketch.

APPENDIX C
POTENTIAL OF AN AXIALLY SYMMETRIC MASS

Consider the integral

$$U = - \iiint_{m_e} \frac{G\, dm_e}{r}$$

defining the potential of a distributed mass M_e at an external point P. (See section 2.1). But,

$$r = \left[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2 \right]^{\frac{1}{2}}$$

where (x, y, z) are the coordinates of P, and (ξ , η , ζ) are the coordinates of dm_e.
Then

$$\frac{\partial U}{\partial x} = - \iiint_{m_e} G \, \frac{(x-\xi)}{r^3} \, dm_e$$

and

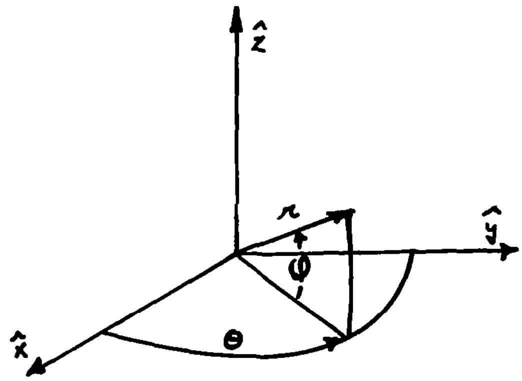
$$\frac{\partial^2 U}{\partial x^2} = - \iiint_{m_e} G \left[\frac{1}{r^3} - \frac{3(x-\xi)^2}{r^5} \right] dm_e$$

By writing the other two second partials and adding the result, it is shown that the potential function of a distributed mass satisfies Laplace's equation in the space external to the mass; i.e.,

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0$$

Or, in terms of spherical coordinates (r, θ, ϕ), Laplace's equation becomes

$$\frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial U}{\partial r} \right) + \frac{1}{r^2 \cos\phi} \frac{\partial}{\partial \phi}\left(\cos\phi \frac{\partial U}{\partial \phi} \right) + \frac{1}{r^2 \cos^2\phi} \frac{\partial^2 U}{\partial \theta^2} = 0 \tag{C1}$$



The method of separation of variables will be used to generate a solution of the forms

$$U = R(r)\,\Theta(\theta) + \Phi(\phi)$$

Substituting this form of the solution in Laplace's equation (C1) yields

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)\Theta\,\Phi + \frac{1}{r^2\cos\phi}\frac{d}{d\phi}\left(\cos\phi\frac{d\Phi}{d\phi}\right)R\,\Theta + \frac{1}{r^2\cos^2\phi}\frac{d^2\Theta}{d\theta^2}R\,\Phi = 0 \qquad \text{(C2)}$$

and separation of variables is obtained by dividing (C2) by $\frac{R\Theta\Phi}{r^2}$ to obtain

$$\frac{\frac{d}{dR}\left(r^2\frac{dR}{dr}\right)}{R} = -\frac{1}{\cos\phi}\,\frac{\frac{d}{d\phi}\left(\cos\phi\frac{d\Phi}{d\phi}\right)}{\Phi} - \frac{1}{\cos^2\phi}\,\frac{d^2\Theta/d\theta^2}{\Theta} \qquad \text{(C3)}$$

Since the left side of (C3) is a function of r alone and the right side a function of ϕ and θ , then for any general set of coordinates r , θ , ϕ these sides must be constant. Let this constant be

$$q = n(n+1)$$

then (C3) is rewritten as:

$$r^2 \frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - n(n+1)R = 0 \qquad (C4)$$

$$\frac{\frac{d^2\Theta}{d\theta^2}}{\Theta} = -q\cos^2\phi - \frac{\cos\phi}{\Phi}\frac{d}{d\Phi}\left(\cos\phi\frac{d\Phi}{d\phi}\right) \qquad (C5)$$

In a similar manner, since the left and right sides of (C5) are functions of ϕ and θ, respectively, then they must be constant. Let this constant be $-p^2$. Then (C5) becomes

$$\frac{d^2\Theta}{d\theta^2} + p^2\Theta = 0 \qquad (C6)$$

and

$$\cos\phi\frac{d}{d\phi}\left(\cos\phi\frac{d\Phi}{d\phi}\right) + (q\cos^2\phi - p^2)\Phi = 0 \qquad (C7)$$

Solution of (C4), (C6), and (C7) then completes the problem of determination of a potential function by separation of variables. Solutions will be sought that are symmetric about the axis $\phi=\pi/2$ since this case very closely approximates the potential of the earth about its axis of rotation.

The general solution of the equation in R (C4) is

$$R = Ar^n + Br^{-(n+1)}, \qquad n \neq -\tfrac{1}{2}$$
$$= Ar^{-\frac{1}{2}} + Br^{-\frac{1}{2}}\ln r, \qquad n = -\tfrac{1}{2}$$

Equation (C6) has a general solution of the form

$$\Theta = C\cos p\theta + D\sin p\theta, \qquad p \neq 0$$
$$= C + D\theta, \qquad p = 0$$

But for the axially symmetric body, the solution is independent of θ, p = 0, D = 0, and $\Theta = C$, a constant. Equation (C7) then becomes

$$\cos\phi \frac{d}{d\phi}\left(\cos\phi \frac{d\Phi}{d\phi}\right) + q \cos^2\phi \, \Phi = 0 \tag{C8}$$

Equation (C 7) can now be rewritten in the form known as Legendre's equation

$$(1-\mu^2) \frac{d^2\Phi}{d\mu^2} - 2\mu \frac{d\Phi}{d\mu} + q\Phi = 0 \tag{C9}$$

by letting $\mu = \sin\phi$. A solution to (C9) is obtained in the form of a power series in μ [convergent for $|\mu| < 1$], i.e.,

$$\Phi = \sum_{k=0}^{\infty} a_k \mu^k \tag{C10}$$

Substitution of (C10) into (C9), however, yields the following recursion relation between the coefficients

$$a_{k+2} = \frac{k(1+k) - q}{(k+1)(k+2)} a_k \quad , \quad k=0,1,\dots$$

Equation (C11) leaves two arbitrary constants open to choice and the general solution to (C9) can be written

$$\Phi = a_0 u_q(\mu) + a_1 v_q(\mu)$$

where $u_q(\mu)$ and $v_q(\mu)$ are series in even and odd powers of μ, respectively. (The value of these series depends on the value chosen for the constant q). Let q be chosen such that n is an integer where $q = n(n+1)$ then for k=n

$$q = k(k+1)$$

and the coefficient a_{k+2} from equation (C10) becomes zero. Consequently, the coefficients a_{k+4} , a_{k+6}, will also be zero; and depending on whether k is even or odd, the even or odd series, $u_q(\mu)$ or $v_q(\mu)$ terminates as a nth order polynomial.

For any other value of q chosen (where n is not an integer), the series does not terminate and can be shown to be divergent by means of Raabe's test. It therefore follows that the only solutions which remain convergent are those even or odd series terminating in a nth order exponent of μ, i.e., for n equal to an integer. These are the series:

$$a_0 u_q \ , \qquad n = 0, 2, 4, 6, \cdots$$

$$a_1 v_q \ , \qquad n = 1, 3, 5, 7, \cdots$$

These polynomials are known as Legendre polynomials when normalized such that the value of the functions is unity for $\mu = 1$. The first few of the Legendre polynomials in μ are:

$$P_0 = 1$$

$$P_1 = \mu$$

$$P_2 = \frac{1}{2}(3\mu^2 - 1)$$

$$P_3 = \frac{1}{2}(5\mu^3 - 3\mu)$$

$$P_4 = \frac{1}{8}(35\mu^4 - 30\mu^2 + 3)$$

$$P_5 = \frac{1}{8}(63\mu^5 - 70\mu^3 + 15\mu)$$

Combining the solution for R, Θ, and Φ then yields

$$U = c\left[A r^n + B r^{-(n+1)}\right] P_n(\mu)$$

Since only functions which vanish as r goes to infinity are of interest with regard to the potential outside of a mass, the solution containing $A r^n$ is of no interest here. Further, since the potential equation $\nabla^2 U = 0$ is linear, linear combinations of the form

$$U = -\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} P_n(\sin\phi)$$

are valid solutions. **This form is** employed in describing the potential of an axially symmetric earth model.

APPENDIX D
EQUATIONS RELATING THE CLASSICAL PARAMETERS OF CONIC MOTION

Equations derived in the text of this monograph and in Appendix B have been utilized to derive other quantities of direct interest in the analysis of elliptic and hyperbolic trajectories. This Appendix summarizes these efforts. The equations presented relate the five independent constants of integration derived in the text to the initial position and velocity vectors, and then proceed to relate the components of position and velocity at other points along the trajectory to the constants (elements) and the true or eccentric anomaly at the new position. If the equations for elliptic and hyperbolic motion are the same, they will not be repeated.

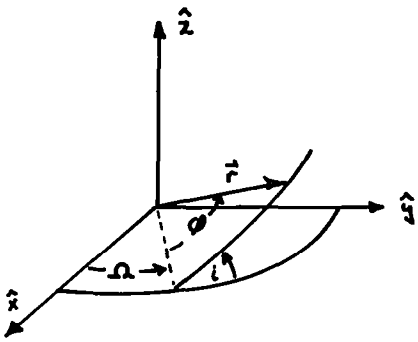
elliptic	hyperbolic
Angular momentum	
$h = \lvert \bar{r} \times \bar{V} \rvert$	
$= r^2 \dot{\theta}$	
$= r V \cos \gamma$	
$= \sqrt{\mu p}$	
Semi-major axis	
$a = r\mu / (2\mu - r V^2)$	
$= h^2 / [\mu (1 - e^2)]$	
$= .5 (r_a + r_p)$	not applicable
$= p / (1 - e^2)$	
$= r_a / (1 + e)$	not applicable
$= r_p / (1 - e)$	
$= r_a r_p / p$	not applicable
$= [T \sqrt{\mu} / (2\pi)]^{2/3}$	

elliptic	hyperbolic
Eccentricity $e = \sqrt{1 - h^2/(\mu a)}$ $= (r_a - r_p)/(r_a + r_p)$ $= 1 - p/r_a$ $= (p/r_p) - 1$ $= \sqrt{1 - p/a}$ $= (r_a/a) - 1$ $= 1 - r_p/a$	 not applicable not applicable not applicable
Semi-latus rectum $p = h^2/\mu$ $= (r\,v\cos\gamma)^2/\mu$ $= a(1-e^2)$ $= (r_a/a)(2a - r_a)$ $= (r_p/a)(2a - r_p)$ $= r_a r_p/a$ $= r_a(1-e)$ $= r_p(1+e)$ $= 2 r_a r_p/(r_a + r_p)$	 not applicable not applicable not applicable not applicable
Orientation elements $i = \cos^{-1}(\hat{h}\cdot\hat{z})$ $0 < i < \pi$ $\Omega = \tan^{-1}[(\hat{h}\cdot\hat{x})/(\hat{h}\cdot\hat{y})]$ $0 < \Omega < \pi/2$ $\hat{h}\cdot\hat{x} > 0$, $\hat{h}\cdot\hat{y} > 0$ $\pi/2 < \Omega < \pi$ $\hat{h}\cdot\hat{x} > 0$, $\hat{h}\cdot\hat{y} < 0$ $\pi < \Omega < 3\pi/2$ $\hat{h}\cdot\hat{x} < 0$, $\hat{h}\cdot\hat{y} < 0$ $3\pi/2 < \Omega < 2\pi$ $\hat{h}\cdot\hat{x} < 0$, $\hat{h}\cdot\hat{y} > 0$ $\omega = \cos^{-1}[\vec{l}\cdot\vec{e}/(l\,e)]$ $0 < \omega < \pi$ $e_z > 0$ $\pi < \omega < 2\pi$ $e_z < 0$ $\vec{l} = (\hat{h}\cdot\hat{y})\hat{x} - (\hat{h}\cdot\hat{x})\hat{y}$ $\vec{e} = (\dot{\vec{r}} \times \vec{h} - \mu\hat{r})/\mu$	

elliptic	hyperbolic
Radius $r = p / (1 + e\cos\theta)$ $= (h/\dot{\theta})^{1/2}$ $= a(1 - e\cos E)$ $= a\sqrt{1-e^2}\,\sin E / \sin\theta$ $= 2\mu a / (aV^2 + \mu)$ $= p\tan\gamma / (e\sin\theta)$	
Velocity $V = \sqrt{\dfrac{\mu(1+e\cos E)}{a(1-e\cos E)}}$ $= (\mu(2a-r)/ra)^{1/2}$ $= h/(r\cos\gamma)$ $= \sqrt{\dfrac{\mu(1+2e\cos\theta+e^2)}{r(1+e\cos\theta)}}$ $= \mu(1+2e\cos\theta+e^2)^{1/2}/h$	$V = \sqrt{\dfrac{\mu(1+e\cosh F)}{a(1-e\cosh F)}}$
Flight path angle $\gamma = \sin^{-1}[e\sin\theta/(1+2e\cos\theta+e^2)^{1/2}]$ $= \cos^{-1}[(1+e\cos\theta)/(1+2e\cos\theta+e^2)^{1/2}]$ $= \cos^{-1}[h/rV]$ $= \tan^{-1}[e\sin E/(1-e^2)^{1/2}]$ $= \tan^{-1}[(1-r/p)\tan\theta]$	$\gamma = \tan^{-1}[e\sinh F/(e^2-1)^{1/2}]$
Eccentric anomaly $E = 2\tan^{-1}\left[\sqrt{\dfrac{1-e}{1+e}}\tan\dfrac{\theta}{2}\right]$ $= \cos^{-1}[(aV^2-\mu)/e(aV^2+\mu)]$ $= \cos^{-1}[(e+\cos\theta)/(1+e\cos\theta)]$ $= \cos^{-1}[(a-r)/(ae)]$ $= \sin^{-1}[(r\sin\theta)/(a\sqrt{1-e^2})]$ $= \sin^{-1}[\sqrt{1-e^2}\sin\theta/(1+e\cos\theta)]$	Hyperbolic anomaly $F = 2\tanh^{-1}\left[\sqrt{\dfrac{e-1}{e+1}}\tan\dfrac{\theta}{2}\right]$ $= \cosh^{-1}[(aV^2-\mu)/e(aV^2+\mu)]$ $= \cosh^{-1}[(e+\cos\theta)/(1+e\cos\theta)]$ $= \cosh^{-1}[(a-r)/(ae)]$ $= \sinh^{-1}[(r\sin\theta)/(a\sqrt{e^2-1})]$ $= \sinh^{-1}[\sqrt{e^2-1}\sin\theta/(1+e\cos\theta)]$

elliptic	hyperbolic
True anomaly $\theta = \cos^{-1}[(\cos E - e)/(1 - e\cos E)]$ $= \cos^{-1}[(p - r)/(e r)]$ $= \sin^{-1}[\dot{r} h/(e\mu)]$ $= \sin^{-1}[a \sin E \sqrt{1-e^2}/r]$ $= \sin^{-1}[p \tan\gamma/(e r)]$ $= 2\tan^{-1}\left[\sqrt{\frac{1+e}{1-e}} \tan\frac{E}{2}\right]$	$\theta = \cos^{-1}[(\cosh F - e)/(1 - e\cosh F)]$ $= \sin^{-1}[a \sinh F \sqrt{e^2-1}/r]$ $= \tan^{-1}\left[\sqrt{\frac{e+1}{e-1}} \tanh\frac{F}{2}\right]$
Radial velocity $\dot{r} = v \sin\gamma$ $= (v^2 - (h/r)^2)^{1/2}$ $= h\tan\gamma/r$ $= h(1 - r/p)\tan\theta/r$ $= e\mu\sin\theta/h$ $= \sqrt{\mu a}\, e \sin E/r$	$\dot{r} = \sqrt{-\mu a}\, e \sinh F/r$
Angular velocity $\dot{\theta} = h/r^2$ $= (a v^2 - \mu)^2 h/(4\mu^2 a^2)$ $= \mu^2(1 + e\cos\theta)^2/h^{3/2}$ $= [\mu(1 + e\cos\theta)/r^3]^{1/2}$	

Position & Velocity Components in 3-D



$$\varphi = \theta + \omega$$
$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$
$$\vec{V} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}$$

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = T_z(-\Omega)\, T_x(-i)\, T_z(-\varphi) \begin{Bmatrix} r \\ 0 \\ 0 \end{Bmatrix}$$

$$\equiv T(\Omega, i, \varphi) \begin{Bmatrix} r \\ 0 \\ 0 \end{Bmatrix}$$

Where $T_x(\beta)$, etc., denotes a positive rotation about the X-axis through the angle β. This notation was adapted in Reference 2.

$$\begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{Bmatrix} = T(\Omega, i, \varphi)\, T_z(90-\gamma) \begin{Bmatrix} V \\ 0 \\ 0 \end{Bmatrix}$$

$$= T'(\Omega, i, \varphi, \gamma) \begin{Bmatrix} V \\ 0 \\ 0 \end{Bmatrix}$$

but for the second case, two successive rotations about the Z-axis are required. Thus, the arguments of the transformation can be added. Thus, letting $\Xi = \varphi + 90 - \gamma$, the transformation T' reduces to

$$T'(\Omega, i, \varphi, \gamma) = T(\Omega, i, \Xi)$$

The explicit form of the transformation T thus enables both position or velocity components to be written as

$$x \text{ (or } \dot{x}) = r \text{ (or } V) [\cos\Omega \cos\alpha - \sin\Omega \cos i \sin\alpha]$$
$$y \text{ (or } \dot{y}) = r \text{ (or } V) [\sin\Omega \cos\alpha + \cos\Omega \cos i \sin\alpha]$$
$$z \text{ (or } \dot{z}) = r \text{ (or } V) [\sin i \sin\alpha]$$
$$\alpha = \varphi \text{ (or } \Xi)$$

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