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GUIDANCE, FLIGHT MECHANICS AND TRAJECTORY OPTIMIZATION

Volume XI - Guidance Equations for Orbital Operations

by G. E. Townsend, D. R. Grier, and A. L. Blackford

Prepared by
NORTH AMERICAN AVIATION, INC.
Downey, Calif.
for George C. Marshall Space Flight Center

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FOREWORD

This report was prepared under contract NAS 8-11495 and is one of a series intended to illustrate analytical methods used in the fields of Guidance, Flight Mechanics, and Trajectory Optimization. Derivations, mechanizations and recommended procedures are given. Below is a complete list of the reports in the series.

Volume I	Coordinate Systems and Time Measure
Volume II	Observation Theory and Sensors
Volume III	The Two Body Problem
Volume IV	The Calculus of Variations and Modern Applications
Volume V	State Determination and/or Estimation
Volume VI	The N-Body Problem and Special Perturbation Techniques
Volume VII	The Pontryagin Maximum Principle
Volume VIII	Boost Guidance Equations
Volume IX	General Perturbations Theory
Volume X	Dynamic Programming
Volume XI	Guidance Equations for Orbital Operations
Volume XII	Relative Motion, Guidance Equations for Terminal Rendezvous
Volume XIII	Numerical Optimization Methods
Volume XIV	Entry Guidance Equations
Volume XV	Application of Optimization Techniques
Volume XVI	Mission Constraints and Trajectory Interfaces
Volume XVII	Guidance System Performance Analysis

The work was conducted under the direction of C. D. Baker, J. W. Winch, and D. P. Chandler, Aero-Astro Dynamics Laboratory, George C. Marshall Space Flight Center. The North American program was conducted under the direction of H. A. McCarty and G. E. Townsend.

TABLE OF CONTENTS

1.0 STATEMENT-OF-THE-PROBLEM

The basic objectives of this monograph are the development of the mathematical formulations of the maneuver in orbit problem and the presentation of solutions which can be utilized in the analysis of space missions and the associated guidance process. These objectives will be realized by isolating those factors which affect the motion of a vehicle and by defining the optimum sequence of events required to produce the desired motion.

The approach to this problem has been to divide the maneuvers which will be considered into two general classes for the purpose of discussion. The first class (referred to as gross maneuvers) contains all of those corrective strategies in which the individual corrections are of such a magnitude as to require that the non-linear equations describing the maneuver be solved explicitly to obtain the required accuracy. Such problems are common in the analysis of orbital transfer, injection into a specific escape trajectory from orbit, etc. This class of maneuvers will be analyzed to demonstrate the manner in which specific trajectories are generated and to provide the reference to which motion will be controlled in subsequent discussions.

The discussion of the gross maneuver begins with the formulation of the problem for the case of two-body motion (that is, for the case where the trajectories are conic sections). This portion of the analysis defines the parameters of the problem and suggests the most efficient combinations of these parameters for use in a given numerical solution. Attention then turns to the construction of the performance index (cost, impulse function) to be used in comparing the various transfers and to an automated means of presenting this index as a function of the parameters of the transfer orbit. This presentation affords the ability to view a broad spectrum of transfers simultaneously for the purpose of locating the neighbors of optimal 2-body transfers. This knowledge is essential in the location of truly optimal transfers since all numerical and analytic formulations of the optimization problem result in a solution which is optimal only with respect to those in its vicinity. This presentation also discusses the special case of the transfer problem obtained by applying a time of transfer constraint (that is the rendezvous problem).

The discussions of the gross maneuver conclude with a presentation of the generation of optimal transfers in the true force field. This section shows that the 2-body solution can be modified through the mechanism of differential corrections so as to satisfy the 2-point boundary problem in the true force field and that optimization of the resultant trajectory can be effected in an efficient manner either numerically or by a combination of analytic and numerical methods.

The second class of maneuvers contains all of the corrective strategies which are sufficiently small as to allow the non-linear equations describing the corrections to be expanded in a first order series in the neighborhood of a pre-selected nominal trajectory without adversely affecting the resultant accuracy in the analysis. This class of maneuvers (referred to as differential corrections or midcourse corrections) is commonly encountered in the evaluation control strategies required to null the effects of injection errors, target vehicle trajectory estimation errors in a rendezvous problem, etc. The presentation of the discussion of this class of maneuvers will progress from a formulation of a simplified guidance process to a discussion of optimal control as measured by several loss functions and will conclude with a series of observations pertaining to the application of this material.

The introductory sections, treating this second class of maneuvers, develop the concepts of fixed and variable time of arrival guidance for the case in which each correction is designed to null some set of components of the terminal error. The specific objective of this material is to develop an awareness of the variations in the formulation which can effect changes in the control requirements and suggest avenues worthy of more involved investigations. These guidance discussions are complimented with a discussion of errors and their effects on the control requirements and terminal accuracy.

The introductory developments are followed by a section which draws heavily on the literature available to present several approaches to the optimal controller problem. The first of the discussions in this section pertain to the "classic" approach (so named because of its nearly universal use). In this approach a quadratic loss function of the form

$$F = \sum_{i=1}^{N} \left[\delta_i^T(t)\, Q_i(t)\, \delta_i(t) + u_{i-1}^T(t-1)\, \gamma_i(t)\, u_{i-1}(t-1) \right]$$

where $\delta(t)$, $u(t)$ are vectors representing the state (e.g., first order position and velocity deviations) of the system and the applied control

$Q(t)$, $\gamma(t)$ are symmetric arrays of weights which express the emphasis between accuracy and control effort.

is utilized to measure the performance of the system in regard to both accuracy and control effort. [This particular form of the loss function has much to recommend its use since partial derivatives taken to define the optimal control are linear, since the general nature of the expression obeys intuitive reasoning regarding the nature of the "loss" ascribed to positive and negative errors of like magnitude and since the stochastic optimal controller (the optimal controller for the complete ensemble of midcourse problems) can be handled with the same ease as its deterministic counterpart provided only first and second statistical moments are employed.] For these reasons, this form of performance loss has been utilized by a large number of researchers in many different applications. Thus, the number of works available are legion, and care must be taken to select from the group a set of papers which are unique in the discussion of various aspects of the problem or more illuminating in the discussion. While the choice is obviously a matter of opinion, it is believed that the works of Kalman (et. al.), Gunckel, Lee, Wonham, Meier, Bellman, Joseph (et. al.), and Kushner constitute such a set. As such, these papers (References 2.1 through 2.15) will form the backbone of the presentation of the quadratic loss approach to optimal control. Other papers discussing various aspects of the problem may be found in the Bibliography.

A newer more involved approach to the stochastic optimal control problem was formulated by Breakwell and others (References 2.16 through 2.21). This formulation employs a different performance index for measuring the control effort. Namely

$$L(u_T) = \sum_{i=1}^{N} \left| E(u_i^T u_i) \right|$$

where E( ) denotes the expected value. This statement-of-cost more closely corresponds to the penalty associated with control effort since the magnitude of the control (for a rocket propulsion system) is a direct function of propellant consumed. The cost is not a precise statement since

$$\left| E(u_i^T u_i) \right| \neq E\left| u_i^T u_i \right|$$

but it does serve to bound the maximum control. This formulation, referred to by the originators as minimum effort control, results in a different weighting (relative to the quadratic loss criteria) of the dependencies between the corrections which are applied at the different times. This difference in turn is responsible for differences in the optimal control policy and the total cost (generally the cost is lower).

The theory of minimum effort control is based on the same linear model of the dynamics and observations processes as was quadratic loss control. Further, the analysis is restricted to the same assumptions regarding the statistics of the errors (both approaches consider only the first and second statistical moments). The nature of the solution, however, is so completely different that the motivation for the approach, a summary of the implicit assumptions and a review of the development will be presented. This review will present observations of the nature of the solution, its physical meaning and motivation for additional research to complete the development.

The final classification of midcourse corrections for the purposes of this monograph will contain all analyses in which higher order terms in the representation of the dynamical and observational processes are included and/or in which non-second order statistical moments are employed in the description of the error sources in the model, the estimation errors, the midcourse correction errors, etc., and/or in which measures of performance loss which cannot be derived from the first two by proper choice of weights are employed. This class of analyses at this time does not contain many members due to the complexities involved in representing the process and the specific nature of the results once particular distribution functions are selected for the errors involved. However, exploration of this family of problems from the stand point of the construction of a unified theory appears to have reached the point that publication of the results is practical. Accordingly, the monograph concludes with the development of a theory from the concepts of decision theory and from Bayes Theorem which is capable of embodying all of the published work on the problem. Further, this work appears capable of lifting assumptions and restrictions implicit in other work and of providing a great deal of insight into the structure of the stochastic optimal control problem.

In summary, it appears that the current state-of-the-art in midcourse guidance, while far from complete, has reached a degree of sophistication which allows analyses to be conducted in a rigorous and efficient manner. Future efforts would , thus, appear to be most useful if they are directed to the development and exploitation of the unified theory for midcourse guidance. This opinion is predicated on the fact that this theory displays the full impact of the loss function on the generation of an optimal control policy and provides a clear interpretation of the effects of constraints on either the control or the state at points along the trajectory.

2.0 STATE-OF-THE-ART

2.1 THE GROSS MANEUVER

2.1.1 Introduction

In this section, the determination of the velocity impulse necessary to implement two types of corrective action designed to make large changes in the orbital parameters is discussed. (The significance of "large" changes is that linear perturbation theory is inadequate to handle the problem.) The first type corresponds to the case where there is no restriction on the time required to make the change from one orbit to another; this problem is termed an "orbital transfer" problem. When a time constraint is added as, for example, when it is necessary that two spacecraft (initially on different orbits) meet, the problem is termed a "rendezvous" problem. This is the second type of action. The orbital transfer problem will be considered first, and the rendezvous problem discussed later as a specialization of the transfer problem with a time constraint.

The general problem of transfer from an arbitrary position on some initial orbit to an arbitrary position on some final orbit has no unique solution. Even in the case where the problem is restricted by specifying the positions on the initial and final orbits where the transfer is to be made, the transfer orbit is not completely determined. Thus, it is not surprising that the transfer from orbit to orbit by many of the possible transfer trajectories can require quantities of fuel so immense as to be beyond the limitations of vehicle proposed for the mission. It is, therefore, of extreme practical importance to locate, from among the class of all transfer trajectories, those which are optimum in the sense of optimum fuel requirements. Indeed, this will be the objective of the material of this section.

The analysis presented will, however, be restricted to impulsive velocity changes since the transfer duration generally exceeds the burning time by a large amount; one impulse will be given to make the transition from the initial orbit to the transfer orbit and a second impulse will affect the change from the transfer orbit to the final orbit. For the rendezvous problem, a three-impulse transfers as well as two impulse transfers will be considered.

Optimizing the fuel required for performing the orbital transfer is equivalent to minimizing the sum of the magnitudes of the velocity change from the initial to transfer orbit and from the transfer to the final orbit. Thus, a measure of performance, called the impulse function, can be defined as

$$I = \left|\bar{V}_I - \bar{V}_I^{\,T}\right| + \left|\bar{V}_{TF} - \bar{V}_F\right| \qquad (1.1)$$

where

$\overline{V}_I$ = velocity vector on the initial orbit at the point of injection to the transfer orbit

$\overline{V}_{TI}$ = velocity vector on the transfer orbit at the point of injection from the initial orbit

$\overline{V}_{TF}$ = velocity vector on the transfer orbit at the point of injection to the final orbit

$\overline{V}_F$ = velocity vector on the final orbit at the point of injection from the transfer orbit.

The problem of minimizing this impulse function can now be formulated in a straightforward manner. However, the solution to the equations, except for special cases, is analytically intractable. Therefore, appeal to numerical techniques (such as numerical solution of the Euler-Lagrange equation, gradient or steepest descent methods, dynamic programming etc.) must be made. Generally, these numerical techniques have a serious limitation in that they can locate only local minima. Thus, without a' priori knowledge concerning the number and location of the minima, the discovery of local minima will not necessarily mean that a satisfactory solution to the problem has been found. One means for obtaining this a' priori information is provided by graphically displaying all possible transfer solutions for the central force problem by contour maps in the "impulse function" space. (References 1.1, 1.2, and 1.3). The location of the optima from these contour maps can then be used at starting points for a numerical solution of a more precisely formulated optimization problem.

Before a detailed formulation of the impulse function and the development of the minimization problem are attempted, however, a digression into the transfer geometry and orbit parameters will be made.

2.1.2 Orbit Description & Transfer Geometry

Six independent quantities specify a two-body orbit in space; however, many sets of elements (functions of these quantities) are adequate to describe the orbit (Reference 1.4). The set which will be used here describes the orientation of the orbit in space by two angles i and Ω (the inclination and longitude of the node, respectively); the shape and size of the orbit by the eccentricity (e) and the semilatus rectum (p); and the orientation of the orbit by the argument of perigee (ω). The position of a point on the orbit is given by the true anomaly (ν), i.e., the angle between the radius and the perigee direction. Now, if the plane of the final orbit is used as a reference plane, its inclination (i) can be taken as zero and the inclination of the initial orbit (i_I) is then the angle between the planes of the initial and final orbits (see Figure 1.1)

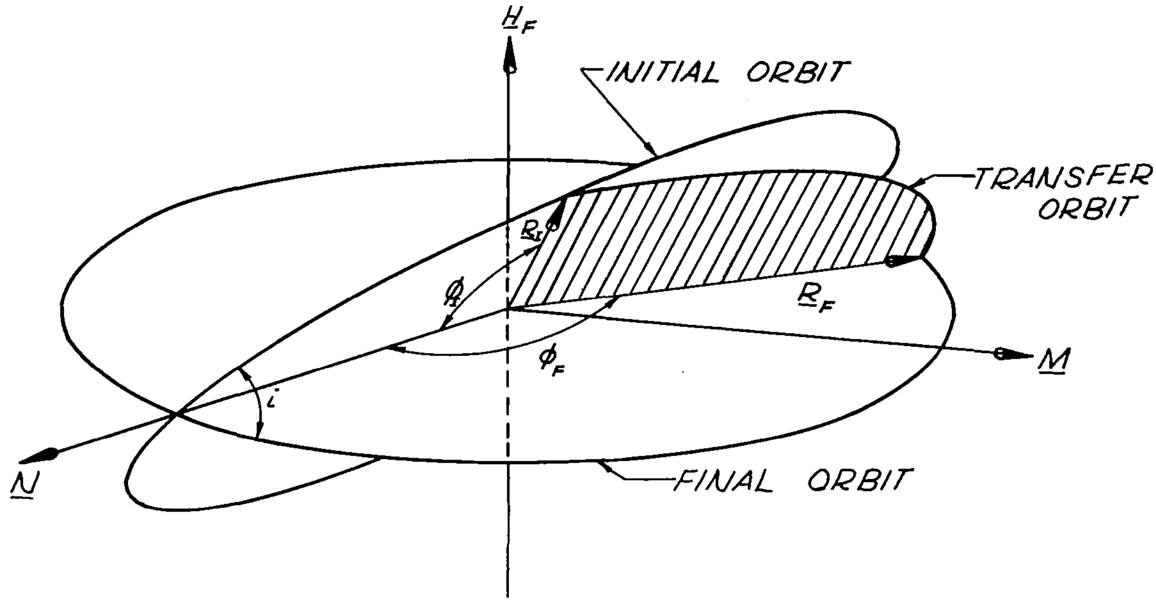


Figure 1.1

The line of intersection of the initial and final orbits can now be used as a reference line from which the location of the node (Ω) can be measured. With this reference, the nodes of the initial and final orbits are both zero. Thus, the elements necessary to define the orbits are

$e_{I,F}$ = eccentricity of the initial (final) orbit

$P_{I,F}$ = semilatus rectum of the initial (final) orbit

i_I = inclination of the initial orbit

$\omega_{I,F}$ = direction of perigee of the initial (final) orbit referenced to the line of intersection of the initial and final orbit planes

$\phi_{I,F}$ = angle of the radius vector to a point on the initial (final) orbit referenced to the line of intersection of the initial and final orbit planes.

The coordinate system in which the calculations will be made is also referenced to the final orbit plane. The unit vectors of the coordinate system are $\overline{N}$, $\overline{M}$, and $\overline{W}_F$ and are defined as follows:

$\overline{N}$ = unit vector along the line of intersection of the initial and final orbit

$\overline{H}_F$ = unit vector along the angular momentum vector of the final orbit

$\overline{M}$ = unit vector normal to $\overline{N}$, in the plane of the orbit such that the set ($\overline{N}$, $\overline{M}$, $\overline{W}_F$) from a right handed system

This (N, M, H_F) coordinate system will be called the reference coordinate system. In terms of the reference coordinate system, the radius vectors toward the departure point on the initial orbit and the arrival point on the final orbit are:

$$\bar{r}_I = \frac{p_I}{1 + e_I \cos(\phi_I - \omega_I)} \begin{pmatrix} \cos \phi_I \\ \sin \phi_I \cos i \\ \sin \phi_I \sin i \end{pmatrix} \tag{1.2}$$

$$\bar{r}_F = \frac{p_F}{1 + e_F \cos(\phi_F - \omega_F)} \begin{pmatrix} \sin \phi_F \\ \sin \phi_F \\ 0 \end{pmatrix} \tag{1.3}$$

$$\bar{r}_T = \frac{p_T}{1 + e_T \cos(\phi_T - \omega_T)} \begin{pmatrix} \cos(\Delta\phi - \phi_T) \cos \phi_F + \sin(\Delta\phi - \phi_T) \cos \alpha \sin \phi_F \\ \cos(\Delta\phi - \phi_T) \sin \phi_F + \sin(\Delta\phi - \phi_T) \cos \alpha \cos \phi_F \\ \sin(\Delta\phi - \phi_T) \sin \alpha \end{pmatrix}$$

where ϕ_T and ω_T are measured from the line of intersection of the initial orbit plane and transfer orbit plane; $\Delta\phi$ is the change in the true anomaly going from the insertion point to the departure point in the transfer orbit, and α is the angle between the nominal to the initial orbit plane and the transfer orbit plane. i.e.,

$$\Delta\phi = \arc \cos \left(\frac{\bar{r}_I \cdot \bar{r}_F}{|\bar{r}_I| \; |\bar{r}_F|} \right)$$

$$\alpha = \arc \cos \left(\frac{(\bar{r}_I \times \bar{r}_F) \cdot H_F}{|\bar{r}_I| \; |\bar{r}_F|} \right)$$

The velocity vectors necessary to compute the impulse function (Equation 1.1) can now be found by differentiating the expressions for the position vectors. These equations will be presented in subsequent discussions as they are required.

The initial and final orbits are assumed to be known, that is, that the eccentricity and semilatus rectum for each are known. However, these quantities are not known for the transfer orbit and must be computed. As the first step, an expression for the eccentricity can be written in terms of the magnitude of the radius vector at the beginning and end of the transfer. This is done by first writing an expression for the two radii from the equation for a conic in polar form

$$p_T = r_I(1 + e_T \cos \nu_T)$$

$$p_T = r_F(1 + e_T \cos(\nu_T + \Delta\phi))$$

where $\Delta\phi$ is the angle between $\bar{r}_I$ and $\bar{r}_F$ and where ν_T is the true anomaly in the transfer trajectory at the time the transfer is initiated. These equations can be subtracted and the expression for e_T determined as

$$e_T = \frac{r_F - r_I}{r_I \cos(\nu_T + \Delta\phi) - r_T \cos(\nu_T)} \tag{1.5}$$

The semilatus rectum can then be determined as

$$p_T = r_F(1 + e_F \cos \nu_T) \tag{1.6}$$

Note that both of the parameters of the transfer orbit are expressed as a function of the variable ν_T.

2.1.3 Impulse Function and Optimization Variables

The velocities necessary to compute the impulse function are now obtained by differentiating Equations 1.2, 1.3, and 1.4 as

$$\bar{V}_I = \frac{p_I^2 \sqrt{\frac{\mu}{p_I}}}{r_I^2 [1 + e_I \cos(\phi_I - \omega_I)]} \begin{pmatrix} -\sin\phi_I \\ \cos\phi_I \cos i \\ \cos\phi_I \sin i \end{pmatrix} - e_I \sin(\phi_I - \omega_I) \sqrt{\frac{\mu}{p_I}} \begin{pmatrix} \cos\phi_I \\ \sin\phi_I \cos i \\ \sin\phi_I \sin i \end{pmatrix}$$

$$\bar{V}_F = \frac{p_F^2 \sqrt{\frac{\mu}{p_T}}}{r_F^2[1 + e_F \cos(\phi_F - \omega_F)]} \begin{pmatrix} \sin \phi_F \\ \cos \phi_F \\ 0 \end{pmatrix} - e_F \sin(\phi_F - \omega_F) \sqrt{\frac{\mu}{p_F}} \begin{pmatrix} \cos \phi_F \\ \sin \phi_F \\ 0 \end{pmatrix}$$

Thus, since $\nu_T = \omega_T$ at the point of injection into the transfer orbit,

$$\bar{V}_{TI} = \frac{p_T^2 \sqrt{\frac{\mu}{p_T}}}{r_I^2[1 + e_T \cos(\omega_T)]} \begin{pmatrix} \sin(\Delta\phi) \cos \phi_F - \cos(\Delta\phi) \cos\alpha \sin \phi_F \\ \sin(\Delta\phi) \sin \phi_I + \cos(\Delta\phi) \cos\alpha \cos \phi_F \\ -\cos(\Delta\phi) \sin\alpha \end{pmatrix} + e_T \sin(\omega_T) \sqrt{\frac{\mu}{p_T}} \begin{pmatrix} \cos(\Delta\phi) \cos \phi_F + \sin(\Delta\phi) \cos\alpha \sin \phi_F \\ \cos(\Delta\phi \sin \phi_F + \sin(\Delta\phi) \cos\alpha \cos \phi_F \\ \sin \Delta\phi \sin\alpha \end{pmatrix}$$

$$V_{TF} = \frac{p_T^2 \sqrt{\frac{\mu}{p_T}}}{r_T^2[1 + e_T \cos(\Delta\phi - \omega_T)]} \begin{pmatrix} -\cos\alpha \sin \phi_F \\ \cos\alpha \cos \phi_F \\ -\sin\alpha \end{pmatrix} - e_T \sin(\Delta\phi - \omega_T) \sqrt{\frac{\mu}{p_T}} \begin{pmatrix} \cos \phi_F \\ \sin \phi_F \\ 0 \end{pmatrix}$$

These quantities can be substituted into Equation 1.1 to yield the impulse function which must be minimized.

$$I = |\bar{V}_I - \bar{V}_{TI}| + |\bar{V}_{TF} - \bar{V}_F| \tag{1.1}$$

The velocity components and, therefore, the impulse functions contain the variables p, e , ω , and ϕ. However, since the initial and final orbits are assumed known, the parameters (with the exception of ϕ) are fixed on those orbits. The variable ϕ locates the points at which the transfer orbit intersects the initial and final orbits. This angle (ϕ) is, thus, a natural choice as an optimization variable. With ϕ_I and ϕ_T chosen as optimization variables there are no unknown quantities remaining to be determined for the initial or final orbits. Attention, therefore, must turn to the transfer orbit. Consider Equations (1.5) and (1.6). These equations are independent in the three unknown of the transfer orbit; that is, one of the three parameters P_T, e_T, ω_T may be chosen for the purposes of optimization (the selection of any one immediately determines the value of the other two). However, numerical experimentation has shown that optimization with respect to the variables e_T, and ω_T is less desirable than that performed with respect to p_T. Reference (1.1) uses a disguised form of ω_T for the variable while References (1.3), (1.5), and (1.6) use p_T as the optimization variable.

Rendezvous Problem

If t_I, t_F, and t_T are the times required to traverse, the true anomaly intervals ϕ_I, ϕ_F, and $\Delta\phi$, then the condition for rendezvous can be written as

$$t_I + t_F = t_T + \tau \tag{1.7}$$

where τ is time required for an object in the final orbit to reach the reference axis (the line of intersection of the initial and final orbits) from the position it occupies when the object in initial orbit crosses the reference axis. As long as the periods of the initial and final orbits are not identical, the value of τ will change, by an increment equal to the difference in the periods of the initial and final orbits, with each revolution of the vehicle in the initial orbit.

The time (t) required to traverse a true anomaly interval ($\Delta\phi$) is given by Kepler's equation

$$nt = E_2 - E_1 - e(\sin E_2 - \sin E_1) \tag{1.8}$$

where E_1 and E_2 are the eccentric anomalies of the initial and final points in the interval and n is the mean motion

$$n = \sqrt{\frac{\mu}{a^3}}$$

The eccentric anomaly is defined by the equation (see Reference 1.4)

$$r = a(1 - e \cos E) = \frac{p}{1 + e \cos v}$$

For the rendezvous problem, Equations (1.5), (1.6), and (1.8) specify the transfer orbit in terms of the variables r_F, r_I and t. An alternate set of equations in these variables is available in the form of Lambert's theorem (Lambert's theorem is probably the more widely used form in investigations of rendezvous problems). Lambert's theorem for elliptic motion involves the simultaneous solution of the following equations (manipulations of the equations presented on the previous pages)

$$r_I + r_F + C = 4a \sin^2 \frac{\alpha}{2}$$

$$r_I + r_F - C = 4a \sin^2 \frac{\beta}{2}$$

$$t = \sqrt{\frac{a^3}{\mu}} \left[(\alpha - \sin \alpha) - (\beta - \sin \beta) \right]$$

where

$$C = |\bar{r}_F - \bar{r}_I|$$

A complete derivation and discussion of Lambert's theorem can be found in Reference 1.4 (starting on Page 48).

The utility of this approach is derived from the fact that this formulation uncouples the solutions for "a" and "e" (i.e., in this technique, the problem can be reduced by direct substitution to the solution for one variable at a time). This feature is extremely important and makes the simultaneous solution of Kepler's equation and an equation involving the radii (r_I, r_F) and the angle between them an unnecessary burden.

The solution of the rendezvous problem can now be accomplished by either of the two methods. Equations (1.7) can be solved for τ as

$$\tau = t_i + t_F - t_T$$

and since the transit times, t_i, are functions of the ϕ_i then τ is also a function of ϕ, i.e.

$$\tau = \tau(\phi_1, \phi_2, \Delta\phi)$$

Thus, finding the optimum two-impulse rendezvous trajectory is a matter of minimizing the impulse function, I, under the constraint τ = constant. As an alternative, Lambert's theorem may be applied directly. In either of these two approaches, however, the transfers can be characterized by the variable τ .

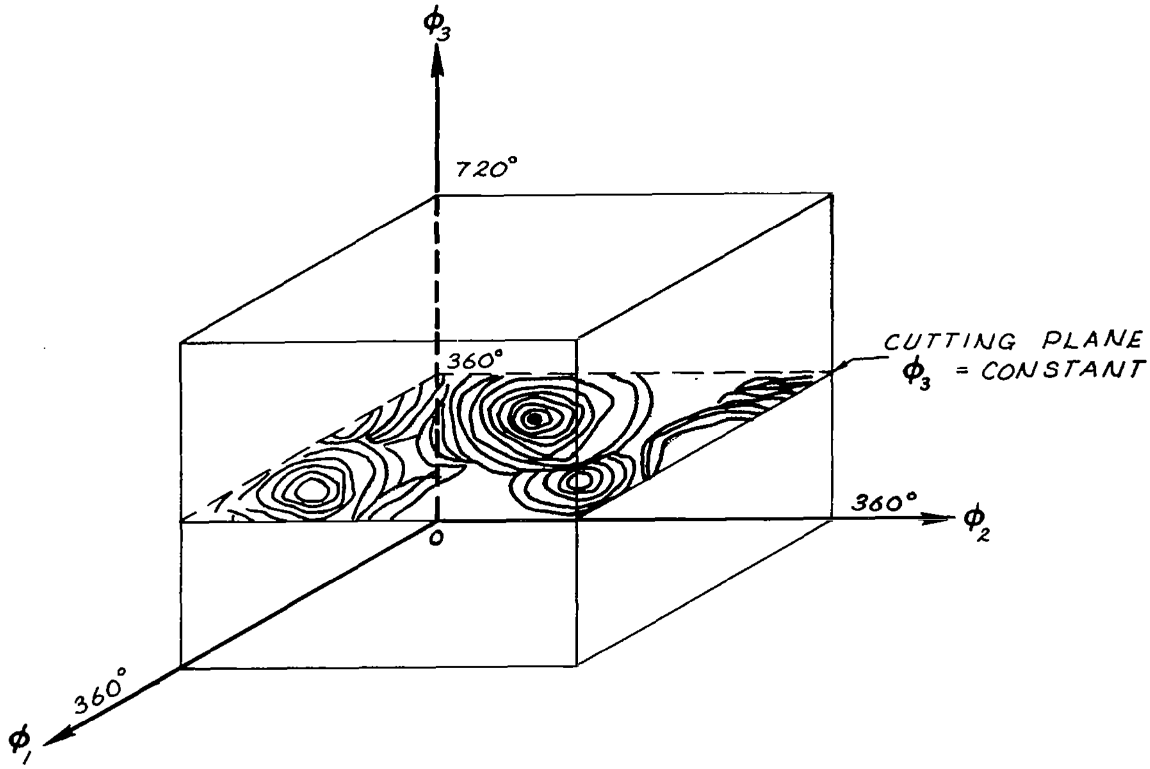
Since the value of τ changes with each revolution in the initial orbit, there will be some value of τ for which a minimum pulse would occur. It is reported in Reference 1.1 that the impulse penalty increases rapidly with (τ - τ_{opt}); thus, only a small portion of the τ range can be covered by a two-impulse rendezvous without prohibitive impulse penalties.

If only two impulse maneuvers are allowed, the change in τ with each revolution of the initial orbit is the difference in the period, (T_F-T_I). If a third impulse is considered, waiting in some intermediate orbit with a period T' is permitted. By this method, τ may be varied between 0 and (T_F-T_I). Reference (1.1) shows that if it is possible to hold the initial orbit for a bounded number of revolutions, the optimum three-impulse rendezvous requires no more impulse than the optimum two-impulse orbital transfer (Further discussion of the three-impulse transfer can be found in References 1.1 and 1.7).

This feature is afforded by applying the 1st and 2nd impulses at the same position vector but displaced in time by some multiple of the orbital period (extension to n-impulses is immediate). Generally, of course, the 3-impulse scheme is less optimum. The exceptions to the general case are discussed in References 1.11, 1.12, and 1.13.

2.1.4 Numerical Solution to the Optimization Problem

The mathematical expressions developed for the minimization problem do not admit an analytic solution except in special simplified cases. Further, conventional numerical search techniques by gradient methods find only the nearest local minimum and provide no indication of the location or relative size of other minima. However, one method used to obtain information on the number and relative size of the minima is to evaluate the impulse function for conic motion for a large number of the possible values of the optimization variables. These data can then be plotted in the function space for contours of the same impulse. Since the function space is three-dimensional a visualization of the space must be made by a series of two-dimensional plots which represent the trace of the surface on planes. An illustration of such a plot is found in Figure 1.2.



This figure is reproduced from Reference 1.1; ω_T was used as the third optimization variable. With this choice of the third variable, the entire function space is contained in a cube whose sides have length 360°.

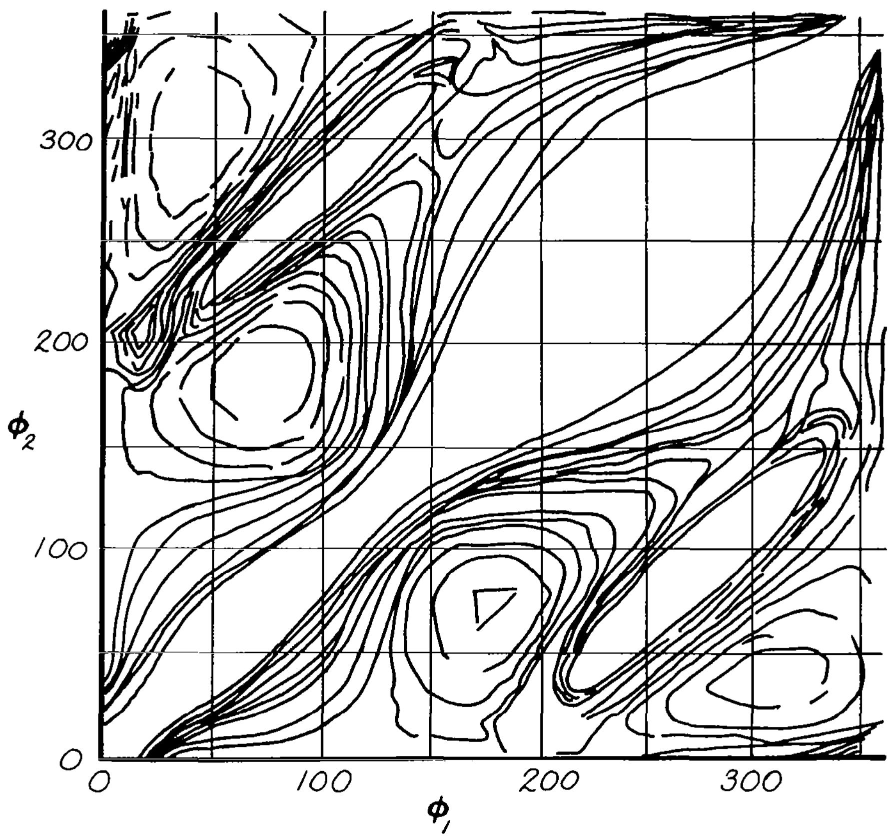
As the complexity of the impulse surfaces increases, more projections on cutting planes are required to obtain a good feeling for the location and importance of the minima. To overcome this difficulty, two of the three variables are chosen at random and the third is optimized. The entire range of the two variables may be covered in this manner and contours of the third variable, representing the optimum values of that variable, plotted (for example in References 1.2, and 1.3 a technique called "p-optimization" is employed.) In this case, the third variable is p, i.e.,

$$I = I(\phi_1, \phi_2, p)$$

If fixed values are assumed for ϕ_1 and ϕ_2 the optimum value of p can be found from the equation

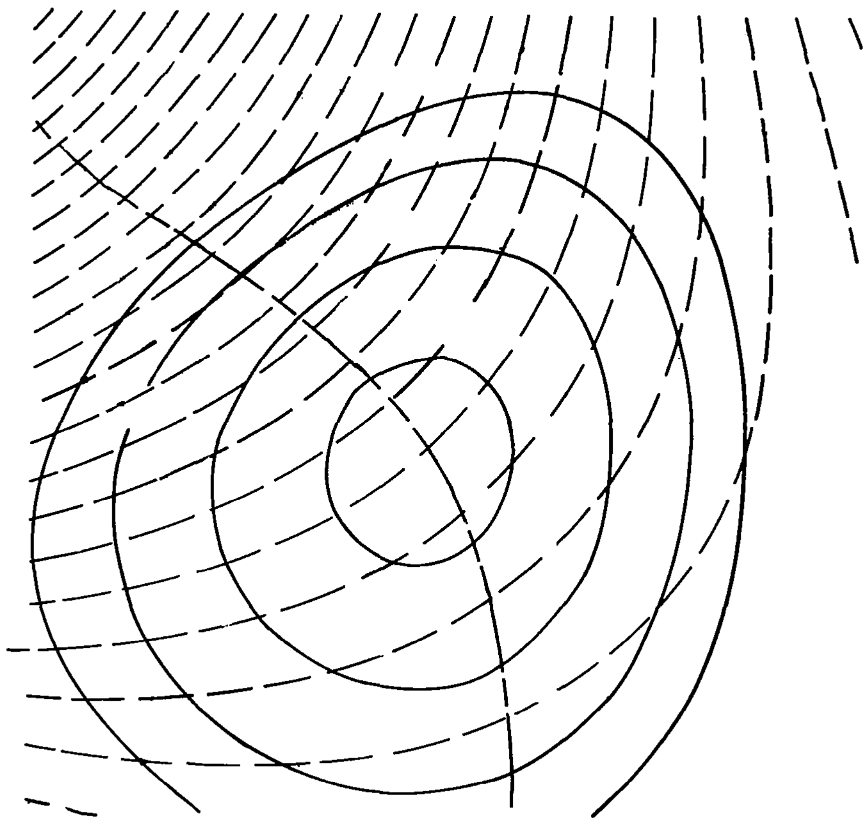
$$\frac{dI}{dp} = 0 \qquad (19)$$

The values of ϕ_1 and ϕ_2 can be varied over their applicable range and a corresponding optimum p determined for each set. A plot can now be made in the ϕ_1 and ϕ_2 plane by connecting points with corresponding values of p. Such a plot has been reproduced below from Reference 1.2.



Lee (Reference 1.8) shows that the equation for the optimization of p (Equation 1.9) is equivalent to an eighth order polynomial whose real roots must include all the values of p for which the impulse is an extremum. By further analysis of this octic, identification of hyperbolic transfers and double minimum was possible.

The problem of achieving actual rendezvous rather than orbital transfer can also be displayed graphically. Since τ is a function of the same three variables as the impulse function, surfaces of constant τ as well as constant impulse can be described in the function space. The visual solution consists of superimposing the τ and impulse contours on the same plot as is done in Figure 1.4.



The contours of constant τ are very regular compared to the complicated shapes of most impulse contours. This fact considerably simplifies the problem of conducting numerical searches since only the impulse surfaces discontinuities cause major difficulties.

Because of the large number of variables involved in the transfer problem, it is difficult to make general statements concerning the number, location, or relative importance of the minima. Reference 1.1 presents data for various types of orbits (i.e., coplanar, inclined asymmetric orbits, etc.) and the effects of perturbations of the parameters on the minima. The data in this reference may be useful to the reader with a specific problem in mind, however, it is felt that specific problems will have to be analyzed on an individual basis using the techniques described here and in the references.

2.1.5 Steepest Descent Solutions

The visual technique described in the last section makes possible the identification of regions of the impulse function which represent optimum transfer. However, the approach does not generally provide (nor was it intended to) the numerical accuracy necessary for the design of space missions. This observation is the result of the fact that some impulse functions have long narrow valleys containing several minima. For these situations, the visual information obtained from the impulse contours is used as a starting point for a numerical search technique such as the method of steepest described below.

The basic idea behind the method of steepest descent is that the desired root (solution) can be found by starting at any point in the neighborhood of that root and steping in the direction of the greatest change of the function. From vector analysis, it is known that the gradient vector is oriented in the direction of the greatest change of the function; thus, the motion should be along the local gradient vector. That is, if the impulse is a function of the three angles ϕ_1, ϕ_2, and ϕ_3, and the starting point is ϕ_1, ϕ_2, ϕ_3, the changes in the ϕ's which should be made to reach the minimum are

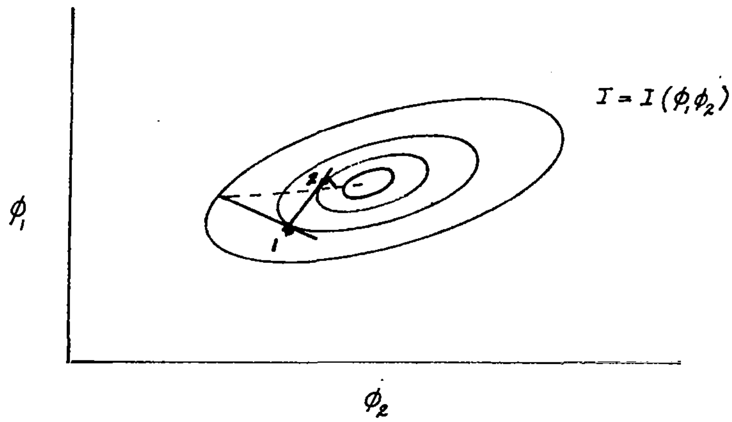
$$\begin{pmatrix} \Delta\phi_1 \\ \Delta\phi_2 \\ \Delta\phi_3 \end{pmatrix} = -k \begin{pmatrix} \dfrac{\partial I}{\partial \phi_1} \\ \dfrac{\partial I}{\partial \phi_2} \\ \dfrac{\partial I}{\partial \phi_3} \end{pmatrix}$$

(In the case where the minimum value of I is known, an estimate of K can be computed explicitly and the method becomes a Newton-Raphson iteration. However, since the minimum value of I is not known in this problem, a numerical search is required.)

The new value of ϕ_1, ϕ_2, ϕ_3 which hopefully, is closer to the minimum is computed as

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} \phi_1' \\ \phi_2' \\ \phi_3' \end{pmatrix} + \begin{pmatrix} \Delta\phi_1 \\ \Delta\phi_2 \\ \Delta\phi_3 \end{pmatrix}$$

The process is now repeated by computing partial derivatives at the new point in the ϕ space. Since the number of computations of the partial derivatives could be excessive in terms of computer time, an alternate to computing new values of the derivatives at each step is to continue in the direction of the original gradient until I reaches a minimum. A new gradient direction is then computed and the process repeated. In effect, this "stepwise method" reduces the three-dimensional problem to a series of one-dimensional problems. A two-dimensional illustration of these two techniques are illustrated in Figure 1.5.



The gradient direction, as shown, is normal to a contour while the local minimum in the gradient direction is attained at the point of tangency to a contour. Discussion of a modified steepest descent solution to the optimum transfer problem is presented in Reference 1.5, a general discussion of steepest descent methods as well as other numerical search methods can be found in Reference 1.9.

2.1.6 Real Force Field Effects

For mission analysis, the formulation of the orbit transfer problem in terms of patched conics should provide adequate accuracy. However, if an actual flight is to be analyzed, it will be necessary to include the effects of the actual force field. Such an analysis would be extremely uneconomical using the real force field because of the large number of trajectories which must necessarily be generated to define the impulse function necessary to locate the optima since these trajectories must be numerically integrated from the initial position to the final position (a closed form solution does not exist, Reference 1.10). However, the process can be drastically simplified if the optimal 2-body trajectories are assumed to lie in the neighborhood of the optimal trajectories for the true force field. This assumption is valid for essentially all trajectories for which the perturbations induced by the noncentral nature of the force field are of the order of a few percent of the total position and velocity vectors.

When the initial conditions defining the optimal 2-body rendezvous trajectory are now integrated in the true force field, the end point (the point at which rendezvous is desired) will not generally coincide with a position on the target orbit containing the target. In fact, the transfer trajectory may not intersect the target orbit at all. Thus, a differential corrections process will be required to drive this first estimate of the transfer trajectory until the end conditions are matched. This differential correction process is exactly equivalent to the two-impulse midcourse guidance formulation presented in Section 2.2.1 when V_{g_1} is interpreted as the correction required in the initial velocity vector to shape the trajectory (V_{g_2} is of no concern). This shaping process continues until some acceptable accuracy has been obtained. The arrays of partial derivatives required in this solution may be generated by finite differences, by integration of the differential equations for the state transition matrix (Reference 2.22) or approximated by those derived for conic motion (the effects of the perturbations on these derivatives will be negligible for most purposes).

The trajectory resulting from this shaping process will not generally be optimum. Thus, a numerical optimization technique based on the Euler-Lagrange equations, dynamic programming or steepest descent must now be attempted. The initial conditions for this process will, of course, be the shaped 2-body optimum.

2.1.7 Application of Optimum Transfer Methods

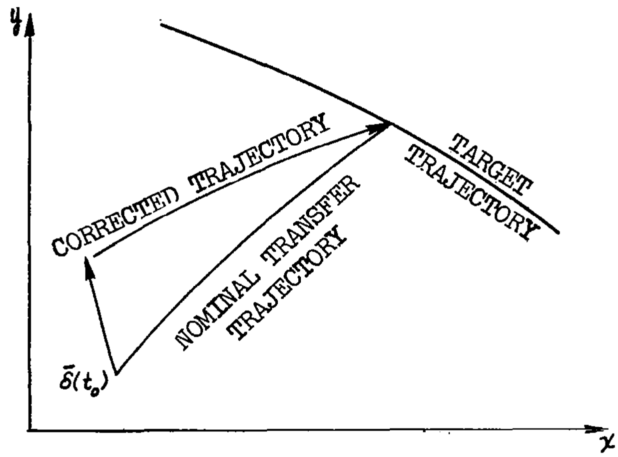
The techniques presented in this section would not be used in an on-board midcourse guidance scheme because of the computational requirements and the necessity of the visual analysis of the data to select an appropriate optimum. Instead, these techniques would fall into the analysis phase of the mission and would be performed before an actual flight. The information obtained by these methods would be reflected in the choice of a reference trajectory or in the selection of abort or alternate mission trajectories. The linear midcourse guidance formulations discussed in subsequent sections of the monograph can be optimized in this manner in both a large and small sense.

2.2 MIDCOURSE GUIDANCE

2.2.1 Two Impulse Fixed and Variable Time of Arrival Guidance

2.2.1.1 Fixed Time of Arrival

In the way of an introduction to the midcourse guidance problem, a simple corrective strategy will be presented. This strategy will be developed for the case in which some prescribed mission objective requires that particular position and velocity vectors be attained at a prescribed epoch. Schematically,



In this analysis the state (state deviation) of the system (represented by the vector $\bar{\delta}(t_o)$ is assumed to be sufficiently small that its time history can be expressed through the mechanism of the state transition matrix (see Reference 2.22 for a complete discussion). Under this assumption and the assumption that all of the parameters used to define the reference trajectory are correct (i.e., no other perturbations than those in position and velocity will be considered). Then

$$\delta(t) \equiv \begin{Bmatrix} \delta\bar{r} \\ \delta\bar{V} \end{Bmatrix} = \begin{bmatrix} \frac{\partial\bar{r}}{\partial\bar{r}_o} & \frac{\partial\bar{r}}{\partial\bar{V}_o} \\ \frac{\partial\bar{V}}{\partial\bar{r}_o} & \frac{\partial\bar{V}}{\partial\bar{V}_o} \end{bmatrix} \begin{Bmatrix} \delta\bar{r}_o \\ \delta\bar{V}_o \end{Bmatrix}$$

$$\equiv \phi(t,t_o)\ \bar{\delta}(t_o) \qquad (2.1)$$

Now, by evaluating $\phi(t,t_o)$ at the terminal time (t_f) and by requiring that

$$\delta\bar{r}\ (t_f) = 0$$

it is possible to evaluate the velocity-to-be-gained ($\bar{V}_g$) which must be applied to the velocity vector at the time a correction is commanded to affect reaiming the vehicle. That is, before the correction

$$\delta \bar{r}(t_f) = \left[\frac{\partial \bar{r}_f}{\partial \bar{r}_o}\right] \delta \bar{r}_o + \left[\frac{\partial \bar{r}_f}{\partial \bar{V}_o}\right] \delta \bar{V}_o$$

while after the correction

$$\bar{0} = \left[\frac{\partial \bar{r}_f}{\partial \bar{r}_o}\right] \delta r_o + \left[\frac{\partial \bar{r}_f}{\partial \bar{V}_o}\right] \left\{\delta \bar{V}_o + \bar{V}_{g_1}\right\}$$

Thus, by subtraction

$$\delta \bar{r}(t_f) \equiv \quad \Xi \; \phi(t_f, t_o)\, \bar{\delta}(t_o) = -\left[\frac{\partial \bar{r}_f}{\partial \bar{V}_o}\right] \bar{V}_{g_1}$$

$$\bar{V}_{g_1} = -\left[\frac{\partial \bar{r}_f}{\partial \bar{V}_o}\right]^{-1} \Xi \; \phi(t_f, t_o)\, \bar{\delta}(t_o) \tag{2.2}$$

where

$$\Xi = \begin{bmatrix} I & I \\ 0 & 0 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This correction produces closure with the desired point in space but it induces an additional perturbation in the velocity at that point. Thus, if a specific velocity is to be attained it is necessary to apply a second correction. This correction can be computed by nulling $\delta \bar{V}_f$

$$\begin{aligned} \bar{V}_{g2} &= -\delta \bar{V}_f \\ &= -\left[\frac{\partial \bar{r}_f}{\partial \bar{r}_o}\right] \delta \bar{r}_o - \left[\frac{\partial \bar{V}_f}{\partial \bar{V}_o}\right] \left\{\delta \bar{V}_o + \bar{V}_{g_1}\right\} \\ &= \Theta \, \phi(t_f, t_o)\, \bar{\delta}(t_o) - \left[\frac{\partial \bar{V}_f}{\partial \bar{V}_o}\right] V_{g_1} \end{aligned} \tag{2.3}$$

where

$$\Theta = \begin{bmatrix} 0 & 0 \\ I & I \end{bmatrix}$$

This guidance concept is extremely simple and may be optimized for the deterministic case by finding the time, represented by t_o, which results in some measure of performance such as

$$F_1 = |\bar{V}_{g1}| + |\bar{V}_{g2}|$$

or

$$F_2 = \bar{V}_{g1}^T \bar{V}_{g1} + \bar{V}_{g2}^T \bar{V}_{g2}$$

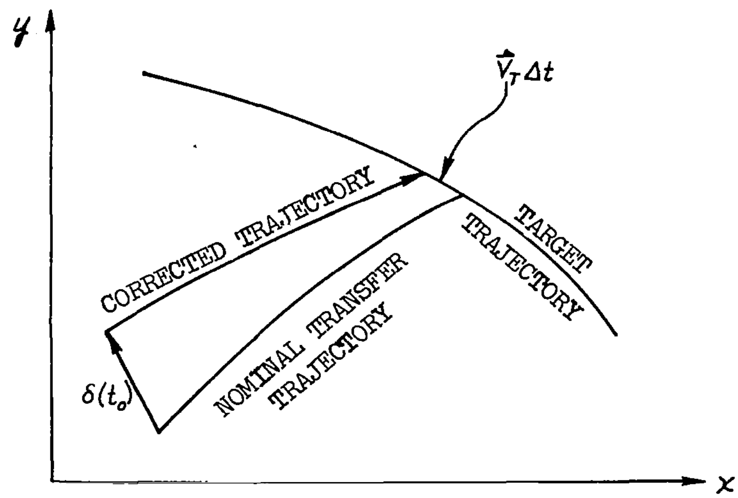
attaining a minimum value. (This aspect of the problem will be deferred until the next discussion to avoid duplication.) However, two assumptions implicit in the development of this strategy:

1. The time of arrival is rigidly constrained
2. The correction designated as V_{g1} is selected so as to null the complete error $\delta \bar{r}(t_f)$ in the absence of other errors (such as application errors, state uncertainties at t_o, and estimation errors for the elements of the target trajectory).

Thus, in general, there is a more optimum means of correcting the errors. The procedure for introducing these objectives will be discussed in subsequent sections.

2.2.1.2 Variable Time of Arrival

The most obvious source of inefficiency in the previous analysis arises from the fact that the time at which the corrected and target trajectories intersected was held absolutely constant. This requirement may be relaxed in most problems of common interest, however, provided that the target itself is still intercepted. This requirement is equivalent to stating that any point on the target trajectory is acceptable as an intercept point, provided the target arrives at the new point at the same time as does the interceptor. Schematically,



The requirement for intercept (assuming a linear theory of motion is adequate) is now

$$\Delta \bar{r}\ (t_f + \Delta t) = \vec{V}_T\ (t_f)\ \Delta t \tag{2.4}$$

This process introduces an additional degree of freedom (Δt) and allows the control to be optimized with respect to this parameter as well as with respect to t_o.

Consider the equations

$$\begin{Bmatrix} \Delta \bar{r}_f \\ \Delta \bar{V}_f \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial \bar{r}}{\partial \bar{r}_o} & \dfrac{\partial \bar{r}}{\partial \bar{V}_o} \\ \dfrac{\partial \bar{V}}{\partial \bar{r}_o} & \dfrac{\partial \bar{V}}{\partial \bar{V}_o} \end{bmatrix} \begin{Bmatrix} \Delta \bar{r}_o \\ \Delta \bar{V}_o \end{Bmatrix} \equiv \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{Bmatrix} \Delta \bar{r}_o \\ \Delta \bar{V}_o \end{Bmatrix}$$

and the situation in which $\Delta \bar{r}_f$ has been measured (or computed) and found (estimated) to be non-zero. The problem as before is to apply a correction (Vg_1) which will produce a zero position error at some time $t_f + \Delta t$, i.e.

$$\begin{Bmatrix} \Delta\bar{r}_f + \bar{V}_T\,\Delta t \\ \Delta\bar{V}_f + \bar{a}_T\,\Delta t \end{Bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{Bmatrix} \Delta\bar{r}_0 \\ \Delta\bar{V}_0 + \bar{V}_{g_1} \end{Bmatrix} + \begin{Bmatrix} \dfrac{\partial\bar{r}_f}{\partial t} \\ \dfrac{\partial\bar{V}_f}{\partial t} \end{Bmatrix}_{t_f} \Delta t$$

or

$$\begin{Bmatrix} \Delta\bar{r}_f + \bar{A}\Delta t \\ \Delta\bar{V}_f + \bar{B}\Delta t \end{Bmatrix} = \phi(t_f, t_0) \begin{Bmatrix} \Delta\bar{r}_0 \\ \Delta\bar{V}_0 + \bar{V}_{g_1} \end{Bmatrix}$$

where

$\bar{V}_T$ = Target's velocity at $t = t_f$

$\bar{a}_T$ = Target's acceleration at $t = t_f$

$\bar{A} = \bar{V}_T - \left.\dfrac{\partial\bar{r}_f}{\partial t}\right|_{t=t_f}$

$\bar{B} = \bar{a}_T - \left.\dfrac{\partial\bar{V}_f}{\partial t}\right|_{t=t_f}$

Now, if $\bar{V}g_1$ is to be applied so as to null

$$\Delta\bar{r}_2 + \bar{A}\Delta t$$

The same functional form of the solution obtained for the fixed time of arrival case will result; i.e.,

$$\Delta\bar{r}_f + \bar{A}\Delta t = -\left[\frac{\partial r_f}{\partial V_0}\right]\bar{V}_{g_1} \equiv -\phi_{12}\,V_{g_1}$$

But as before

$$\Delta\bar{r}_f = \phi_{11}\,\Delta\bar{r}_0 + \phi_{12}\Delta\bar{V}_0$$

Thus,

$$\begin{aligned} \bar{V}_{g_1} &= -\{[\phi_{12}^{-1}\,\phi_{11}\,I]\,\bar{\delta}(t_0) + \phi_{12}^{-1}\,\bar{A}\,\Delta t\} \\ &\equiv -\{K_1\,\bar{\delta}(t_0) + \bar{K}_2\,\Delta t\} \end{aligned} \tag{2.5}$$

where for convenience K_1 and K_2 have been defined as

$$K_1 = [\phi_{12}^{-1}\phi_{11} \quad I]$$

$$K_2 = \phi_{12}^{-1}\bar{A}$$

Upon arrival, a second impulsive correction will be required as in the fixed time of arrival concept. This correction will be

$$\begin{aligned}\bar{V}_{g2} &= -[\Delta\bar{V}_f + \bar{B}\Delta t] \\ &= -[(\phi_{21} \mid \phi_{22})\,\bar{\delta}(t_0) + \phi_{22}\,\dot{V}_{g1} + \bar{B}\Delta t] \\ &= -[(\phi_{21} \mid \phi_{22})\,\bar{\delta}(t_0) - \phi_{22}(K_1\bar{\delta}(t_0) + \bar{K}_2\Delta t) + \bar{B}\Delta t]\end{aligned}$$

but

$$\begin{aligned}(\phi_{21}\ \phi_{22}) - \phi_{22}K_1 &= (\phi_{21}\ \phi_{22}) - (\phi_{22}\phi_{12}^{-1}\phi_{11} \mid \phi_{22}) \\ &= (\phi_{21} - \phi_{22}\,\phi_{12}^{-1}\,\phi_{11})\ \phi_{22} \\ &\equiv -K_3\end{aligned}$$

Thus,

$$\begin{aligned}\bar{V}_{g2} &= K_3\,\bar{\delta}(t_0) + [\phi_{22}\,\bar{K}_2 - \bar{B}]\Delta t \\ &\equiv K_3\,\bar{\delta}(t_0) + \bar{K}_4\,\Delta t\end{aligned} \tag{2.6}$$

where

$$K_3 = -[\phi_{21} - \phi_{22}\,\phi_{12}^{-1}\phi_{11} \mid O]$$

$$\bar{K}_4 = \{\phi_{22}\,\phi_{12}^{-1}\bar{A} - \bar{B}\}$$

To this point no constraint or condition has been employed to determine the value of Δt. Thus, the final step requires that a scalar comparison function be specified which can be utilized to judge the relative performance of different guidance strategies. Three such functions are

a) $|\bar{V}_{g1}| = min$

b) $|\bar{V}_{g2}| = min$

c) $|\bar{V}_{g1}| + |\bar{V}_{g2}| = min$

Each of these functions will be considered in turn.

2.2.1.2.1 Δt by Minimization of the First Impulse

As the first step in this solution it is recognized that minimization of $|V_{g_1}|$ is equivalent to minimization of $|\bar{V}_{g_1}|^2$. This observation allows the problem to be formulated in a more convenient manner by eliminating radicals in comparison function; thus consider,

$$
\begin{aligned}
F_1 = |\bar{V}_{g_1}|^2 = \bar{V}_{g_1}^T \bar{V}_{g_1} &= \bar{\delta}^T(t_0) K_1^T K_1 \bar{\delta}(t_0) + \Delta t^2 \bar{K}_2^T \bar{K}_2 \\
&\quad + \bar{\delta}(t_0)^T K_1^T \bar{K}_2 \Delta t + \Delta t \bar{K}_2^T K_1 \bar{\delta}(t_0) \\
&\equiv \bar{\delta}^T(t_0) K_1^T K_1 \bar{\delta}(t_0) + \Delta t^2 \bar{K}_2^T \bar{K}_2 + 2\Delta t \bar{K}_2^T K_1 \bar{\delta}(t_0)
\end{aligned}
\tag{2.7}
$$

Now for F_1 to attain a minimum with respect to Δt,

$$
\frac{\partial F_1}{\partial \Delta t} = 0
$$

But,

$$
\frac{\partial F_1}{\partial \Delta t} = 2\Delta t \bar{K}_2^T \bar{K}_2 + 2\bar{K}_2^T K_1 \bar{\delta}(t_0)
$$

thus,

$$
\Delta t = -\frac{\bar{K}_2^T K_1}{\bar{K}_2^T \bar{K}_2} \bar{\delta}(t_0)
\tag{2.8}
$$

Under this substitution the corrections assume the form

$$
\begin{aligned}
\bar{V}_{g_1} &= -\left[K_1 \bar{\delta}(t_0) - \bar{K}_2 \Delta t\right] \\
&= \left[-K_1 + \bar{K}_2 \frac{\bar{K}_2^T K_1}{\bar{K}_2^T \bar{K}_2}\right] \bar{\delta}(t_0) \\
&\equiv G_1 \bar{\delta}(t_0)
\end{aligned}
\tag{2.9}
$$

and

$$
\begin{aligned}
\bar{V}_{g2} &= \left[K_3 - \bar{K}_4 \frac{\bar{K}_2^T K_1}{\bar{K}_2^T \bar{K}_2}\right] \bar{\delta}(t_0) \\
&= G_2 \bar{\delta}(t_0)
\end{aligned}
\tag{2.10}
$$

2.2.1.2.2 Δt by Minimization of the Second Impulse

The same approach employed in the previous paragraph will yield the required solution for this case. Thus, a function F_2 is formed as:

$$
\begin{aligned}
F_2 &= |V_{g2}|^2 = \bar{V}_{g2}^T \bar{V}_{g2} \\
&= \bar{\delta}(t_0) K_3^T K_3 \bar{\delta}(t_0) + \Delta t^2 \bar{K}_2^T \bar{K}_4 + 2\Delta t \bar{K}_4^T K_3 \bar{\delta}(t_0)
\end{aligned}
\tag{2.11}
$$

and its partial derivative with respect to Δt is set equal to zero to yield

$$
\Delta t = - \frac{\bar{K}_4^T K_3}{\bar{K}_4^T \bar{K}_4} \bar{\delta}(t_0) \tag{2.12}
$$

Substitution of this result now yields $\bar{V}g_1$ and $\bar{V}g_2$ as

$$
\bar{V}_{g1} = \left[-K_1 + \bar{K}_2 \frac{\bar{K}_4^T K_3}{\bar{K}_4^T \bar{K}_4} \right] \bar{\delta}(t_0) \equiv G_3 \bar{\delta}(t_0) \tag{2.13}
$$

$$
\bar{V}_{g2} = \left[K_3 - \bar{K}_4 \frac{\bar{K}_4^T K_3}{\bar{K}_4^T \bar{K}_4} \right] \bar{\delta}(t_0) \equiv G_4 \bar{\delta}(t_0) \tag{2.14}
$$

2.2.1.2.3 Δt by Minimization of the Total Impulse

The final case of interest requires a slightly different approach since the solution obtained by minimizing the sum of two magnitudes is not in general equivalent to minimizing the sum of the squares of these magnitudes. Thus, the comparison function F_3 is formed as

$$
\begin{aligned}
F_3 &= \left(|\bar{V}_{g1}|^2\right)^{1/2} + \left(|V_{g2}|^2\right)^{1/2} \\
&= \sqrt{\bar{V}_{g1}^T \bar{V}_{g1}} + \sqrt{\bar{V}_{g2}^T \bar{V}_{g2}}
\end{aligned}
$$

Now once again the partial of F with respect to Δt is formed and the result equated to zero.

$$\frac{\partial F_3}{\partial \Delta t} = 0 = \frac{\Delta t\, \bar{K}_2^T \bar{K}_2 + \bar{K}_2^T K_1\, \bar{\delta}(t_0)}{|\bar{V}_{g1}|} + \frac{\Delta t\, \bar{K}_4^T \bar{K}_4 + \bar{K}_4 K_3\, \bar{\delta}(t_0)}{|\bar{V}_{g2}|}$$

or

$$\frac{\Delta t\, \bar{K}_2^T \bar{K}_2 + \bar{K}_2^T K_1\, \bar{\delta}(t_0)}{|\bar{V}_{g1}|} = - \frac{\Delta t\, \bar{K}_4^T \bar{K}_4 + \bar{K}_4 K_3\, \bar{\delta}(t_0)}{|\bar{V}_{g2}|}$$

This equation or an equivalent form obtained by squaring both sides and substituting $\bar{V}_{g_i}^T \, \bar{V}_{g_i}$ for $|V_{g_i}|^2$ must now be solved for Δt. In general, a solution can be found by numerical techniques though the processes may be involved.

However, for most guidance mechanizations and studies of common interest an analytic expression is desired for Δt as a function of $\delta(t_0)$ so that the matrices which define $\bar{V}g_1$ and $\bar{V}g_2$ can be written explicitly in terms of $\delta(t_0)$. This fact requires that a different measure of the performance of the system be adopted and results in a less optimal (though much simpler) scheme. Though many measures can be constructed which produce the desired result, the simplest of these measures is

$$F_3' = \bar{V}_{g1}^T V_{g1} + \bar{V}_{g2}^T V_{g2} \tag{2.15}$$

This function is a positive measure of the control and would be equivalent to the index F_3 were it not for differences in the weighting of the dependence of the second correction on the first. However, experimentation for special cases has shown that the control generated for this index is generally comparable though slightly less efficient than the more precise statement of the problem.

Minimization of $F_3{}^1$ with respect to Δt can now be accomplished by expanding the respective scalar products

$$F_3' = \bar{\delta}^T(t_0)(K_1^T K_1 + K_3^T K_3)\bar{\delta}(t_0)$$
$$+\Delta t^2(\bar{K}_2^T \bar{K}_2 + \bar{K}_4^T \bar{K}_4) + 2\Delta t(\bar{K}_2^T K_1 + \bar{K}_4^T K_3)\bar{\delta}(t_0)$$

Thus,

$$\frac{\partial F_3'}{\partial \Delta t} \equiv 0 = 2\Delta t\,(\bar{K}_2^T \bar{K}_2 + \bar{K}_4^T \bar{K}_4) + 2(\bar{K}_2^T K_1 + \bar{K}_4^T K_3)\,\bar{\delta}(t_o)$$

or

$$\Delta t = -\frac{\bar{K}_2^T K_1 + \bar{K}_4^T K_3}{\bar{K}_2^T \bar{K}_2 + \bar{K}_2^T \bar{K}_2}\,\bar{\delta}(t_o) \qquad (2.16)$$

Substituting this relation into the expressions for the corrections $\bar{V}g_1$ and $\bar{V}g_2$ now yields

$$\bar{V}_{g1} = \left[-K_1 + \bar{K}_2\,\frac{\bar{K}_2^T K_1 + \bar{K}_4^T K_3}{\bar{K}_2^T \bar{K}_2 + \bar{K}_4^T \bar{K}_4}\right]\bar{\delta}(t_o) \equiv G_5\,\bar{\delta}(t_o) \qquad (2.17)$$

$$\bar{V}_{g2} = \left[K_3 - \bar{K}_4\,\frac{(\bar{K}_2^T K_1 + \bar{K}_4^T K_3)}{\bar{K}_2^T \bar{K}_2 + \bar{K}_4^T \bar{K}_4}\right]\bar{\delta}(t_o) \equiv G_6\,\bar{\delta}(t_o) \qquad (2.18)$$

2.2.1.3 Observations of the Form of the Guidance Gains

In all three of the previous cases a linear relation for Δt in terms of the state deviation $\bar{\delta}(t_o)$ was obtained (for the assumed quadratic performance index). Thus, both of the corrections assume the form

$$\bar{V}_{gi} = \beta_{ij}\,\bar{\delta}(t_o) \qquad i = 1,2$$

where the subscript j on the gain matrix β denotes the type of performance utilized to define the array. Further, this form of the solution is exactly comparable to the fixed time of arrival derived earlier. This fact will be employed in subsequent discussions to divorce these sections from a specific guidance logic. That is, future discussions will assume that a choice of the guidance concept has been made and that the proper form of the gain (β) is available.

2.2.1.4 Error Analysis

The formulations presented on the previous pages will yield the required corrections to effect a prescribed change in the position and velocity vectors under the assumption that the state, $\bar{\delta}(t_o)$, is known. In all cases of interest, however, $\bar{\delta}(t_o)$ will not be known (rather, an estimate will be available), the corrections will be imprecisely applied, and the exact state of the target trajectory (as in the rendezvous problem) will be uncertain. The net result of these errors will be a random variation in the commanded corrections which will be a function of the errors themselves. Thus, it is necessary to establish the mathematical framework relating these errors to the correction and to errors at points along the trajectory or in the terminal state.

To accomplish the desired result and introduce the effects of additive errors it is necessary to focus attention on the correction equations in the following form:

$$\bar{V}_{g1} = -\phi_{12}^{-1}\{\Delta\hat{r}_f + A\Delta t\}$$

$$\hat{V}_{g2} = -\{\Delta V_f + \phi_{22}\hat{V}_{g1}^{*} + B\Delta t\}$$

$$\Delta t = \Upsilon\hat{\delta}(t_o)$$

where Υ depends upon the choice of the guidance scheme as discussed in preceding sections, where the "hat" above a quantity indicates that the variable is an estimate, and where the vector notation has been dropped for simplicity. Now define the estimates $\Delta\hat{v}_f$, $\Delta\bar{v}_f$ and $\hat{\delta}(t_o)$ in terms of their true values and additive noise as

$$\Delta\hat{r}_f = \Delta\bar{r}_f + \bar{\eta}_{T,o} + \bar{\eta}_{V,o} \tag{2.19a}$$

$$\Delta\bar{V}_f = \Delta\bar{V}_f + \bar{\nu}_{T,f} + \bar{\nu}_{V,f} \tag{2.19b}$$

$$\hat{V}_{g1}^{*} = \hat{V}_{g1} + \bar{\psi} \tag{2.19c}$$

$$\hat{\delta}(t_o) = \bar{\delta}(t_o) + \bar{\epsilon}_{V,o} \tag{2.19d}$$

where $\bar{\eta}$, $\bar{\nu}$ are 3-vectors of errors in the terminal position and velocity

$\bar{\psi}$ is the application error for $\bar{V}g_1$

$\bar{\epsilon}$ is the error in the estimate of the state at (t_o)

and where the subscripts (T, O), (V,O) denote errors in the target or vehicle based upon information available at the time t_o. Similarly for (T,f), (V,f).

$$\bar{\eta}_{V,0} = \Xi \phi_V \bar{\epsilon}_{V,0}$$

$$\bar{\nu}_{V,f} = \Theta \bar{\epsilon}_{V,f}$$

$$\bar{\eta}_{T,0} = \Xi \phi_T \bar{\epsilon}_{T,0}$$

$$\bar{\nu}_{T,f} = \Theta \bar{\epsilon}_{T,F}$$

where

$$\Xi = \begin{bmatrix} I & I \\ O & O \end{bmatrix}$$

$$\Theta = \begin{bmatrix} O & O \\ I & I \end{bmatrix}$$

Thus,

$$\Delta \hat{r}_f = \Xi [\phi_V \bar{\delta}(t_o) + \phi_T \bar{\epsilon}_{T,0} + \phi_V \bar{\epsilon}_{V,0}]$$

$$\Delta \hat{V}_f = \Theta [\phi_V \bar{\delta}(t_o) + \bar{\epsilon}_{T,f} + \bar{\epsilon}_{V,f}]$$

and

$$\hat{V}_{g_1} = -\phi_{12}^{-1} \Xi \left[(\phi_V + AT) \bar{\delta}(t_o) + \phi_T \epsilon_{T,0} + (\phi_V + AT) \epsilon_{V,0} \right] \tag{2.20}$$

Now, if the additive noise processes are independent (that is, if the estimation errors for the target orbit and the transfer orbit are not functions of one another) and if all known biases have been compensated for by adjusting the respective trajectories, then the first two statistical moments are:

$$E(\hat{V}_{g_1}) = E\left[-\phi_{12}^{-1} \Xi (\phi_V + AT) \delta(t_o)\right] = \bar{V}_{g_1}$$

and

$$E\left\{\left[\hat{V}_{g_1} - E(\hat{V}_{g_1})\right]\left[\hat{V}_{g_1} - E(\hat{V}_{g_1})\right]\right\} \equiv V_1$$

$$= \phi_T \, E(\bar{\epsilon}_{T,0} \; \bar{\epsilon}^T_{T,0}) \, \phi_T^T + (\phi_V + AT) \, E(\bar{\epsilon}_{V,0} \; \bar{\epsilon}^T_{V,0})(\phi_V + AT)^T$$

$$\equiv \phi_T \, P_{T,0} \, \phi_T^T + (\phi_V + AT) P_{V,0} \, (\phi_V + AT)^T \tag{2.21}$$

This equation displays the effect of estimation errors in both the target and transfer trajectories on the variation of the estimated correction. Note the absence of any reference to errors occurring after the time at which the correction is commanded.

The second correction is considered in an exactly equivalent manner. First, Vg_2 is written in terms of the independent error sources of the problem

$$\begin{aligned}\hat{V}_{g2} = -\Big\{ & \Theta \, \bar{\delta}\,(t_f) + \Theta \, \bar{\epsilon}_{T,f} + \Theta \, \bar{\epsilon}_{V,f} \\ & - \phi_{22} \, \phi_{12}^{-1} \, \Xi \left[(\phi_V + AT) \, \delta(t_0) + \phi_T \, \bar{\epsilon}_{T,0} \right. \\ & \left. + (\phi_V + AT) \, \bar{\epsilon}_{V,0} + \psi\right] + BT\left[\bar{\delta}(t_0) + \bar{\epsilon}_{V,0}\right]\Big\}\end{aligned} \tag{2.22}$$

But

$$\bar{\delta}(t_f) = \phi \, \bar{\delta}(t_0)$$

so that

$$\begin{aligned}\hat{V}_{g2} = -\Big\{ & \left[\Theta\phi - \phi_{22} \, \phi_{12}^{-1} \, \Xi(\phi_V + AT) + BT\right] \bar{\delta}(t_0) + \Theta(\bar{\epsilon}_{T,f} + \bar{\epsilon}_{V,f}) \\ & + (\phi_{22} \, \phi_{12}^{-1} \, \Xi)\left[\phi_T \, \bar{\epsilon}_{T,0} + \left[\phi_V + (A+B)T\right] \bar{\epsilon}_{V,0} + \bar{\psi}\right]\Big\} \\ \equiv \; & \alpha_1 \, \bar{\delta}(t_0) + \alpha_2 \, \bar{\epsilon}_{T,0} + \alpha_3 \, \bar{\epsilon}_{V,0} + \alpha_4 \, \bar{\psi} - \Theta(\bar{\epsilon}_{T,f} + \bar{\epsilon}_{Vf})\end{aligned} \tag{2.23}$$

where

$$\alpha_1 = -\Theta\phi + \phi_{22}\ \phi_{12}^{-1}\Xi(\phi_V + AT) - BT$$

$$\alpha_2 = -\phi_{22}\ \phi_{12}^{-1}\Xi\phi_T$$

$$\alpha_3 = -\phi_{22}\ \phi_{12}^{-1}\Xi\left[\phi_V + (A+B)T\right]$$

$$\alpha_4 = \alpha_2\ \phi_T^{-1}$$

Now

$$E(\hat{V}_{g2}) = \alpha_1\ \bar{\delta}(t_o)$$
$$\equiv \bar{V}_{g2}$$

and

$$E\left[(\hat{V}_{g2} - E(\hat{V}_{g2}))(\hat{V}_{g2} - E(\hat{V}_{g2}))^T\right] \equiv V_2$$

Thus, upon expansion,

$$\begin{aligned} V_2 = {} & \alpha_2 P_{T,0}\ \alpha_2^T + \Theta P_{T,f}\ \Theta^T - (\Theta + \alpha_2) R_{T,0F} (\Theta + \alpha_2)^T \\ & + \alpha_3 P_{V,0}\ \alpha_3^T + \Theta P_{V,f}\ \Theta^T - (\Theta + \alpha_5) R_{V,0F} (\Theta + \alpha_3)^T \\ & + \alpha_4 Q\ \alpha_4^T \end{aligned} \tag{2.24}$$

where $R_{T\ OF}$ is the auto correlation function of the estimation errors corresponding to the two times t_o, t_f for the target vehicles $\left[E(\epsilon_{T,O}\epsilon_{TF}^T)\right]$

and where $Q = E(\psi\psi^T)$ - covariance matrix for the application errors in V_{g_1}

The result of the errors expressed by V_1 and V_2 can now be evaluated in terms of the variance in the terminal state. This step is accomplished by referring to

$$\Delta\bar{r}_f + \bar{A}\Delta t = -\phi_{12}\ \bar{V}_{g_1}$$

and defining

$$\begin{aligned} \Delta R_f &= \text{error in the terminal position} \\ &= -\phi_{12}\ \delta\bar{V}_{g_1} \\ &= -\phi_{12}\ (\bar{V}_{g_1} - \hat{V}_{g_1}) \end{aligned}$$

Thus,

$$E\left[\Delta R_f \Delta R_f^T\right] = \phi_{12}\, V_1\, \phi_{12}^T$$
$$R \equiv \phi_{12}\, V_1\, \phi_{12}^T \tag{2.25}$$

Similarly, noting that

$$\Delta \bar{V}_f + \bar{B}\Delta t = -\bar{V}_{g2}$$

and defining the error in the terminal velocity as ΔV_f allows the covariance matrix for errors in the terminal velocity to be written as

$$E\left[\Delta V_f\ \Delta V_f^T\right] = V_2$$
$$V \equiv V_2 \tag{2.26}$$

Since the terminal state, $\delta(t_f)$, is not equal to the desired state, the possibility exists that the mission objectives cannot be satisfied by employing the two correction concepts without imposing severe correction requirements. In this case, a decision should be made based upon the elements of R as to whether the resultant despersions will be acceptable. If not, subsequent corrections should be considered. These intermediate corrections can be computed in exactly the same manner as those outlined on the previous pages with two important diffference.

1) The reference trajectory about which the control is being exercised is the corrected trajectory resulting from the previous correction. (That is, the problem can be restarted using the commanded trajectory resulting from the previous correction as the nominal and the errors in the correction as the state.) However, the error sources affecting the performance of the system are the same as those existing in the earlier analysis.

2) Since the total correction concept requires that the second of the two impulses be applied at the terminal state, a composite correction equivalent to the vector sum of the corrections $(\bar{v}_{g2})_2$ is required.

The net result of this process will be a correction strategy which can be commanded in real time using very simple computational algorithms. The policy will not be "optimum" in the sense of the comparison function employed, however, since the feedback matrices defining Vg_1 and Vg_2 are each predicated on the assumption that exactly <u>two</u> corrections are being utilized. Thus, this source of inefficiency must be added to the list of items serving as motivation for a more encompassing analysis of the midcourse guidance problem.

The analysis of the first two moments of the statistical distributions for $\overline{V}g_1$ and $\overline{V}g_2$ in general will not completely describe a random process. However, for the special case where all of the errors are Gaussian and where the relationships between all of the variables are linear, the analysis is complete since the resultant distributions are also Gaussian. This fact was demonstrated in an earlier monograph (Reference 2.22).

These results also have other applications. One such application is to the problem of defining the probable amount of fuel required for correction of a planned trajectory for the situation where estimates of the error processes are available. This application is realizable by considering the state, $\delta(t_0)$, to be an independent random variable resulting from injection guidance errors. Under the assumption,

$$V_1^* \doteq V_1 + SGS^T \tag{2.27a}$$

$$V_2^* = V_2 + TGT^T \tag{2.27b}$$

where

$$S = \phi_{12}^{-1}\,\Xi\,(\phi_V + A\Upsilon)$$

$$T = \Theta\,\phi - \phi_{22}\,S + B\Upsilon$$

$$G = E\left[\bar{\delta}(t_0)\,\bar{\delta}(t_0)^T\right]$$

and where the asterisk used as a superscript denotes that this equation defines the probable requirements for the corrections. Interpretation of the numbers in these arrays is now possible in terms of the magnitudes of the probable corrections by considering

$$E(V_{g1}^*)^2 = E\left[(V_{g1}^*)^T(V_{g1}^*)\right]$$

$$\alpha_1 E(V_{g1}^*)^2 = \alpha_1\,Tr\,(V_1^*)$$

and

$$\alpha_2 E(V_{g2}^*)^2 = \alpha_2\,Tr\,(V_2^*)$$

where $Tr(\ )$ denotes the trace operation and where α_1 and α_2 define the probabilities which can be associated with the estimates (see Reference 2.22). This capability is enhanced greatly by the fact that the trace operation is commutative (i.e. $Tr\,(A+B) = Tr(A) + Tr(B)$) thus allowing the contributions and sensitivities of each independent error source to the performance of the total system to be assessed. This capability is especially valuable in the performance of preliminary design studies since it affords the possibility of generating much of the data in a parametric form.

2.2.2 Optimal Control for Quadratic Cost

2.2.2.1 Introduction

The midcourse guidance problem for most spatial missions is characterized by control intervals which are generally short relative to the total flight phase. This fact means that a suitable model for this process can be constructed by assuming that the corrections themselves are impulsive, however, no assumption will be made regarding the nature of the observation process used to define the state. Rather, it is assumed that the state has been measured and is known. The effects of errors in the process will be introduced subsequently.

The problem is now to determine a series of corrections which will minimize some suitable measure of performance for the system. To accomplish this objective, the mechanism of Dynamic Programming (Reference 2.3) will be employed as follows. Assume first that the loss function for the system is a linear combination of the losses resulting from each control action. That is

$$J_N = \sum_{i=1}^{N} F_i$$

Further, assume that the F_i are functions of the state of the system and the control action (thus, by proper selection of the control, the F_i and consequently J_N can be minimized). The functional form of the F is unimportant at the present time. It is only necessary to indicate that

$$F_i = F_i\left[\delta_i\,(\delta_0; u_0, \ldots, u_{i-1}),\, u_{i-1}\right]$$

and to note that the value of the performance loss will change as any of the controls $i\,(0 \leq n < i-1)$ changes. (The fact that the subscripts used to denote the state and the control differ by one is a matter of convention dictated by the lag between a specified control and the state just prior to the next correction and by recalling that δ_0 is specified whereas u_0 is free to be chosen.) For the sake of simplicity in the presentation, the vector notation will be dropped from δ , u . . . etc. This deletion will not cause problems provided normal matrix - vector operations are considered for all of the products and inverses. Now, consider the sequence of control times

$$\begin{array}{cccccccccc} \bullet & \bullet & --- & \bullet & \bullet & \bullet & --- & \bullet \\ 0 & 1 & & n-1 & n & n+1 & & N \end{array}$$

and focus attention on the epoch n (i.e., the n plus first control action). The loss associated with the duration corresponding to the remaining corrections is now

$$J_{N-n+1} = \sum_{i=n}^{N} F_i(\delta_i, u_{i-1})$$

(Note that the subscripts on performance loss increase in a direction opposite from that assumed for the control epochs.) At this point, the specific value of J_{N-n+1} which is minimum, I_{N-n+1} is defined as

$$I_{N-n+1} = \underset{\text{ALL } u_K}{\text{MIN}} J_{N-n+1} \qquad n \leq K \leq N$$

This notation means that the variable J_{N-n+1} is minimized with respect to all of the controls which can be applied from the present epoch to that of mission completion. But, this minimum loss can be re-written using the additive property assumed for the performance index as

$$I_{N-n+1} = \underset{u_{N-1}}{\text{MIN}} \left[F_n(\delta_n, u_{n-1}) + \underset{u(n)}{\text{MIN}} \cdots \underset{u(N-1)}{\text{MIN}} \sum_{i=n+1}^{N} F_i(\delta_i, u_{i-1}) \right]$$

$$\underset{u_{N-1}}{\text{MIN}} \left[F_n(\delta_n, u_{n-1}) + I_{N-n} \right] \qquad (2.28)$$

This equation is the "principle of optimality" of Dynamic Programming. It states in mathematical form that the optimum control strategy satisfies the condition that all corrective actions, regardless of the state of the system, must be optimum with respect to the state resulting from all preceding controls.

Having rederived this basic principle, attention can now turn to the choice of a functional form for $F_i(x_i, u_{i-1})$. Discussions in the introductory developments of the midcourse guidance problem presented two such functions and arguments which revealed problems in applying these two functions. Based upon these observations the quadratic loss function will be employed as the measure of performance for the system under consideration. That is

$$F_i(x_i, u_{i-1}) = \delta_i^T Q_i \delta_i + u_{i-1}^T \gamma_i u_{i-1} \qquad (2.29)$$

The reformulation of the problem in terms of the minimization of a sum of vector magnitudes will be considered later. Before leaving this discussion it is noted that many other loss functions can be selected for most problems. However, few are of general interest to the midcourse guidance problem, therefore, little or no work has been reported for them.

As the final preparatory step in the development of the optimal control, one further assumption will be made. This assumption is that the times corresponding to the various corrections are all known (that is, the matrices relating the states and controls at two successive correction times are constants). This assumption will be relaxed in subsequent discussions. However, relaxation will not be considered until this special case is understood.

The developments presented in this section of the monograph are based in large part on the work prepared by Kalman (Reference 2.3) and Gunckel (Reference 2.14). Thus, to a large degree the notation will be similar. In contrast to these works and the works of others who have been concerned with the performance of more general linear systems, the discussions to be presented here pertain only to the midcourse guidance problem. This loss of generality is felt justified in the light of the objective of this monograph and the excellent nature of the references.

2.2.2.2 Optimization of Deterministic Systems

2.2.2.2.1 Formulation

The loss having been defined and the mechanism provided for performing the analysis, emphasis can now turn to the development of the optimum control. This development is accomplished by substituting the relationship for the state of the system

$$\delta_N = \phi_{N,N-1}\ \delta_{N-1} + \Gamma_{N,N-1}\, u_{N-1} \tag{2.30}$$

into the loss function, solving for the control which minimized this last leg* of the problem and employing the optimality principle to provide the relationships for the remaining corrections. First,

$$\begin{aligned} F_N &= (\phi_{N,N-1}\ \delta_{N-1} + \Gamma_{N,N-1}\, u_{N-1})^T Q_N\, (\phi_{N,N-1}\ \delta_{N-1} + \Gamma_{N,N-1}\, u_{N-1}) \\ &\quad + u_{N-1}^T \cdot \gamma_N\, u_{N-1} \\ &= \delta_{N-1}^T\, \phi_{N,N-1}^T\, Q_N\, \phi_{N,N-1}\, \delta_{N-1} + 2u_{N-1}^T\, \Gamma_{N,N-1}^T\, Q_N\, \phi_{N,N-1}\, \delta_{N-1} \\ &\quad + u_{N-1}^T\, (\gamma_N + \Gamma_{N,N-1}^T\, Q_N\, \Gamma_{N,N-1})\, u_{N-1} \end{aligned} \tag{2.31}$$

* The last leg of the problem is considered first so as to divorce the analysis from consideration of anything which has occured prior to the epoch denoted t_{N-1}. This procedure is suggested by the principle of optimality itself.

But for this segment

$$J_{N-N+1} \equiv J_1 = F_N$$

and, therefore,

$$I_1 = \underset{u_{N-1}}{\mathrm{Min}} \left[F_N\right]$$

Now, if J_1 is to attain a minimum with respect to u_{N-1}, the condition

$$\frac{\partial J_1}{\partial u_{N-1}} = 0$$

must be satisfied. That is

$$2\,\Gamma^T_{N,N-1} Q_N \phi_{N,N-1} \delta_{N-1} + 2\left[\Gamma^T_{N,N-1} Q_N \Gamma_{N,N-1} + \gamma_N\right] u_{N-1} = 0$$

or

$$\begin{aligned} u_{N-1} &= -\left[\Gamma^T_{N,N-1} Q_N \Gamma_{N,N-1} + \gamma_N\right]^{-1} \Gamma^T_{N,N-1} Q_N \phi_{N,N-1} \delta_{N-1} \\ &\equiv K_1 \delta_{N-1} \end{aligned} \tag{2.32}$$

where K_1 is the gain (feedback) matrix required to effect the correction in an optimal fashion.

The generalization of this result for the preceding corrections is now realized by applying dynamic programming. First, substituting the solution for u_{N-1} into the equation for J_1 yields

$$\begin{aligned} I_1 = \; & \delta^T_{N-1}\Big[\phi^T_{N,N-1} Q_N \phi_{N,N-1} + 2K_1^T \Gamma^T_{N,N-1} Q_N \phi_{N,N-1} \\ & + K_1^T (\gamma_N + \Gamma_{N,N-1} Q_N \Gamma_{N,N-1}) K_1 \Big] \delta_{N-1} \\ \equiv \; & \delta^T_{N-1} P_1 \delta_{N-1} \end{aligned}$$

Thus, from the optimum principle,

$$I_2 = \underset{u_{N-2}}{MIN}\left[\delta_{N-1}^T Q_{N-1} \delta_{N-1} + u_{N-2}^T \gamma_{N-1} u_{N-2} + \delta_{N-1}^T P_1 \delta_{N-1}\right]$$

$$= \underset{u_{N-2}}{MIN}\left[\delta_{N-1}^T (Q_{N-1} + P_1)\delta_{N-1} + u_{N-2}^T \gamma_{N-1} u_{N-2}\right]$$

But, Q_{N-1} and P_1 are not functions of u_{N-1} and the functional form of this expression is identical to that which was minimized to determine u_{N-1}. Thus,

$$u_{N-2} = -\left[\Gamma_{N-1,N-2}^T (Q_{N-1} + P_1)\, \Gamma_{N-1,N-2} + \gamma_{N-1}\right]^{-1} \cdot \Gamma_{N-1,N-2}^T (Q_{N-1} + P_1)\, \phi_{N-1,N-2}\, \delta_{N-2}$$

$$\equiv K_2\, \delta_{N-2}$$

And, the corresponding loss (cost) is

$$I_2 = \left[(\phi_{N-1,N-2} + \Gamma_{N-1,N-2} K_2)\,\delta_{N-2}\right]^T (Q_{N-1} + P_1)\left[\phi_{N-1,N-2} + \Gamma_{N-1,N-2} K_2\right]\delta_{N-2} + \delta_{N-2}^T K_2^T \gamma_{N-1} K_2\, \delta_{N-2}$$

$$= \delta_{N-2}^T \Big\{ \phi_{N-1,N-2}^T (Q_{N-1} + P_1)\phi_{N-1,N-2} + K_2^T\left[\Gamma_{N-1,N-2}^T (Q_{N-1} + P_1)\Gamma_{N-1,N-2} + \gamma_{N-1}\right] K_2 + 2\, K_2^T\, \Gamma_{N-1,N-2}^T (Q_{N-1} + P_1)\phi_{N-1,N-2} \Big\}\, \delta_{N-2}$$

But, the last term in this equation for I_2 can itself be written as

$$2K_2^T\left[\Gamma_{N-1,N-2}^T (Q_{N-1} + P_1)\phi_{N-1,N-2}\right] = -2K_2^T\left[\Gamma_{N-1,N-2}^T (Q_{N-1} + P_1)\Gamma_{N-1,N-2} + \gamma_{N-1}\right] K_2$$

by simple manipulation of the definition of K_2. Thus,

$$I_2 = \delta_{N-2}^T \Big\{ \phi_{N-1,N-2}^T (Q_{N-1} + P_1)\phi_{N-1,N-2} - K_2^T\left[\Gamma_{N-1,N-2} (Q_{N-1} + P_1)\Gamma_{N-1,N-2} + \gamma_{N-1}\right] K_2 \Big\}\, \delta_{N-2}$$

$$\equiv \delta_{N-2}^T P_2\, \delta_{N-2}$$

Finally, these equations can be expressed in recursive form for the general point, n, as follows:

$$F^*_{N-n} = \Gamma^T_{n,n-1}\left[Q_N + P_{N-n}\right] \Gamma_{n,n-1} + \gamma_n \tag{2.33a}$$

$$K_{N-N+1} = -\left[F^*_{N-n}\right]^{-1} \Gamma_{n,n-1}\left[Q_n + P_{N-n}\right] \phi_{N,n-1} \tag{2.33b}$$

$$P_{N-n+1} = \phi^T_{n,n-1}\left[Q_n + P_{N-n}\right] \phi_{n,n-1} - K^T_{N-n+1} F^*_{N-n} K_{N-n+1} \tag{2.33c}$$

$$u_n = K_{N-n+1} \delta_n \tag{2.33d}$$

where the initial conditions for the P array must be

$$P_{N-N} = P_o \equiv 0$$

The process for the special case where the control times are known is now completely specified provided the weights (γ_n and Q_n) are given. Thus, the solution can be obtained as follows:

1) Starting at the Nth epoch, work backward in time defining F, K and P based upon assumed elements for the various arcs of the reference trajectory (these elements define Γ and ϕ)

2) Starting at the initial state, work forward evaluating the control (using the known gains, K) and translating the state using the equation

$$\delta_n = \phi_{n,n-1} \delta_{n-1} + \Gamma_{n,n-1} u_{n-1}$$

3) Repeat steps 1 and 2 until the matrices $\Gamma_{n,n-1}$ and $\phi_{n,n-1}$ for the two passes agree with each other and with the values used in the preceding pass. At this time, the control is consistent with the arcs of the reference trajectory.

4) The state at any point can now be computed as

$$\delta_n = \left[\prod_{i=1}^{n} \left(\phi_{i,i-1} + \Gamma_{i,i-1} K_{N-i+1}\right)\right] \delta_o \tag{2.34}$$

where $\prod_{i=1}^{n}$ denotes the product operation (analogous to $\sum_{i=1}^{n}$).

This last equation shows a very important fact. The terminal state (or any other) is a function of the N gains (in a complex fashion -- these gains in turn are a function of the times of the corrections and the weighting matrices, Q and γ) and the initial state. Thus, if both the initial state and the correction epochs are specified, the terminal state is also specified. This fact, though not unexpected, is unfortunate since this value of the terminal state may or may not agree with physical constraints applied to the problem. Further, the magnitudes of the corrective actions may or may not satisfy constraints (such as $u_n \leq u_n = \max$, $u_i \geq u_i = \min$, etc.) imposed on the control. While some freedom exists in satisfying both types of constraints, by varying the free parameters the actual process for ataining these objectives will not be simple for a particular problem. To illustrate the point, assume that

$$|\delta_N| > |\delta_{N_{MAX}}|$$

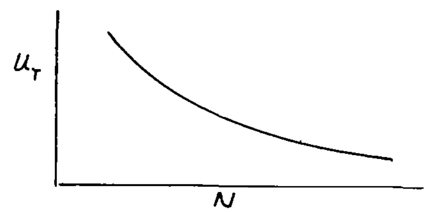
$$|u_i| > |u_{i_{MAX}}| \qquad 1 \leq i \leq N$$

For this case, both conditions can be relaxed providing a larger number of corrections can be applied. The result of the increased frequency of the control will generally

1) drive the terminal error in the general direction of zero

2) reduce the magnitudes of the commanded corrections

3) reduce the magnitude of the total control. This observation must be tempered with the observation that the total control has the general appearance



Thus, optimization of the total control by increasing the number of impulses eventually ceases to pay off.

For other cases, this approach may make the problem worse. For example, consider the problem where the terminal error is too large; but one or more of the corrections is smaller than the minimum which can be applied. (This situation could exist if the corrections were applied by firing a variable number of fixed impulses.) For this case, it is necessary to vary the entire gain structure by adjusting all of the free parameters (N, t_i, Q_i, γ_i). In fact, this is the only logical means of defining these quantities to effect a given type of control. The alternative to this search process is to adjoin the equations of constraint directly to the loss function and to resolve for a new set of gains. To illustrate this latter approach, consider the case of a terminal constraint on some linear combination of the elements of the state which can be treated simply as:

$$J = \sum_{i=1}^{N} \left(\delta_i^T Q_i \, \delta_i + u_{i-1}^T \, \gamma_i \, u_{i-1}\right) + \delta_N^T H^T \Lambda H \delta_N$$

$$= \sum_{i=1}^{N} \left(\delta_i^T Q_i' \, \delta_i + u_{i-1} \gamma_i \, u_{i-1}\right) \tag{2.35}$$

where

$$Q_i' = Q_i \qquad i \neq N$$

$$= Q_i + H^T \Lambda H \qquad i = N$$

H = transformation relating δ to some set of parameters which will be constrained (for example, perigee height, orbital inclination ...)

Λ = A square array of undetermined multipliers whose dimension is that of the vector $H\delta_N$ (the vector of constraint parameters)

Now, the gains can be generated in exactly the same manner employed before. However, for this case, the last gain weight, Q_i, is not free (arbitrary) and must be determined so as to satisfy the constraint. This process is iterative in nature, and very little was found in the literature regarding the terminal constraint problem relative to its solution or difficulties encountered in a set of numerical computations.

Unfortunately, nothing was located in the literature which treated the bounded control aspects of this particular problem. This statement should not be taken to mean that no literature exists; rather that within the effort available, none could be located. Further, since the schedule for the preparation of this monograph precludes the amount of research necessary to develop such a theory, no further discussion of the bounded control problem for deterministic linear systems will be provided.

2.2.2.2.2 Simple Example. To illustrate the application of the "optimal" control equations presented in the preceding sections and establish the relationship between these discussions and those presented for the fixed and variable time of arrival concepts, a particular example will be presented. In this example, the loss function utilized for the purposes of comparing various controls will be selected such that the square of the position error is minimum; no weight will be ascribed to the amount of control required to accomplish this objective.

For this sample

$$J_N = \Delta \bar{r}_N^T \Delta \bar{r}_N$$

$$= \delta_N^T \left[\begin{array}{c|c} I & O \\ \hline O & O \end{array}\right] \delta_N$$

where the partitions of this matrix are each 3 x 3. Now, comparison of this relation for J_N and its general quadratic form results in the following definitions

$$Q_N = \left[\begin{array}{c|c} I & O \\ \hline O & O \end{array}\right]$$

$$\gamma_N = 0$$

and

$$F_o = \Gamma^T_{N,N-1}\, Q_N\, \Gamma_{N,N-1}$$

$$K_1 = -\left[F_o\right]\Gamma^T_{N,N-1}\, Q_N\, \phi_{N,N-1}$$

$$u_{N-1} = K_1\, \delta_{N-1}$$

But, for the case of impulsive corrections

$$\delta_N = \phi_{N,N-1}\, \delta_{N-1} + \Gamma_{N,N-1}\, \Delta V_{N-1}$$

$$= \left[\begin{array}{c|c} \phi_{11} & \phi_{12} \\ \hline \phi_{21} & \phi_{22} \end{array}\right]_{N,N-1} \delta_{N-1} + \left[\begin{array}{c} \phi_{12} \\ \hline \phi_{22} \end{array}\right]_{N,N-1} \Delta V_{N-1}$$

Thus,

$$\Gamma_{N,N-1} = \left[\begin{array}{c} \phi_{12} \\ \hline \phi_{22} \end{array}\right]_{N,N-1}$$

Substitution of this relation now yields (dropping the subscripts N and N-1 for convenience)

$$F_o = \left[\phi^T_{12} \;\; \phi^T_{22}\right] \begin{bmatrix} I & O \\ O & O \end{bmatrix} \begin{bmatrix} \phi_{12} \\ \phi_{22} \end{bmatrix}$$

$$= \phi^T_{12}\, \phi_{12}$$

$$K_1 = -(\phi^T_{12}\, \phi_{12})^{-1} (\phi^T_{12} \,|\, \phi^T_{22}) \left[\begin{array}{c|c} I & O \\ \hline O & O \end{array}\right] \left[\begin{array}{c|c} \phi_{11} & \phi_{12} \\ \hline \phi_{21} & \phi_{22} \end{array}\right]$$

$$= -\phi^{-1}_{12} \left[I \,|\, (\phi^T_{12})^{-1} \phi_{22}\right] \begin{bmatrix} \phi_{11} & \phi_{12} \\ O & O \end{bmatrix}$$

$$= -\phi^{-1}_{12} \left[\phi_{11} \;\; \phi_{12}\right]$$

But, this gain is exactly that which was derived for the fixed time of arrival two-impulse guidance concept. Thus, the simple nature of the cost function for the previous analysis can be fully interpreted.

Continuation of this process will yield the gains for correction epochs prior to the next to last. First, P_0 is updated as

$$
\begin{aligned}
P_1 &= \begin{bmatrix} \phi_{11}^T & \phi_{21}^T \\ \phi_{12}^T & \phi_{22}^T \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \\
&\quad - \begin{bmatrix} \phi_{11}^T \\ \phi_{12}^T \end{bmatrix} (\phi_{12}^{-1})^T \phi_{12}^T \phi_{12} \phi_{12}^{-1} \begin{bmatrix} \phi_{11} & \phi_{12} \end{bmatrix} \\
&= \left[\begin{array}{cc|cc} \phi_{11}^T & \phi_{11} & \phi_{11}^T & \phi_{12} \\ \hline \phi_{12}^T & \phi_{11} & \phi_{12}^T & \phi_{12} \end{array}\right] - \left[\begin{array}{cc|cc} \phi_{11}^T & \phi_{11} & \phi_{11}^T & \phi_{12} \\ \hline \phi_{12}^T & \phi_{11} & \phi_{12}^T & \phi_{12} \end{array}\right] = 0
\end{aligned}
$$

Now, since $P_1 = 0$, the remaining gains are of the same mathematical structure as K_1 (the array $\phi = \phi_{n,n-1}$). That is

$$
K_i = -\left\{ \phi_{12} \left[\phi_{11} \mid \phi_{12} \right] \right\}_{n,n-1}
$$

Note that this result differs from that discussed earlier in that the intent of the cost function is different for all but the last correction. This difference arises from the fact that the sample problem is attempting to minimize the summation of the square of the position errors at all of the correction epochs as well as the terminal epoch. This objective is contrasted with that of minimizing the terminal error alone. This distinction is extremely important in the construction of the cost function for a particular problem.

2.2.2.3 Optimization of Stochastic Systems

2.2.2.3.1 Stochastic Control. As was the case in the analyses of the two impulse total correction guidance concept, the first observation of the solution (of the results obtained in the previous section) is that the state of the system is generally not known. Thus, a reformulation in terms of the total expectation (i.e., the expected value over the entire ensemble of trajectories) of the performance index is in order (other redefinitions of this index which admit the fact that the state is a random variable are possible; however, this approach allows the problem to be stated in a manner analogous to the deterministic case). This reformulation is the subject of subsequent paragraphs.

The first step in this development is to show that an equivalent form of the principle of optimality exists by employing the total expectation (taken over the total ensemble) of the performance index

$$J_{N-n+1} = \underset{ALL\ u_K}{MIN}\ E \sum_{i=n}^{N} F_i(\delta_i, u_{i-1})$$

and the corresponding minimum is

$$I_{N-n+1} = \underset{ALL\ u_K}{MIN}\ E \sum_{i=n}^{N} F_j(\delta_i, u_{i-1}) \qquad n \leq K \leq N$$

Thus, interchanging the order of differentiation and expectation

$$I_{N-n+1} = E\left\{\underset{ALL\ u_K}{MIN} \sum_{i=n}^{N} F_i(\delta_i, u_{i-1})\right\}$$

$$= E\left\{\underset{u_{N-1}}{MIN}\left[F_i(\delta_i, u_{i-1}) + \underset{ALL\ u_m}{MIN} \sum_{i=n+1}^{N} F_i(\delta_i, u_i)\right]\right\} \qquad n+1 \leq m \leq N$$

$$= E\left\{\underset{u_{n-1}}{MIN}\left[F_i(\delta_i, u_{i-1}) + I_{N-n}\right]\right\}$$

Finally, once again changing the order of differentiation and integration yields

$$I_{N-n+1} = \underset{u_{n-1}}{Min}\ E\left[F_i(\delta_i, u_{i-1}) + I_{N-n}\right] \tag{2.36}$$

This form of the principle of optimality will be used for the balance of this development.

Now, as before, the second step is to focus attention on the last leg of the trajectory. But the stochastic development requires that the total expectation for this leg be written as

$$E(F_N) = E_\delta E_D \left[F_N/\delta\right]$$

where δ and D denote the state and the data, respectively. Now,

$$E(F_N) = \int_\delta \int_D F_N\ f(\delta)\ f(D/\delta)\ dD\, d\delta$$

where f( ) denotes the probability density function of the argument. Finally, interchanging the order of integration and employing the fact that

$$f(\delta)\ f(D/\delta) = f(D)\ f(\delta/D)$$

yields

$$\begin{aligned} E(F_N) &= \int_D \int_\delta \left[F_N\, f(\delta/D)\, d\delta\right] f(D)\, dD \\ &\equiv \int_D B\left[\delta_N, u_{N-1}(D)\right] f(D)\, dD \\ &= E_D\, B\left[\delta_N, u_{N-1}(D)\right] \end{aligned} \tag{2.37}$$

where $B\left[\delta_N, u_{N-1}(D)\right]$ is the Bayes function or the a' posteori risk discussed in a previous monograph (Reference 2.22). Thus, under the assumption that the order of differentiation and expectation can be interchanged, the minimization of $E(F_N)$ can be accomplished.

$$\underset{u_{N-1}}{Min}\ E(F_N) = E_D\ \underset{u_{N-1}}{Min}\ B\left[\delta_N, u_{N-1}(D)\right]$$

But this objective is achieved by focusing attention on the Bayes function

$$B\left[\delta_N, u_{N-1}(D)\right] = E_\delta\left[F_N(\delta_N, u_{N-1}/D)\right]$$

and by making the following substitution

$$\delta_N = E_\delta(\delta_N/D_{N-1}) + \bar{\delta}_N = \hat{\delta}_N + \bar{\delta}_N$$

where

$$E(\hat{\delta}_N \hat{\delta}_N^T/D_{N-1}) = 0$$

$$E(\hat{\delta}_N/D_{N-1}) = 0$$

$\hat{\delta}_N =$ optimal estimate of the stated based on data obtained through t_{n-1} (from the orthonality of the optimal estimate and the estimation error and the fact that the optimal estimate is unbiased).

Under this set of definitions, the particular loss function being considered transforms to

$$\begin{aligned}
E(F_N) &= E_D\left(B[\;]\right) \\
&= E_D\left(\hat{\delta}_D^T Q_N \hat{\delta}_N + u_{N-1}^T \gamma_N u_{N-1}\right) + E_D E_\delta\left(\bar{\delta}_N^T Q_N \bar{\delta}_N\right) \\
&= E_D\left[\left(\hat{\delta}_{N-1}^T \phi_N^T + u_{N-1}^T \Gamma_N^T\right) Q_N \left(\phi_N \delta_{N-1} + \Gamma_N u_{N-1}\right) + u_{N-1}^T \gamma_N u_{N-1}\right] \\
&\qquad + E_D E_\delta\left(\delta_N Q_N \bar{\delta}_N\right) \\
&= E_D\left[\hat{\delta}_{N-1}^T Q_N \hat{\delta}_{N-1} + u_{N-1}^T\left(\Gamma_N^T Q_N \Gamma_N + \gamma_N\right) u_{N-1}\right] + E_D E_\delta\left(\bar{\delta}_N Q_N \bar{\delta}_N\right) \\
&\equiv E_D\left[F_N(\hat{\delta}_{N-1}, u_{N-1})\right] + E_D E_\delta\left(\bar{\delta}_N Q_N \bar{\delta}_N\right)
\end{aligned} \tag{2.38}$$

That is, the expected cost can be separated into two parts. The first part $E_D(\hat{F}_N)$, is a function of the control and the estimated state $(u_{N-1}, \hat{\delta}_{N-1})$ and the second part is a function of the error in the estimated state. (This property is not general; on the other hand, neither is it restricted to quadratic loss functions).

Now, since the second term is independent of the control

$$\underset{u_{N-1}}{MIN}\ E(F_N) = \underset{u_{N-1}}{MIN}\ E_D(\hat{F}_N)$$

$$= E_D\ \underset{u_{N-1}}{MIN}\ (\hat{F}_N)$$

(again assuming the interchange of expectation and differentiation).

But

$$\underset{u_{N-1}}{MIN}\ (\hat{F}_N) \Rightarrow \hat{u}_{N-1} = K_1\ \hat{\delta}_1 \tag{2.39}$$

where

$\hat{u}_{N-1}$ = optimal control

$\hat{\delta}_{N-1}$ = $E(\delta_{N-1}/D_{N-1})$

K_1 = the gain matrix for the N minus first control as evaluated for the deterministic system

D_{N-1} = the data vector for all epochs through that of N-1

This result states that the optimal control is a linear function of the optimal estimate of the state. However, the exact nature of the control as related to the statistical distributions of the data and the state has not been stated nor is it required. That is, $\hat{u}_{N-1}$ may be

$$\hat{u}_{N-1} = E(u_{N-1}/D_{N-1})$$

but this possibility has not been established here and is not required to define the optimal control.

To this point, many questions have been raised by the interchange of the expectation and minimization operations, even in the case where only one leg of the total trajectory was being considered. However, the problem is drastically compounded at this point since the results of the analysis must be generalized employing the Principle of Optimality. While this application appeals to reason, for the case where the total loss function is composed of a summation of terms which depend on single values of the state and control (i.e., these variables are uncoupled), it is not rigorous. Indeed, Striebel (Reference 2.17) has been concerned with this type of problem and has indicated that such an application, while possible for this type of function, is not possible for many cases. However, the principle can be applied as Gunckel (Reference 2.14) and others have done. This fact allows the results for the first leg of the stochastic process to be generalized exactly as were the results for the first leg in the deterministic process. The results are:

$$F^*_{N-n} = \Gamma^T_{n,m-1}\left[Q_n + P_{N-n}\right]\Gamma_{n,n-1} + \gamma_n \tag{2.40a}$$

$$K_{n-n+1} = -\left[F^*_{N-n}\right]^{-1}\Gamma^T_{n,n-1}\left[Q_n + P_{N-n}\right]\phi_{n,n-1} \tag{2.40b}$$

$$P_{n-n+1} = \phi^T_{n,n-1}\left[Q_n + P_{N-n}\right]\phi_{n,n-1} - K^T_{n-n+1} F^*_{N-n} K_{N-n+1} \tag{2.40c}$$

$$\hat{u}_n = K_{N-n+1}\hat{\delta}_N \tag{2.40d}$$

Thus, under the assumptions outlined, the optimal control process in the case of linear systems with quadratic cost is a linear function of the estimated state. That is, the optimum controller for this stochastic problem is obtained by cascading the optimal estimation and the deterministic controller.

The loss function (performance index) for the preceding discussions has now been optimized with respect to the controls applied at the correction epochs $t_0 \ldots t_N$. Thus, this index is a function only of these epochs, (the weighting matrices (Q_i , γ_i) are parameters of this problem and may be varied arbitrarily since no constraints have been employed. Thus, optimization with respect to these choices is also possible), i.e.,

$$J_N = \sum_{i=0}^{N} E\left[F_i(t_{i+1}, t_i)\right]$$

Now, if a table of values of F_i (t_{i+1}, t_i) is constructed for a grid of assumed times, the correction epochs can be optimized using Dynamic Programming by locating the path through the table which has the smallest possible cost. This optimization is accomplished by progressing <u>forward</u> in time with

$$I_n(t_N) = \underset{t_n}{MIN}\,[F_n(t_n, t_{n-1}) + I_{n-1}(t_{n-1})] \tag{2.41}$$

$$F(t_n, t_{n-1}) = \delta_{n-1}^T\,[\phi_{n,n-1}^T Q_n \phi_{n,n-1} + 2K_{N-n+1}^T \Gamma_{n,n-1}^T Q_n \phi_{n,n-1}$$
$$+ (\gamma_n + \Gamma_{n,n-1}^T Q_n \Gamma_{n,n-1})]\,\delta_{n-1}$$

$$I_o(t_o) = 0$$

A complete discussion of this type of solution is presented in any reference on Dynamic Programming. Thus, no further discussion will be devoted to this phase of the analysis at this time.

2.2.2.3.2 The Terminal State. In contrast to the equations relating the state at points along the trajectory to the initial state as presented in Section 2.2.2.2, the state at such points will also be a function of the estimation and application errors occuring at all of the preceding correction epochs. This fact can be shown by considering the state just prior to the second correction, i.e.,

$$\begin{aligned} \delta_1 &= \phi_{1,0}\,\delta_0 + \Gamma_{1,0}\,K_N\,(\delta_0 + \epsilon_0 + \psi_0) \\ &= (\phi_{1,0} + \Gamma_{1,0}\,K_N)\,\delta_0 + \Gamma_{1,0}\,K_n\,(\epsilon_0 + \psi_0) \\ &= A_{1,0}\,\delta_0 + B_{1,0}\,(\epsilon_0 + \psi_0) \end{aligned} \tag{2.42}$$

where

ϵ_0 = the estimation error at the time of the first correction

ψ_0 = the application error in the first correction

Similarly, the state just prior to the third correction will be

$$\begin{aligned} \delta_2 &= A_{2,1}\,\delta_1 + B_{2,1}\,(\epsilon_1 + \psi_1) \\ &= A_{2,1}\,A_{1,0}\,\delta_0 + A_{2,1}\,B_{1,0}\,(\epsilon_0 + \psi_0) + B_{2,1}\,(\epsilon_1 + \psi_1) \end{aligned}$$

or in general

$$\delta_n = \left[\prod_{i=1}^{n} A_{i,i-1}\right]\delta_o + \sum_{j=1}^{n}\left[\prod_{i=1}^{n} C_{i,i-1}\right]B_{N,n-j}(\epsilon_{n-j}+\psi_{n-j}) \tag{2.43}$$

where

$$C_{1,0} = I$$

$$C_{i,i-1} = A_{i,i-1}$$

But this equation can be rewritten by constructing a composit vector to represent the errors in this model. That is, if

$$D_j \equiv \left[\prod_{i=1}^{J} C_{i,i-1}\right] B_{n,n-j}$$

$$\nu_j = \epsilon_{n-j} + \psi_{n-j}$$

then

$$\sum_{j=1}^{n} D_j\, \nu_j = \bar{D}^T \bar{\nu}$$

where

$$\bar{D}_n^T = \left[D_1 \,\vdots\, D_2 \,\vdots\, D_3 \,\vdots\, \cdots \,\vdots\, D_n\right] \qquad \bar{\nu}_n = \begin{Bmatrix} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_n \end{Bmatrix}$$

and where the vector ν is uncorrelated with the true state δ_o, (since the estimation errors and the correction errors themselves are uncorrelated with the state). Under this substitution, δ_n becomes

$$\begin{aligned}\delta_n &= \left[\prod_{i=1}^{n} A_{i,i-1}\right]\delta_o + \bar{D}_n^T\, \bar{\nu}_n \\ &= G_n\, \delta_o + \bar{D}_n^T\, \bar{\nu}_n\end{aligned} \tag{2.44}$$

Now, since the expected value of δ_N over the entire ensemble of trajectories which might be flown is

$$E(\delta_n) = G_n\, E(\delta_o) + \bar{D}_n^T E(\bar{\nu}_n) = 0$$

the second moments can be written as

$$E(\delta_n - E(\delta_n))(\delta_n - E(\delta_n))^T = G_n P_o G_n^T + \bar{D}_n^T R_n \bar{D}_n \qquad (2.45)$$

where

$$P_o = E(\delta_o \delta_o^T)$$

$$R_n = E(\nu_n \nu_n^T)$$

2.2.2.3.3 Terminal Constraints in the Stochastic Problem. The procedure for applying constraints to the stochastic problem is a considerably more involved process than the comparable problem in deterministic systems. This difference arises from the fact that the estimation and control functions are contributing to the error in the terminal state. Thus, if a constraint of the form

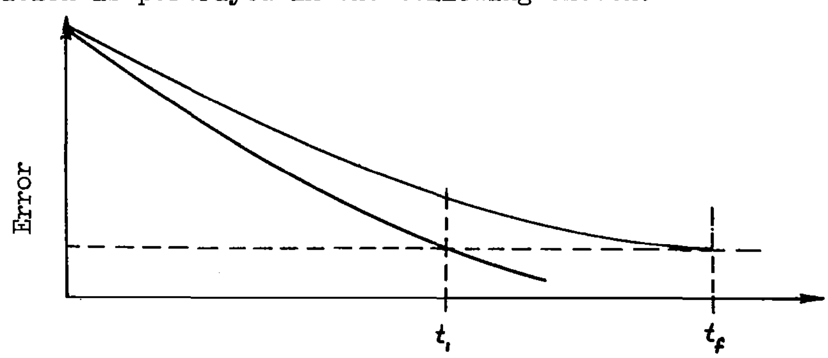
$$E(H \delta_N \delta_N^T H^T) \leq C_N \qquad (2.46)$$

(that is, if the error in some linear function of the state at the final point is constrained) is applied to the problem, a necessary condition for satisfying this constraint is that the function $H\delta_N$ be estimable to the required precision.

This apparently trivial statement can be utilized to advantage in a particular solution as follows: First, the estimation process is simulated to define the capability of the navigation loop and implications in so far as the amount of data and sensor accuracy required. After this requirement is satisfied, there will exist a range of times at which the state will be known well enough to satisfy the constraint

$$E(H \delta_N \delta_N^T H^T) \leq C_N$$

This situation is portrayed in the following sketch.



Thus, any time in the interval $[t_1, t_f]$ can be utilized for the purpose of adjusting three of the components of the terminal state. This correction, alone, may be adequate for the purpose at hand; however, if another correction is required at arrival to adjust the velocity, it can be made.

This restriction on the choice of the times at which the final corrections can be made, thus allows the inequality constraint to be satisfied. The control policy can now be optimized as before. That is, the gains for the control loop can be evaluated under the assumption the times for the corrections are known. Then, the times for the corrections can be optimized by applying Dynamic Programming to construct the smallest cost path through the series of corrections subject to the constraint that at least one correction be in the interval $[t_1, t_f]$. It is interesting to note that the resultant solution for this problem is not always the most apparent solution. That is, the n-stage control process can achieve a minimum (lowest expended control) without requiring that the final control itself attain a minimum, however, there will be a range of times in this interval over which the required control effort for the N-1st correction will increase continuously as time approaches t_f. Thus, in general, the optimum policy will be to null as much of the terminal error as is required at a time very close to the time denoted t_1. This fact can be employed to advantage in the construction of a nearly optimum initial solution. The alternative to this process is the inclusion of a series of undetermined multipliers in the criterion of performance which can be defined so as to satisfy the constraints. (This approach is taken in the discussion of minimum effort control in subsequent sections.) Unfortunately, however, the introduction of the multipliers and the fact that the mean value of the state on a given trajectory is non-zero compound the problem to the extent that a completely new gain structure is required each time an additional piece of data is required. This behavior makes constraints in many problems computationally out of the question with a more rigorous formulation.

2.2.3 Minimum Effort Control*

2.2.3.1 Introduction

The problem of maneuvering a space vehicle during the midcourse phase of a lunar or planetary transfer has been analyzed extensively in the literature during the past six or seven years. Probably the most complete treatment to date is the "minimum effort" control of Striebel and Breakwell (Reference 2.22) in which the "expected value" of the fuel to be used during the midcourse phase of the transfer is minimized, subject to a variance constraint placed on one component of the terminal state (e.g., the peri-apse altitude at the target planet). This analysis, which appeared originally in 1963, has been modified and extended in several subsequent papers (References 2.16 to 2.21). The purpose of these notes is to outline the minimum effort strategy for a vehicle with an impulsive propulsion system and to show how the theory can be used in an on-board control mechanization. The analysis closely parallels that given in Reference (2.21) and Section 2.8 of Reference (2.16).

Unlike most other optimal trajectory problems, the midcourse correction problem is stochastic in nature. If the vehicle was placed exactly on its design trajectory, then no midcourse maneuvering would be required. However, due to the inevitability of errors at injection, the vehicle's path deviates from the design condition. Furthermore, the extent of deviation is imperfectly known since any measurement made to determine the deviation will be contaminated with noise. Hence, the problem is not one of making some minimum fuel correction maneuver which will bring the vehicle to the correct terminal condition, but rather, of making corrections which will most probably minimize the fuel consumption, while at the same time keeping some statistical measure of the dispersion at the terminal point within reasonable bounds.

From considerations of fuel economy, all correction maneuvers should be made early in the flight, since small expenditures of fuel here will result in large changes in the terminal state. However, the trajectory errors which the correction maneuvers are to remove are known very poorly initially and with steadily improving accuracy as measurements and sightings are taken. Hence, corrections made early have a high probability of being wrong, while those made later are more accurate but require more fuel. It is this trade-off between terminal accuracy and fuel consumption that the minimum effort theory seeks to determine.

It should be mentioned that the optimal correction strategy resulting from an application of the minimum effort theory is <u>not</u> a strategy that would be used on board a vehicle during an actual planetary transfer. Rather, it serves as a design tool by which an "average" fuel requirement can be determined subject to a specified terminal accuracy condition. This shortcoming of the theory is due to the fact that in calculating the gains and correction times, no use is made of the actual sighting and measurement data which are

*This section of the report was prepared by J. E. McIntyre as a review of the work of J. V. Breakwell, C. Striebel et. al. (References 2.16 - 2.21).

gathered during the flight, but only of the types of measurements and the times at which they are to be made. However, as will be shown, it is relatively simple to extend the theory to include on-board mechanization; although, in this case, the problem becomes much more computationally involved.

2.2.3.2 Minimum Effort Control As A Preliminary Design Tool

2.2.3.2.1 Problem Statement. It is assumed that the actual trajectory of the vehicle is sufficiently close to the design condition to justify the use of the linear dynamic model

$$\dot{X} = AX + Gu \tag{3.1}$$

≠ (see footnote)
where X is an n dimensional state vector, A is an nxn time varying matrix, u is an r dimensional control vector and G is an nxr time varying matrix. Usually X will denote the vehicle's position and velocity, and hence, will be six dimensional (or four dimensional if the motion is restricted to be planar).

During the course of the transfer maneuver, observations and measurements are made in order to better estimate the state of the system. This operation is represented by the observation equation

$$y = Mx + \varepsilon \tag{3.2}$$

where y is an m dimension vector of observed minus computed residuals, M is an mxn time varying matrix, and ε is a Gaussian white noise with mean zero and covariance R (i.e., $E(\varepsilon(t)\varepsilon(\tau)) = R(t)\,\delta(t-\tau)$). These measurements may be made continously or at discrete instants of time.

The initial state of the system, X_0, is not known exactly. Rather, X_0 is assumed to be a Gaussian random variable with zero mean and covariance matrix V_0; that is

$$\left.\begin{aligned} E(X_0) &= 0 \\ E(X_0 X_0^T) &= V_0 \end{aligned}\right\} \text{ at } t = 0 \tag{3.3}$$

where E denotes the total expectation operator. The midcourse phase of the transfer is to terminate at a specified time T at which point the state (X(T)) is to satisfy a variance condition to be described next.

Now, let Z(T) denote any linear function of the terminal state which is to be constrained. i.e.,

$$Z(T) = HX(T) \tag{3.4}$$

≠ The state deviation will be expressed using the variable X rather than δ for this discussion to conform to the notation of the original references where H is a constant sxn matrix and where $s \leq n$. Three different types of terminal constraints will be considered:

(1) $TR \; E\{ZZ^T\} \leq C$; C is a scalar (3.5A)

(2) $(E\{ZZ^T\})_{ii} \leq C_i$; i = 1, s (3.5B)

(3) $(E(ZZ^T))_{ii} \leq C_i$; i = 1, s < s (3.5C)

In the first case, the symbol TR denotes the trace of the matrix $E(ZZ^T)$. Hence the sum of the diagonal elements of $E\{ZZ^T\}$ is required to be less than or equal to some number C. In the second case, the individual diagonal elements of $E\{ZZ^T\}$, that is, $(E(ZZ^T))_{ii}$ i = 1, s are required to satisfy inequality conditions while in case (3) only the first S_1 diagonal elements of $E\{ZZ^T\}$ are constrained. To clarify the physical meaning of these constraints, some examples will be considered.

These three possible forms for the constraint by no means exhaust the number of choices. Rather, they are introduced simply to indicate the physical situations that can be represented by constraints of the form of Equations (3.5A) to (3.5C). In the following development, it is only required that H be some constant matrix with dimensions less than or equal to nxn, where n is the number of components in the state vector (X).

But, in order to satisfy the terminal constraint of Equation (3.5), some control action will be required; that is, the vector u in Equation (3.1) must be non-zero over some portion of the flight. This control action will be measured by a number referred to as the characteristic velocity and will be given by the equation

$$\Delta V = \int_0^T |u| \, dt = \int_0^T \sqrt{\sum_{i=1}^{r} u_i^2} \, dt \tag{3.6}$$

For a chemical propulsion system, this number is a direct function of the fuel consumed during the maneuver and will vary as a function of both the vector u and the particular realization of the random variables X_o and ε. The problem to be considered is the determination of that control time history which satisfies the terminal constraint of Equation (3.5) and at the same time minimizes the expected value of the characteristic velocity; that is

$$J = E\left\{\Delta v\right\} = E\int_0^T \sqrt{\sum_{i=1}^{r} u_i^2} \; dt = MIN \tag{3.7}$$

Of the possible control functions u(t), attention will be focused only on those functions which correspond to impulsive velocity corrections; that is, the control u(t) must take the form

$$u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_r(t) \end{pmatrix} = \sum_{i=1}^{N} v(t)\, \delta(t-t_i) = \sum_{i=1}^{N} \begin{pmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_r(t) \end{pmatrix} \delta(t-t_i) \tag{3.8}$$

where $\delta(t-t_i)$ denotes the Dirac delta function; t_i denotes the times at which impulses are applied: N denotes the number of corrections; and $v(t)$ denotes the direction and magnitude in which corrections are to be made. This restriction [Equation (3.8)] is imposed on the physical grounds that the propulsion system will be of sufficient thrust level to allow the change in the state of the system to be accomplished in a time which is short compared to the total mission time. By substituting Equation (3.8) into Equation (3.7), the criterion function becomes

$$J = E\int_0^T |u|\, dt = E\sum_{i=1}^{N} |v(t_o)| = E\sum_{i=1}^{N} \sqrt{v_1^2(t_i) + v_2^2(t_i) + \ldots v_r^2(t_i)}$$

and the optimization problem becomes one of determining the number N, correction times t_i and vectors $v(t_i)$ which minimize this expression.

2.2.3.2.2 The Expectation Operator. In most papers dealing with stochastic systems, there is usually some ambiguity in regard to the meaning of the expectation operator E. This ambiguity results from the fact that the same symbol, namely E, is used to denote expectation conditioned on different types and amounts of information. To avoid this difficulty, the following symbol convention is adopted:

(1) The quantity $E(\xi)$ will denote the expected or average value of ξ, where the averaging is conducted over the random variables X_o and $\varepsilon(\tau)$ $\{all\ \tau \in (0,T]\}$ Thus

$$E(\xi) = \int_{-\infty}^{\infty} \xi(X_o, \varepsilon(\tau))\, p(X_o, \varepsilon(\tau))\, dX_o\, d\varepsilon(\tau) \qquad \{all\ \tau \in (0,T]\} \tag{3.9}$$

where

$$d\varepsilon(\tau) = d\varepsilon(\tau_1)\, d\varepsilon(\tau_2) \ldots \qquad \{all\ \tau_i \in (0, T]\}$$

Alternately, by use of the observation equation

$$y = Mx + \varepsilon$$

the random variable ε can be replaced by the random variable Y with

$$E(\xi) = \int_{-\infty}^{\infty} \xi[x_o, y(\tau)]\, p[x_o, y(\tau)]\, dx_o\, dy(\tau) \qquad \{all\ \tau \in (0, T]\} \tag{3.10}$$

and in what follows this alternate form [Equation (3.10)] will be used.

(2) The quantity $\overset{t}{E}(\xi)$ will denote the average value of ξ where the averaging is conducted over the variables X(t) and Y(τ) $\{all\ \tau\epsilon(t,T]\}$ Thus

$$\overset{t}{E}(\xi) = \int_{-\infty}^{\infty} p[x(t), y(\tau)]\ \xi[x(t), y(\tau)]\ dx(t)\ dy(\tau)\ \{all\ \tau\epsilon(t,T]\} \quad (3.11)$$

Note that x(t) is a random variable whose distribution can be calculated from a knowledge of the distributions of X_o and Y(τ) $\{all\ \tau\epsilon\ (0,t]\}$ and the control applied on $[o,t)$.

(3) The quantity $\overset{t}{E}\{\xi/\mathcal{Y}(t_1), \mathcal{U}(t_1)\}$ will denote the expected value of ξ where the averaging is conducted over the random variables X(t) and Y(τ) $\{all\ \tau\epsilon(t,T]\}$, but where the distributions of X(t) and Y(τ) are conditioned on the observations $\mathcal{Y}(t_i)$ and the control $\mathcal{U}(t_i)$ with

$$\mathcal{Y}(t_1) = y(t) \quad ; \{0 < t \leq t_1\} \quad (3.12)$$

$$\mathcal{U}(t_1) = u(t); \ \{0 \leq t \leq t_1\} \quad (3.13)$$

Thus

$$\overset{t}{E}(\xi/\mathcal{Y}(t_1), \mathcal{U}(t_1)) = \int_{-\infty}^{\infty} \xi[x(t), y(\tau)]\ p[x(t), y(\tau)/\mathcal{Y}(t_1), \mathcal{U}(t_1)]\ dx(t)\, dy(\tau) \quad (3.14)$$

$$\{all\ \tau\epsilon[t,T]\}$$

As will be observed, this nomenclature leaves much to be desired. However, it is consistent and will prove useful in what follows.

Before proceeding, it is well to establish one important identity. Since the probability density $P(X_1, X_2)$ can be written

$$p(x_1, x_2) = p(x_1/x_2)\, p(x_2) \quad (3.15)$$

it follows directly that

$$E\{f(x_1, x_2)\} = E\left\{E\{f(x_1, x_2)/x_2\}\right\} \quad (3.16)$$

where the first E symbol on the right operates on the variable X_2, and the second, on the variable X_1. For clarity, this expression could also be written as

$$E\{f(x_1,x_2)\} = E\{E\{f(x_1\ x_2)/x_2\}\} = \underset{x_2}{E}\{\underset{x_1}{E}\{f(x_1\ x_2)/x_2\}\} \qquad (3.17)$$

This result can be used to demonstrate

$$\begin{aligned} \overset{t_1}{E}\{\xi/\mathcal{Y}(t_1), \mathcal{U}(t_1)\} &= \overset{t_1}{E}\{\overset{t_2}{E}\{\xi/\mathcal{Y}(t_2), \mathcal{U}(t_2)\}/\mathcal{Y}(t_1), \mathcal{U}(t_1)\} \\ &= \underset{\mathcal{Y}(\tau)}{\overset{t_1}{E}}\{\overset{t_2}{E}\{\xi/\mathcal{Y}(t_2), \mathcal{U}(t_2)\}/\mathcal{Y}(t_1), \mathcal{U}(t_1)\} \qquad t_1 < \tau \leq t_2 \end{aligned} \qquad (3.18)$$

where as above, the first expectation in the right operates on the variables $\mathcal{Y}(\tau)\ \tau \in (t_1, t_2]$. This expression will be used again and again in the following sections.

2.2.3.2.3 The Filter Equations. For the state system

$$\dot{x} = Ax + Gu$$

with X_o a Gaussian random variable satisfying

$$E(x_o) = 0$$

$$E(x_o x_o^T) = V_o$$

and with the observation equation

$$y = Mx + \varepsilon$$

it has been shown by Kalman and others that the distribution of X(t), conditioned on the observations $\mathcal{Y}$ (t) and control $\mathcal{U}$(t), is Gaussian with mean $\hat{X}(t)$($\hat{x}(t) = \overset{t}{E}\{x(t)/\mathcal{Y}(t), \mathcal{U}(t)\}$) and covariance $v(t)\left(v(t) = \overset{t}{E}\{[x(t) - \hat{x}(t)]\,[x(t) - \hat{x}(t)]^T/\mathcal{Y}(t), \mathcal{U}(t)\}\right)$

Further, X(t) and V(t) satisfy differential or algebraic equations depending on whether discrete or continuous observations are taken. These equations are

(1) Continuous Observations

$$\dot{\hat{x}} = A\hat{x} + Gu + VM^TR^{-1}(y - M\hat{x})\ ; \quad \hat{x}(0) = 0 \qquad (3.19)$$

$$\dot{V} = AV + VA^T - VM^TR^{-1}MV \quad ; \quad V(0) = 0 \qquad (3.20)$$

(2) Discrete Observations

$$\left.\begin{aligned} \hat{x}(t_2) &= \Phi(t_2, t_1)\, \hat{x}(t_1) \\ V(t_2) &= \Phi(t_2, t_1)\, V(t_1)\, \Phi^T(t_2, t_1) \end{aligned}\right\} \quad \begin{aligned} &\text{between} && (3.21) \\ &\text{observations} && (3.22) \end{aligned}$$

$$\left.\begin{aligned} \hat{x}^+ &= \hat{x}^- + V^- M^T (R + M V^- M^T)^{-1} (y - M\hat{x}^-) \\ V^+ &= V^- - V^- M^T (R + M V^- M^T)\, M V^- \end{aligned}\right\} \quad \begin{aligned} &\text{across} && (3.23) \\ &\text{observations} && (3.24) \end{aligned}$$

The matrix Φ is once again the state transition matrix [Reference (2.23)] and satisfies

$$\frac{d\Phi}{dt}(t, \tau) = A(t)\, \Phi(t, \tau) \tag{3.25}$$

or

$$\frac{d\Phi}{d\tau}(t, \tau) = -\Phi(t - \tau) A(\tau) \; ; \quad \Phi(t, t) = I \tag{3.26}$$

The observation process, continuous or discrete, effects the algebra involved in reaching the final result, but not the result itself. However, for convenience, it will be assumed that continuous measurements are made with Equations (3.19) and (3.20) holding. The discrete case will be treated later on.

At any time (t) the conditional mean[$\hat{X}(t)$] is computed from Equation (3.19) and this value can be used to predict the terminal state which would result if no control action were applied on [t, T] . Let $\hat{X}(T,t)$ denote this predicted value. Then it follows that

$$\hat{x}(T, t) = \Phi(T, t)\, \hat{x}(t) \tag{3.27}$$

This predicted quantity plays a central role in the computation of the optimal control since the terminal variance constraint is:

$$z(T) = H\, x(T)$$

Let $\hat{Z}(T,t)$ denote the predicted value of Z(t) at time t which would result if no control were applied on the interval [t, T] . i.e.,

$$\hat{z}(T, t) = H \hat{x}(T, t) \tag{3.28}$$

In the next section, it will be shown that the optimal control to be applied at time t, u_{opt} (t), is a function only of the variable $\hat{Z}(T,t)$; that is

$$u_{opt}(t) = u(\hat{z}(T,t)) \quad .$$

2.2.3.2.4 <u>Functional Form of the Optimal Control</u>. Following the procedures of the Calculus of Variations, the problem of minimizing

$$J = E(\Delta V) = E\int_0^T |u|\, dt \tag{3.29}$$

subject to a terminal constraint on the quantity

$$E\left(z(T)\, z^T(T)\right) \quad ; \quad z(T) = H\, x(T)$$

is reduced to the problem of minimizing the modified functional

$$J_0 = E\left\{\Delta V + Tr\, \Lambda\, z(T)\, z^T(T)\right\} = E\left\{\int_0^T |u|\, dt + Tr\, \Lambda\, z(T)\, z^T(T)\right\} \tag{3.30}$$

where Λ is an SXS constant diagonal matrix of Lagrange multipliers selected so that the specified terminal variance condition is satisfied. The particular form of the matrix Λ will depend on the type of terminal constraint which is imposed. For example, if X is a six dimensional vector and H is the matrix

$$H = \begin{pmatrix} 1, 0, 0, 0, 0, 0 \\ 0, 1, 0, 0, 0, 0 \end{pmatrix}$$

the terminal constraint of Equation (3.5A) becomes

$$E\left\{x_1^2(T) + x_2^2(T)\right\} = E\left(z_1^2(T) + z_2^2(T)\right) \leq C$$

and the quantity to be adjoined to Equation (3.29) to form (3.30) is

$$\lambda E\left\{z_1^2(T) + z_2^2(T)\right\} = E\left\{\lambda\left[z_1^2(T) + z_2^2(T)\right]\right\}$$

which is equivalent to $E\{Tr\, \Lambda\, z z^T\}$ provided

$$\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

Alternately, if the terminal constraint of Equation (3.5B) is imposed, the quantity to be adjoined to Equation (3.29) is

$$\lambda_1 E z_1^2(T) + \lambda_2 E z_2^2(T)$$

which will equal $E \; Tr \, \Lambda \, z z^T$ if

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

The demonstration that the control u(t) which minimizes J_o in Equation (3.30) is a function only of $\hat{Z}(T,t)$ begins by resorting to the principal of optimality of Dynamic Programming which states

> "An optimal sequence of decisions in a multistage decision process problem has the property that whatever the initial stage, state and decision are, the remaining decisions must constitute an optimal sequence of decisions for the remaining problem, with the stage and state resulting from the first decision considered as initial conditions."

From this principle, it follows that if $u_{opt}(t) \{t \epsilon [0, T]\}$ is the control which minimizes J_o, then $u_{opt}(t) \{t \epsilon [t_i, T]\}$ must also minimize

$$\overset{t_i}{E}\left\{\int_{t_i}^{T} |u| dt + Tr \, \Lambda \, z z^T\right\} \tag{3.31}$$

where the distribution of $X(t_i)$ needed in evaluating this expectation is developed from the distributions of X_o and $Y(\tau)$ $\{\tau \epsilon (0, t_i]\}$ and from the control $u_{opt}(\tau)$ applied on the interval $[o, t_i)$. By using Equation (3.16), this equation can be rewritten as

$$\underset{\substack{y(\tau) \\ 0 < \tau \leq t_i}}{E} \left\{ \overset{t_i}{E}\left\{\int_{t_i}^{T} |u| dt + Tr \, \Lambda \, z z^T / \mathcal{Y}(t_i), \, \mathcal{V}(t_i)\right\}\right\} \tag{3.32}$$

Now, the control at any time t can depend only on $\mathcal{Y}(t)$ (that is, on the observations that have been made up to time t) since there is no other information on which the control can be based. From this condition, it follows that the control u which minimizes the expression in Equation (3.32) also minimizes

$$J_{t_i} = \overset{t_i}{E}\left\{\int_{t_i}^{T} |u| dt + Tr \, \Lambda \, z z^T / \mathcal{Y}(t_i), \, \mathcal{V}(t_i)\right\} \tag{3.33}$$

In summary then, if U_{opt} (t) for $t\epsilon[o,T)$ minimizes J_o in Equation (3.30), then from the principle of optimality, U_{opt} (t) for $t\epsilon\ [t_i, T]$ also minimizes the quantity J_{t_i} in Equation (3.33). For impulsive control action where

$$u(t) = \sum_{i=1}^{N} v(t)\,\delta(t-t_i)$$

the quantities J_o and J_{t_i} reduce to

$$J_o = E\left\{\sum_{i=1}^{N} |v(t_i)| + Tr\,\Lambda\, z(T)\, z^T(T)\right\} \tag{3.34}$$

$$J_{t_i} = \overset{t_i}{E}\left\{\sum_{j=1}^{N} |v(t_j)| + Tr\,\Lambda\, z(T)\, z^T(T)/\mathcal{Y}(t_i),\, V(t_{i-1})\right\} \tag{3.35}$$

Thus, if L_{t_i} is

$$L_{t_i} = \underset{\substack{u(t_k)\\ k=1,N}}{MIN}\ J_{t_i} \tag{3.36}$$

then the minimum value of J_o is given by

$$MIN\ J_o = \underset{\substack{y(\tau)\\ 0<\tau\le t_1}}{E}\left\{L_{t_i}\right\} \tag{3.37}$$

The demonstration that the optimal control, $v(t_i)$, is a function of $\hat{Z}$ (T, t_i) only, proceeds inductively as follows:

(1) Show that

$$v(t_N) = v(\hat{z}(T, t_N^-))$$

$$L_{t_N} = L_{t_N}(\hat{z}(T, t_N^-))$$

(2) Assume that

$$v(t_{i+1}) = v(\hat{z}(T, t_{i+1}^-))$$

$$L_{t_{i+1}} = L_{t_{i+1}}(\hat{z}(T, t_{i+1}^-))$$

(3) Show that

$$v(t_i) = v(\hat{z}(T, t_i^-)) \quad *$$

$$L_{t_i} = L_{t_i}(\hat{z}(T, t_i^-))$$

*The symbol $\hat{Z}(T, t_i^-)$ denotes the value of $\hat{Z}(T,t)$ at the time immediately preceding the ith impulse.

The proof parallels that given in References (2.17) and (2.32).

Now consider Equation (3.36)

$$S_{t_N} = \underset{\nu(t_N)}{MIN} \overset{t_N}{E}\left\{ |\nu(t_N)| + Tr\Lambda z(T) z^T(T) / \mathcal{Y}(t_N), \mathcal{V}(t_{N-1}) \right\} \tag{3.38}$$

But, as stated earlier, $\nu(t_N)$ can depend only on $\mathcal{Y}(t_N)$; hence, this quantity can be moved outside of the expectation operator to provide

$$S_{t_N} = \underset{\nu(t_N)}{MIN} \left\{ |\nu(t_N)| + \overset{t_N}{E}\left[Tr\Lambda z(T) z^T(T) / \mathcal{Y}(t_N), \mathcal{V}(t_{N-1}) \right] \right\} \tag{3.39}$$

where

$$z(T) = Hx(T)$$

or

$$z(T) = H\left\{ \Phi(T, t_N) x(t_N^-) + G\nu(t_N) \right\}$$

Now, the distribution of $X(t_N)$ conditioned on $\mathcal{Y}(t_N)$ and $\mathcal{V}(t_{N-1})$ is Gaussian with mean $\hat{X}(t_N^-)$ and covariance $V(t_N^-)$; thus it follows that the distribution of $Z(T)$, conditioned on $\mathcal{Y}(t_N)$ and $\mathcal{V}(t_{N-1})$, is also Gaussian with mean and covariance given by

$$m = \overset{t_N}{E}\left(z(T) / \mathcal{Y}(t_N), \mathcal{V}(t_{N-1}) \right) = \hat{z}(T, t_N^-) + HG\nu(t_N)$$

$$\overset{t_N}{E}\left\{ (z(T) - m)(z(T) - m)^T / \mathcal{Y}(t_N), \mathcal{V}(t_{N-1}) \right\} = H\Phi V \Phi^T H^T$$

But since H, Φ, V and G are known functions, it follows that the second term on the right of Equation (3.39) is a function only of $\hat{Z}(T, t_N^-)$ and $\nu(t_N)$. Performing the minimization over $\nu(t_N)$, then provides $\nu(t_N)$ as a function of $\hat{Z}(T, t_N^-)$ only; hence S_{t_N} is a function of $\hat{Z}(T, t_N^-)$

Assuming that $S_{t_{i+1}}$ and $\nu(t_{i+1})$ depend only on $\hat{Z}(T, t_{i+1}^-)$, the the quantity S_{t_i} can be written as

$$S_{t_i} = \underset{\substack{\nu(t_k) \\ k=i,N}}{MIN} E\left\{ \sum_{k=i}^{N} |\nu(t_k)| + Tr\Lambda z(T) z^T(T) / \mathcal{Y}(t_i), \mathcal{V}(t_{i+1}) \right\}$$

$$= \underset{\nu(t_i)}{MIN} \overset{t_i}{E}\left\{ |\nu(t_i)| + \underset{\substack{\nu(t_k) \\ k=i,N}}{MIN} |\nu(t_k) + Tr\Lambda z(T) z^T(T) / \mathcal{Y}(t_i), \mathcal{V}(t_{i-1}) \right\}$$

But the identity in Equation (3.18) transforms this equation to

$$S_{t_i} = \underset{\nu(t_i)}{MIN}\left[|\nu(t_i)| + \underset{\substack{y(\tau)\\ t_i < \tau \le t_{i+1}}}{\overset{t_i}{E}}\left\{\underset{\substack{\nu(t_k)\\ k=i,N}}{MIN}\ \overset{t_{i+1}}{E}\left[\sum_{k=i+1}^{N}|\nu(t_k)| + Tr\Lambda zz^T/y(t_{i+1}), \mathcal{V}(t_i)\right]/y(t_i), \mathcal{V}(t_{i-1})\right\}\right]$$

with the final result

$$S_{t_i} = \underset{\nu(t_i)}{MIN}\left\{|\nu(t_i)| + \underset{\substack{y(\tau)\\ t_i < \tau \le t_{i+1}}}{\overset{t_i}{E}}\left[S_{t_{i+1}}/y(t_i), \mathcal{V}(t_{i-1})\right]\right\} \tag{3.40}$$

Now, from the assumption $S_{t_{i+1}} = S_{t_{i+1}}(\hat{z}(T, t_{i+1}^-))$, the right hand side of Equation (3.40) can be evaluated once the conditional distribution of $\hat{Z}(T, t_{i+1}^-)$ is known as a function of $y(\tau)$ for τ on the interval $[t_i, t_{i+1}]$. This distribution will be evaluated next.

First note that the quantity $Z(T, t_{i+1}^-)$ is given by

$$\hat{z}(T, t_{i+1}^-) = H\Phi(T, t_{i+1})\,\hat{x}(t_{i+1}^-) \tag{3.41}$$

and that $X(t)$ for $t \in [t_i^+ t_{i+1}^-]$ satisfies the differential equation /Equation (3.19/

$$\dot{\hat{x}} = A\hat{x} + VM^TR^{-1}(y - M\hat{x}) = A\hat{x} + VM^TR^{-1}(M[x-\hat{x}]) + \varepsilon \tag{3.42}$$

with

$$\hat{x}(t_i^+) = \hat{x}(t_i^-) + G\,\nu(t_i) \tag{3.43}$$

Now, the distribution of $X(t)$ on $[t_i^+, t_{i+1}^-]$, conditioned on $y(t_i)$ and $\mathcal{V}(t_{i-1})$ is Gaussian since both X and ε are Gaussian, and since the equation is linear. Furthermore, it is simple to show that $\hat{Z}(T, t_{i+1}^-)$ is also Gaussian, with mean and covariance

$$m = \overset{t_i}{E}\left\{\hat{z}(T, t_{i+1}^-)/y(t_i), \mathcal{V}(t_{i-1})\right\} = z(T, t_i^-) + H\Phi(T, t_i)\,G\nu(t_i) \tag{3.44}$$

$$\overset{t_i}{E}\left\{(\hat{z}(T, t_{i+1}^-) - m)(\hat{z}(T, t_{i+1}^-) - m)^T/y(t_i), \mathcal{V}(t_{i-1})\right\} \tag{3.45}$$
$$= \int_{t_i}^{t_{i+1}} H\Phi(T, t)\,VM^TR^{-1}MV\Phi^TH^Tdt$$

Thus, the second term on the right of Equation (3.40) is a function of ν (t_i) and $\hat{Z}$ (T, t_i) only. Minimization now yields the desired result

$$\nu(t_i) = \nu(\hat{z}(T, t_i^-))$$

$$S_{t_i} = S_{t_i}(\hat{z}(T, t_i^-))$$

and the proof is complete.

2.2.3.2.5 Additional Simplifications. It was shown in the previous section that the control ν which minimizes

$$J = E\left[\sum_{i=1}^{N} |\nu(t_i)|\right] = E\sum_{i=1}^{N} \sqrt{(\nu(t_i))^2} \qquad (3.46)$$

is a function only of the quantity $\hat{Z}(T,t)$ [i.e., the predicted value of Z(t) conditioned on y (t)] and the requirement that no control is applied on [t, T] . However, even with this knowledge, the determination of the optimal control is very difficult. The usual procedure, at this point, is to make the following two simplifications:

(1) The optimizing criterion J in Equation (3.46) is replaced by the condition K where

$$K = \sum_{i=1}^{N} \sqrt{E\,\nu^2(t_i)} \qquad (3.47)$$

As remarked in Reference (2.16), it can be shown using the Schwartz inequality that

$$K \geq J$$

and hence, minimizing K provides an upper bound on the expected fuel required to successfully complete the midcourse maneuver.

(2) The control ν (t_i) is required to be a linear function of $\hat{Z}$ (T, t_i)

$$\nu(t_i) = -B_i\,\hat{z}(T, t_i) \qquad (3.48)$$

Thus, the optimization problem becomes one of determining the matrices B_i. Note that ν is an r dimensional vector and that Z is an s dimensional vector; hence, the matrices B_i are of dimension rxs.

By employing these two simplifications, the stochastic optimization problem can be reduced to deterministic form and, in certain cases, solutions can be developed.

2.2.3.2.6 Reduction to Deterministic Form. Under the condition that the control depends linearly on $\hat{Z}(T,t)$ with

$$v(t_i) = -B_i \hat{z}(T, t_i^-) \tag{3.49}$$

the optimizing criterion K in Equation (3.47) becomes

$$K = \sum_{i=1}^{N} \sqrt{E\{[B_i z(T,t_i)]^T [B_i z(T,t_i)]\}}$$

But,

$$(B\hat{z})^T (B\hat{z}) = TR\left\{\hat{z} z^T B^T B\right\}$$

Thus, it follows that

$$K = \sum_{i=1}^{N} \sqrt{TR\{E(\hat{z}(T,t_i)\, \hat{z}^T(T,t_i)\, B_i^T B_i\}} \tag{3.50}$$

Using Equations (3.44), (3.45) and (3.49) and the identity

$$E(\xi) = \underset{\substack{y(\tau) \\ 0 < \tau \leq t_i}}{E} \left\{ \overset{t_i}{E} \left\{ \xi / y(t_i),\ v(t_{i-1}) \right\} \right\}$$

it now follows that $\hat{Z}(T,t)$ is a Gaussian random variable with zero mean and with covariance satisfying the differential equation

$$\begin{aligned} \dot{p} &= H\Phi(T,t)\, V M^T R^{-1} M V \Phi^T H^T \\ p(0) &= 0 \end{aligned} \tag{3.51}$$

on intervals between impulses and the equation

$$p_i^+ = (I - H\Phi_i G B_i)\, p_i^- (I - H\Phi_i G B_i)^T \tag{3.52}$$

at the times t_i. Where

$$P_i = P(t_i)$$

$$\Phi_i = \Phi(T, t_i)$$

$$I = \text{Unit Matrix}$$

The constraint at the terminal point is now placed on the quantity $E(ZZ^T)$.

$$E(zz^T) = E\left\{\left(z-\hat{z}(T,T)\right)\left(z-\hat{z}(T,T)\right)^T\right\} + E\left\{z(T,T)\hat{z}^T(T,T)\right\}$$
$$= HV(T)H^T + P(T) \qquad (3.53)$$

But differentiation of Equation (3.53) and substitution of Equations (3.51) and (3.20), leads to the result that the quantity $H\Phi(T,t)V(t)\Phi^T(T,t)H^T + P(t)$ is a constant on invervals between switches. Hence, the terminal condition in Equation (3.53) reduces to

$$E(zz^T) = H\Phi_N V_N \Phi_N^T H^T + P_N^+ \qquad (3.54)$$

where

$$V_N = V(t_N),\ P_N^+ = P(t_N^+),\ \Phi_N = \Phi(T, t_N)$$

Collecting the results of Equations (3.50), (3.51), (3.52) and (3.54), it follows that the stochastic problem of minimizing the functional in Equation (3.47) is equivalent to the deterministic problem of minimizing

$$K = \sum_{C=1}^{U} \sqrt{Tr\, P_i^- B_i^T B_i} \qquad (3.55)$$

subject to the conditions

$$P_i^+ = (I - H\Phi_i GB_i)\, P_i^- (I - H\Phi_i GB_i) \qquad (3.56)$$

$$P_{i+1}^- = P_i^+ + Q_i \ ; \quad Q_i = \int_{t_i}^{t_{i+1}} H\Phi V M^T R^{-1} M V \Phi^T H^T dt \qquad (3.57)$$

$$P_0 = 0 \qquad (3.58)$$

and some terminal constraint [depending on the particular form of Equation (3.5) which is used] on the quantity

$$E(zz^T) = H\Phi_N V_N \Phi_N^T H^T + P_N^T \qquad (3.59)$$

2.2.3.2.7 Formulation Using Maxima-Minima Theory. The problem of minimizing the quantity K in Equation (3.55) subject to the constraints of Equations (3.56) to (3.59) is one that can be analyzed using the maxima-minima theory of the Differential Calculus. Following a procedure identical to that used in the development of Equation (3.30) [see paragraphs following Equation (3.30)], the modified function K_0 is formed where

$$K_0 = \sum_{i=1}^{N} Tr\, P_i^- B_i^T B_i + \sum_{i=1}^{N-1} Tr \left\{ \Lambda_i \left[P_{i+1}^- - Q_i - (I - H\Phi_i G B_i) P_i^- (I - H\Phi_i G B_i)^T \right] \right\} + Tr\, \Lambda_0 (P_1^- - Q_0) - Tr \left\{ \Lambda_N \left[H\Phi_N V_N \Phi_N^T H^T + (I - H\Phi_N G B_N) P_N^- (I - H\Phi_N G B_N)^T \right] \right\} \tag{3.60}$$

The Λ_i $i = 0,N$ are Lagrange multiplier (SXS) matrices which are symmetric since the constraints (specifically P and V) are sy metric. The matrix Λ_N is, in addition, diagonal with the number of different diagonal elements depending on the specific form of Equation (3.5) which is used. Also, since the constraints of Equation (3.5) are inequality constraints, the Kuhn-Tucker Theorem* can be used to demonstrate that the diagonal elements of Λ_N are less than or equal to zero. Thus,

$$\Lambda_N = \text{negative semi-definite matrix} \tag{3.61}$$

Setting the first variation δK_0 to zero with respect to variations in P_i and B_i (i=1,N) provides

$$\frac{P_i^- B_i^T}{\sqrt{Tr\, P_i^- B_i^T B_i}} + 2\left[\Lambda_i (I - H\Phi_i G B_i) P_i^-\right]^T H\Phi_i G = 0 \tag{3.62A}$$

$$\frac{B_i^T B_i}{\sqrt{Tr\, P_i^- B_i^T B_i}} + \Lambda_{i-1} - (I - H\Phi_i G B_i)^T \Lambda_i (I - H\Phi_i G B_i) = 0 \tag{3.26B}$$

But, these equations can be combined to yield

$$\frac{B_i^T}{\sqrt{Tr\, P_i^- B_i^T B_i}} = -2\Lambda_{i-1} H\Phi_i G \tag{3.63}$$

$$\Lambda_{i-1} = (I - H\Phi_i G B_i)^T \Lambda_i = 0 \tag{3.64}$$

*See Hadley, G., "Nonlinear and Dynamic Programming," Addison-Wesley (1963)

The solution of these equations in conjunction with the constraints of Equations (3.56) to (3.59) can be accomplished iteratively if the time t_i at which the impulses are to be applied are given. For example, if the terminal constraint of Equation (3.5A) is imposed with

$$\Lambda_N = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda_N & \vdots \\ 0 & \vdots\ldots & 0 \end{pmatrix} \qquad \lambda_N \leq 0$$

the following iterative scheme could be used:

(1) Guess λ_N

(2) Guess the elements of P_N^+ under the condition that the terminal constraint is satisfied. Since P is an SXS symmetric matrix and since the constraint of Equation (3.5A) is scalar, this would involve guessing $S\frac{(S+1)}{2}$ -1 quantities

(3) Determine P_N^-, B_N and Λ_{N-1} from a knowledge of P_N^+ , Λ_N and Equations (3.63), (3.64) and (3.56)

(4) Set i = N-1

(5) Compute P_i, B_i and Λ_{i-1} from a knowledge of P_{i+1}^- , B_{i+1}^- and Λ_i and using Equations (3.64), (3.63), (3.56) and (3.57)

(6) Test i=1 . If no, set i=i-1 and go to Step (5)

(7) Compute P_o from Equation (3.57)

(8) Test if $P_o \equiv 0$. If yes, solution has been achieved. Since P_o is an SXS symmetric matrix, the condition that $P_o = 0$ is equivalent to $\frac{S(S+1)}{2}$ scalar conditions. But from Steps (1) and (2), $\frac{S(S+1)}{2}$ conditions have been guessed to begin with. So if $P_o \neq 0$,

(9) Use the $\frac{S(S+1)}{2}$ conditions that $P_o \neq 0$ to correct the $\frac{S(S+1)}{2}$ guessed quantities and go back to Step (3).

In correcting the guessed quantities in Step (8), a first or second order method such as a gradient or Newton Raphson technique could be used. However, in such an approach, the rate of convergence and the solution developed depend on both the starting condition and the number S (P is an SXS matrix). For S=2, the problem could be solved without too much difficulty. For S > 2, a great deal of trouble should be expected as in the case of quadratic cost, once a solution has been achieved for a particular set of correction times t_i , i = i,N, additional iteration would be required to determine the optimal times. Hence, even for S=2, the problem appears rather formidable. On the other hand, for S=1 the solution can be developed in a rather straightforward manner as will be shown later.

2.2.3.2.8 Case of Discrete Observations. In the development to this point, it has been assumed that observations were taken continuously with the continuous filter equations, Equations (3.19) and (3.20) holding. An examination of the deterministic form of the stochastic problem which is given in Equations (3.55) to (3.59), indicates that the continuous filter assumption is used only in evaluating the quantity

$$Q_i = \int_{t_i}^{t_{i+1}} H\Phi V M^T R^{-1} M V \Phi^T H^T dt$$

But, this expression can be integrated to give

$$Q_i = H\Phi_i V_i M^T R^{-1} M V_i \Phi_i^T H^T - H\Phi_{i+1} V_{i+1} M^T R^{-1} M V_{i+1} \Phi_{i+1}^T H^T$$

If this expression is used to evaluate Q_i, then the deterministic problem of Equations (3.55) to (3.59) is the same for both continuous and discrete obsevations. Only the manner in which the covariance V is evaluated is different; in the continuous case, Equation (3.20) is used; Equations (3.22) and (3.24) are used in the discrete case.

2.2.3.2.9 One Dimensional Problem. In the special case in which H is a row vector, (i.e., S=1), the variables $\hat{Z}(T,t)$, P_i and Λ_i become scalars and the solution to the minimum fuel transfer takes a particularly simple form. For this case, the criterion function of Equation (3.55) and the constraints of Equations (3.56) to (3.59) reduce to

$$K = \sum_{i=1}^{N} \sqrt{P_i^- B_i^T B_i} \tag{3.65}$$

$$P_i^+ = P_i^- (1 - H\Phi_i G B_i)^2 \tag{3.66}$$

$$P_{i+1}^- = P_i^+ + Q_i \quad ; \quad Q_i = \int_{t_i}^{t_{i+1}} H\Phi V M^T R^{-1} M V \Phi^T H^T dt \tag{3.67}$$

$$P_o = O \tag{3.68}$$

$$E(zz^T) = H\Phi_N V_N \Phi_N^T H^T + P_N^T \leq C \tag{3.69}$$

where C in Equation (3.69) is some specified scalar.

As noted, the criterion function K is no longer the expected characteristic velocity, but rather a quantity which is greater than or equal to it. The expected characteristic velocity corresponds to the criterion function J which is given in Equation (3.46) by

$$J = E\left\{ \sum_{i=1}^{N} |v(t_i)| \right\}$$

Substituting into this equation the expression for the control

$$v(t_i) = -B_i \hat{z}(T, t_i) \qquad ; \; B_i = r \text{ dimensional vector} \tag{3.70}$$

provides the result

$$J = E\left\{ \sum_{i=1}^{N} \sqrt{\hat{z}(T, t_i)^2 B_i^T B_i} \right\} \tag{3.71}$$

Further, since $\hat{Z}(T,t_i)$ is a normal random variable with zero mean and co-variance P_i , it is a simple matter to show that

$$E\left\{ \sum_{i=1}^{N} \sqrt{z(T, t_i)^2} \right\} = \sqrt{\frac{2}{\pi}} \sum_{i=1}^{N} \sqrt{P_i^-} \tag{3.72}$$

Thus, substituting this expression into (3.71) yields

$$J = \sqrt{\frac{2}{\pi}} \sum_{i=1}^{N} \sqrt{P_i^- B_i^T B_i} \tag{3.73}$$

Note that for this case (S=1) the criterion function J (the expected characteristic velocity) differs from the criterion function K by a simple constant. This result leads to two conclusions:

(1) In the scalar case, the control v (t_i) which minimizes the criterion function K also minimizes the criterion J.

(2) In the scalar case, the quantity K exceeds the expected characteristic velocity by approximately 25%. Thus, K is not a very good approximation for J at least in the scalar case. Whether a similar situation exists in the vector case can not be concluded.

From Conclusion (1), it follows that the optimizing conditions of Equations (3.63) and (3.64) hold for both J and K. These equations reduce to

$$\frac{B_i^T}{\sqrt{P_i^- B_i^T B_i}} = -2 \Lambda_{i-1} H \Phi_i G \tag{3.74}$$

$$\Lambda_{i=1} = (1 - H\Phi_i GB)\Lambda_i \tag{3.75}$$

Now forming the dot product of Equation (3.74) with itself, and multiplying both sides by P_i yields

$$2\Lambda_{i-1} = \pm\frac{1}{\sqrt{P_i^-(H\Phi_i G)(H\Phi_i G)^T}}$$

But, since from Equation (3.61) Λ_N is negative, an examination of (3.62) indicates that all Λ_i are negative.

Hence

$$2\Lambda_{i-1} = \frac{-1}{\sqrt{P_i^-(H\Phi_i G)(H\Phi_i G)^T}} \tag{3.76}$$

Substitution of Equation (3.76) into (3.74) now provides

$$\frac{B_i^T}{\sqrt{B_i^T B_i}} = \frac{H\Phi_i G}{\sqrt{(H\Phi_i G)(H\Phi_i G)^T}} \tag{3.77}$$

Thus, control corrections are to be made in the $(H\Phi_i G)^T$ direction (i.e., the direction of maximum sensitivity) as indicated by the equation

$$\hat{z}(T, t_i^+) = \hat{z}(T, t_i)\left\{1 - H\Phi_i G B_i\right\}$$

Now let

$$D_i = \sqrt{(H\Phi_i G)(H\Phi_i G)^T} \tag{3.78}$$

and

$$B_i = \frac{k_i}{D_i}\left[H\Phi_i G\right]^T \tag{3.79}$$

Combining Equations (3.79), (3.75) and (3.76) provides

$$k_i = \frac{1}{D_i}\left\{1 - \sqrt{\frac{P_{i+1}^- D_{i+1}^2}{P_i^- D_i^2}}\right\} \tag{3.80}$$

But from Equations (3.79), (3.66) and (3.67)

$$P_{i+1}^- = Q_i + (1 - D_i k_i)^2 P_i^-$$

Thus, substituting this expression into Equation (3.80) yields

$$k_i = \frac{1}{D_i}\left\{1 - \sqrt{\frac{Q_i D_{i+1}^2}{P_i^- \; D_i^2 - D_{i+1}^2}}\right\} \tag{3.81}$$

Also, at the final impulse, Equations (3.65), (3.69) and (3.79) provide

$$k_N = \frac{1}{D_N}\left\{1 - \sqrt{\frac{C - H\Phi_N V_N \Phi_N^T H^T}{P_N^-}}\right\} \tag{3.82}$$

Collecting results, it follows that the control $v(t_i)$ is given by

$$v(t_i) = -B_i \hat{z}(T, t_i)$$

with

$$B_i = \frac{k_i}{D_i}(H\Phi_i G)^T \quad ; \quad D_i = \sqrt{(H\Phi_i G)(H\Phi_i G)^T} \tag{3.83}$$

and with the k_i computed using Equations (3.81) and (3.82). Finally, substitution of (3.83) into (3.73) yields the minimum expected value of the characteristic velocity as

$$J = \sqrt{\frac{2}{\pi}} \sum_{i=1}^{N} k_i \sqrt{P_i^-} \tag{3.84}$$

From the previous discussions, it is apparent that given the correction times t_i, the optimal solution can be developed directly without iteration. As for the determination of the optimal times, F. Tung in Reference (2.21) has developed a search routine based on Dynamic Programming by which these times can be readily computed.

2.2.3.3 On-Board Minimum Effort Mechanization for the One-Dimensional Problem

In the previous section, it was shown that the optimal control $v(t_i)$ is given by

$$v(t_i) = B_i \hat{z}(T, t_i^-) = -B_i H\Phi_i \hat{x}(t_i^-)$$

where

$$B_i^T = \frac{k_i}{D_i}(H\Phi_i G), \quad D_i = \sqrt{(H\Phi_i G)(H\Phi_i G)^T}$$

and where the gains k_i can be computed using Equations (3.81) and (3.82). Despite the apparent feedback form of these equations, they are not suitable for an "on-board" mechanization due to the fact that the gains k_i and the correction times t_i will vary as the particular realization of the random process varies. For example, in the analysis of Sections 2.2.3.2.7 and 2.2.3.2.9, the average value of $\hat{X}(t_i)$, where the averaging is conducted over all possible observations $Y(\tau)$ for $\tau \leq t_i$ $\{$i.e. $\underset{0 \leq \tau \leq t_i}{\underset{y(\tau)}{E}} \hat{X}(t_i)\}$, is zero. This result greatly simplifies the calculations. However, for a vehicle on an actual planetary trajectory, it would be unlikely that $\hat{X}(t_i)$, as calculated from the observational data (rather than the expected value of this data) would be zero, and if it where not, the gains and correction times would have to be changed accordingly.

One possible approach to the on-board mechanization problem would be to recalculate the quantities k_i and t_i as the random process itself unfolds; that is, after each observation is taken. As indicated in Section 2.2.3.2.9, these quantities can be computed quickly and easily. Hence, if observations are made at discrete instants and are sufficiently separated (say by a few minutes or more) then the flight computer could be used to adjust the correction schedule after each observation. Thus, an on-board control mechanization might take the following form:

(1) Take observation $Y(\tau)$ at time τ. Set $t_i = \tau$

(2) Compute the distribution of $\hat{Z}(T, t_i)$ conditioned on this observation, all previous observations and all control applied prior to this time.

(3) Note the time of next observation, say $t_1 + \Delta t$

(4) Compute optimum correction schedule on $[t_i, T]$ based on the distribution in Step (2) and the required terminal condition.

(5) Test to determine if a correction is to be made on the interval $[t_1, t_1 + \Delta t]$. If yes, make the appropriate correction.

(6) Take next observation $Y(\tau + \Delta t)$. Set $t_1 = t_1 + \Delta t$

(7) Go to Step (2).

The feasibility of this scheme would depend on the flight computer's ability to accomplish this process. In Section 2.2.3.6, it was shown that this is a relatively simple task in the one dimensional problem when the random variables $\hat{X}(t)$ and $\hat{Z}(T,t)$ have zero mean value. However, in the non-zero mean case, the calculations are considerably more difficult, as will be shown next. For convenience, the time interval $[\tau, T]$ indicated in the stepwise procedure above is taken as $[0, T]$.

Consider the problem of minimizing the expected value of the fuel

$$J = E\int_0^T |u|\,dt$$

subject to the scalar terminal constraint

$$E(zz^T) \leq C \quad , z = Hx(T)$$

The observation equation is again given by

$$y = Mx + \epsilon$$

and the state system by

$$\dot{x} = Ax + Gu$$

but now, the initial state X_o is a Gaussian random variable with non-zero mean

$$Ex_o = \hat{x}_o$$

and covariance

$$E\left\{[x_o - \hat{x}_o][x_o - \hat{x}_o]^T\right\} = V_o$$

As in the zero mean case, the impulsive controller is assumed. i.e.,

$$u(t) = \sum_{i=1}^{N} v(t)\,\delta(t - t_i)$$

Thus, it can be shown that

$$v(t_i) = v(\hat{z}(T, t_i))$$

Again, letting

$$v(t_i) = -B_i\,\hat{z}(T, t_i)$$

it is relatively straightforward matter to show that

$$J = E\sum_{i}^{N} \sqrt{\hat{z}(T, t_i^-)^2 B_i^T B_i} \tag{3.85}$$

where $\hat{Z}(T,t_i)$ is a Gaussian random variable with mean m_i and covariance P_i satisfying

$$P_0 = 0$$
$$m_0 = H\Phi(T,0)\hat{x}_0 \tag{3.86}$$

$$m_i^+ = m_i^-(I - H\Phi_i G B_i)$$
$$P_i^+ = P_i^-(I - H\Phi_i G B_i)^2 \tag{3.87}$$

$$P_{i+1}^- = P_i^+ + Q_i$$
$$m_{i+1}^- = m_i^+ \tag{3.88}$$

Hence, the terminal constraint takes the form

$$E(zz^T) = H\Phi_N V_N \Phi_N^T H^T + P_N^T + (M_N^+)^2 \leq C \tag{3.89}$$

If the initial mean (m_o) is zero, then from Equation (3.87) $m_i = 0 \quad i=1,N$. In this case

$$E\sum^{N}\sqrt{\hat{z}(T,t_i^-)^2} = \sqrt{\tfrac{2}{\pi}}\sum_{i=1}^{N}\sqrt{P_i^-} \tag{3.90}$$

and the problem becomes that treated in Section 2.2.3.2.9. For the non-zero mean case, however,

$$J = E\sum_{i=1}^{N}\sqrt{(z(T,t_i^-))^2} = \sum_{i=1}^{N}\left[\left\{\sqrt{\tfrac{2}{\pi}P_i^-}\; e^{\frac{[m_i^-]^2}{2P_i}} + m_i^- \,\mathrm{erf}\left[\frac{m_i}{2P_i^-}\right]\right\}\sqrt{B_i B_i^T}\right] \tag{3.91}$$

where erf [.] denotes the error function. The minimization of this quantity subject to the constraints of Equations (3.86) to (3.89) is considerably more difficult than that experienced in the zero mean case of Equation (3.90) even though it can be shown that the matrices B_i again satisfy

$$B_i^T = -\frac{k_i}{D_i}\,(H\Phi_i G) \tag{3.92}$$

The difficulty arises from the fact that the computation of the gains must be conducted iteratively. Thus, an on-board mechanization which is based on a repeated and rapid solution of this problem does not appear to be computationally feasible.

On the other hand, if the criterion function, J in Equation (3.85), is replaced by the criterion function K where

$$K = \sum \sqrt{E z(T,t_i)^2 B_i^T B_i} \tag{3.93}$$

the optimization problem can be solved directly. While the resulting control does not minimize the expected characteristic velocity, it does minimize a quantity which is always greater than or equal to the expected characteristic velocity and this should be adequate. Letting $\omega_i = p_i + m_i^2$ the criterion function becomes

$$K = \sum \sqrt{\omega_i^- B_i^T B_i}$$

with the constraints

$$\omega_0 = H\Phi(T,0)\hat{x}_0$$

$$\omega_i^+ = \omega_i^-(1 - H\Phi_i G B_i)^2$$

$$\omega_{i+1}^- = \omega_i^+ + Q_i$$

$$H\Phi_N V_N \Phi_N^T H^T + \omega_N^T \leq C$$

This is exactly the problem treated in Section 2.2.3.2.9 and the solution developed there holds when the variable P is replaced by ω . Thus, an on-board mechanization which was not feasible using the criterion function J, is relatively simple for the criterion function K.

2.2.4 A Unified Approach to Statistical Optimization

2.2.4.1 Introduction

It is quite apparent that within space flight technology there exists a set of significant problems which must be classified as problems in statistical optimization; therefore, space flight technology must necessarily encompass methods of statistical optimization. Of particular concern are the problems of optimum navigation and guidance procedures which must be considered on a statistical basis since the natural environment is characterized by random phenomena whose effects can be significantly deleterious to mission success if they are ignored. Thus, it becomes highly desirable to establish either a general theory of (or, at a minimum, a unified approach to) statistical optimization. Of these alternatives, a general theory is the more desirable since, supposedly, all that can be generally known would be contained within the theory. However, this objective represents a rather significant endeavor which characteristically yields results that remain somewhat obscure and which are of doubtful direct usefulness in the development of methods of solution whose ultimate purpose is the application to problems subject to constraints on the state and/or the control. Thus, from the standpoint of space flight technology, it appears that the preferred approach is the formulation of a unified treatment of statistical optimization which directs most of the attention to those aspects of the problem which are of direct interest. In this sense, the unified approach is a general theory of a class of problems with particular characteristics. Obviously, the development of a unified approach is a far less ambitious undertaking than the development of a general theory; however, the former can prove to be far more fruitful than the latter when measured in terms of results directly applicable to the problems involved in the application of methods.

Of course, a significant amount of effort has been devoted to methods of statistical optimization and their application to navigation and guidance (i.e., the problems of optimum estimation and control, respectively). These efforts, however, have not produced a unified approach to the problem since the significant aspects of the problems encountered in application are not clearly understood (in some cases these aspects are not included in the formulation). Rather, each effort is characterized by a set of restrictive assumptions which precludes, at the outset, the ability to analyze the effects of certain unavoidable discrepancies in the actual utilization of the methods. Usually, efforts directed to removing certain restrictions require the imposition of others to be successful. The resulting situation is that a set of singular problems is considered, each of which is distinguished by a particular combination of the restrictions which define the set. Without doubt, the efforts to date represent a significant and meaningful contribution to the problems of statistical optimization; however, if further progress is to be made, then efforts must be directed toward a unified approach which encompasses present results and extends the methods of statistical optimization over a domain which contains all problems of particular concern in their least restrictive form.

Unfortunately, such a unified approach is not presently available; however, from preliminary studies concerning the possibility of a unified approach, it can be concluded that such an approach is highly feasible and lies within the general principles of statistical decision theory. In the following discussion, the formulation of a unified approach to statistical optimization with primary application to space flight technology is considered. The primary purpose of the discussion is to set forth the framework from which the unified approach should evolve and to assess the results which should be obtained or can be expected with reasonable certainty. It will be shown that the probability of success is sufficiently high and the return sufficiently important to warrant efforts which are directed toward the development of a unified approach to these problems as opposed to continued efforts intended to develop particular solutions to restricted problems. This conclusion is based upon the fact that the latter effort will produce very little in the way of new and/or significant results (i.e., the results will be variations of the previous results).

2.2.4.2 General Considerations

There exist several general considerations of a unified approach to statistical optimization as applied to space flight which are of particular concern. These considerations will be discussed at the outset since they can be interpreted as general requirements for a unified approach; in these discussions, the basic precept is that the problem of optimum control, as contrasted with that of optimum estimation, is the objective of the data collection and processing functions.

Present efforts directed to the solution of the problems of statistical optimization view the problems of navigation and guidance as distinct. These efforts usually refer to these processes as optimal estimation and control, respectively. In this terminology, the estimation problem is concerned with the determination of the optimum estimate of the system state, given a control, as a function of a set of available observations which are functions of the state and unknown random phenomena (e.g., measurement errors). The control problem is concerned with the determination of the optimum control for the system, given the optimum estimate of the state. Thus, the optimum estimate and control of the state are usually determined separately, i.e., one is determined, given the other. In the actual situation, observation data are available and it is sufficient for most problems to determine the optimum control as a function of the available data. In these problems, it is essentially irrelevant if the optimum control is a function of the optimum estimate of the state. That is, the optimum estimate is necessary if, and only if, it is required to determine the optimum control. However, it is not generally necessary to determine the optimum estimate to determine the optimum control even if the latter can be shown to be an explicit function of the former since, if $u^* = f[\delta^*(D)]$, then $u^* = g(D)$, (where u^*, δ^*, and D denote the optimum control, optimum estimate of the state and observation data, respectively); hence, the optimum control is an explicit function of D and the determination of $\delta^*(D)$ is not essential to determine u^*.

In the search for optimum solutions, particular emphasis is usually placed upon determining solutions which are linear functions of the observation data. Further, in the case of the optimum control problem, a linear function of the optimum estimate of the state is usually sought. The primary motivations for these steps are, first, linear solutions are generally easier to mechanize and, second, it is usually easier to obtain solutions under restrictions of linearity. However, the resulting solutions can not generally be considered as optimum solutions since the optimum solution is a member of the class of both linear and nonlinear functions and since this optimum will generally not be contained within the sub-class of linear functions. Thus, the "optimum" solution which is restricted to a linear function is actually a "sub-optimum" solution. This "sub-optimum" solution may be adequate for certain problems; however, it is not generally known how "sub-optimum" a "sub-optimum" solution is. Thus, it is conceivable that a "second-order" approximation to the optimum solution will result in a greatly reduced total cost. This fact leads to the conclusion that the adequacy of a sub-optimum solution cannot be assessed unless information is available concerning the structure of the optimum solution. But, if this structure is known, any approximation to the optimum solution can be assessed and an optimum design can be achieved where mechanization is included as a trade-off parameter along with the usual performance criteria.

In problems where an optimum solution is sought there must exist some criterion of optimality, i.e., a measure of certain significant parameters of the problem which is to be extremelized. For all problems of interest, this measure is a monotonically non-decreasing function of either the system state, the control, the estimation error or any combination of these parameters. (The literature most frequently defines this measure as a monotonically increasing function of the quadratic forms of the state, the control and the estimation error vectors). This function is referred to as a "loss" function and is denoted by $L(\delta, \epsilon, u)$. The customary approach to the stochastic problem has been to define the criterion of optimality using the same loss function as that used for the deterministic case but "averaged" using some "suitable" averaging operation, (usually the total expectation). The optimum solution is then defined as the set of parameters which minimizes this averaged value of the loss function (e.g., in the optimum control problem, the control which minimizes the expected value of the loss function is the optimum control.) In general, the optimum solution is the set of parameters for which the minimum expected loss is achieved and the solution is specified by

$$MIN\left\{E\left[L(\delta,\epsilon,u)\right]\right\} = \bar{L}(\delta^*,\epsilon^*,u^*) = \bar{L}^*$$

where E ( ) denotes the expection operation and * denotes the optimum solution. This cost is a function of functions of δ , ϵ and u(again, the cost functions are usually quadratic forms of δ , ϵ , and u).

It is obvious that the exact form of the loss function must be some appropriate measure of the parameters and must properly reflect their effects upon mission success. Furthermore, it is required that minimization of the expected value of the loss function is compatible with or assures the probability of mission success within the design goals or mission objectives.

It becomes apparent that there exist questions concerning optimum solutions which can be relegated to the following aspects of the problem:

(1) The exact form of the loss function as a function of the system state (δ), state estimation error (ϵ) and system control (u).

(2) The adequacy of the optimum solution in terms of achieving required system states within control constraints (e.g., achieving terminal states within an allowable total velocity correction). That is, the optimum control must also be sufficient in terms of mission accomplishment with the desired probability of success.

(3) The minimization of the expected value of the loss function as a fundamental criterion for an optimum solution.

(4) The use of sub-optimum solutions which are restricted to linear functions without information to assess their performance or results relative to the optimum solutions or better approximations to the optimum solution.

Based upon these comments, it can be concluded that the primary objective of the unified approach to statistical optimization should be the development of a general structure of the solution capable of reflecting the interrelationships of those aspects of the problem that are of particular concern in space flight technology.

2.2.4.3 A Synopsis of Present Results

Considerable effort has been devoted to problems of statistical optimization with particular applications to space flight. This class of problems is concerned with the optimum control where the fuel used for velocity corrections is an essential consideration in the optimum solution. A review of present results is given below for the primary purpose of developing an understanding of the problem in its most general and least restrictive formulation.

The majority of effort has been devoted to problems wherein the optimum solution is specified by the minimization of the expected value of quadratic loss functions. The most general form of this problem, which has been considered, is the optimum control problem with a loss function which is a linear function of the quadratic forms of the system state and control vectors, δ and u, respectively; i.e.,

$$L(x,u) = \sum_{i=1}^{N} \left(\delta_i^T Q_i \delta_i + u_{i-1}^T \gamma_i u_{i-1} \right)$$

where a discrete sequence of N control points at times t_i is assumed and Q_i and γ_i are symmetric, positive definite matrices for all i. The optimum control is that for which $E[L(\delta,u)]$ is a minimum. The solution for this problem is given in section 2.2.2.3 for the general linear system, i.e., $\delta_{i+1} = \phi_i \delta_i + \Gamma_i u_i$ (where ϕ_i and Γ_i are independent of the system state δ for all i). The determination of the optimum solution is facilitated by the linearity of the loss function in terms of the quadratic forms of δ and u. (That is, the loss function does not contain cross products of the states and controls or of states (controls) at different points along the trajectory). This particular loss function will be referred to as a "linear-quadratic" loss function since it is linear in the quadratic forms of δ and u. In this case, the optimum control was shown to be a linear function of the optimum estimate of the state; moreover, the relationship between the optimum control and the optimum estimate of the state was shown to be the same as that for the deterministic case. This result is independent of the statistical distributions involved. However, this independence does not imply that the optimum control is a linear function of the observation data; rather, the optimum control is a linear function of the conditional expectation of the state, given the observation data (the conditional expectation can be a nonlinear function of the observation data). On the other hand, if the statistical distributions are Gaussian, then the optimum estimate and, hence, the optimum control are linear functions of the observation data. These results are rather significant; however, in the midcourse guidance problem a direct measure of velocity corrections is more significant than the quadratic form of u; thus, a more meaningful loss function would be written as

$$L(\delta, u) = \sum_{i=1}^{N} \left(\delta_i^T Q_i \delta_i + \sqrt{u_{i-1}^T \gamma_i u_{i-1}} \right)$$

The expected value of this loss function contains the expected value of the total velocity correction for the special case of $\gamma_i = I$, i.e.,

$$E\left[L(x,u)\right] = E \sum_{i=1}^{N} \delta_i^T Q_i \delta_i + E \sum_{i=1}^{N} \sqrt{u_{i-1}^T u_{i-1}}$$

$$= \sum_{i=1}^{N} E(\delta_i^T Q_i \delta_i) + \sum_{i=1}^{N} E|u_{i-1}|$$

$$E\left[L(x,u)\right] = \sum_{i=1}^{N} E(\delta_i^T Q_i \delta_i) + E(\Delta V)$$

where

$$E(\Delta V) = E \sum_{i=1}^{N} \sqrt{u_{i-1}^T \; u_{i-1}}$$

denotes the total expected velocity requirement. The latter loss function, which is nonlinear in the quadratic form of u, is more meaningful than the linear-quadratic loss function since the total velocity correction is of direct concern in the design of a space flight mission. Unfortunately, an explicit solution for the "nonlinear" loss function cannot be readily obtained (as was the case for the "linear" loss function) due to the nonlinearity in the quadratic form of u. Moreover, the effect of the specific form of the loss function is not indicated in the known solution for the "linear" loss function. Thus, the effect of a change in loss function upon the optimum control cannot be explicitly stated; furthermore, the effect upon $E(\Delta V)$ of using the optimum control for the "linear" loss function is not known since it is difficult to extrapolate knowledge from the solution for a particular problem to more general cases.

Recent efforts have been devoted to solving the optimum control problem where the formulation of the problem is augmented in two significant aspects. First, constraints are imposed upon certain statistical parameters of the terminal state, i.e., a constraint is imposed upon the covariance matrix of specified components of the terminal state vector. Second, modified forms of the quadratic loss function which more closely represent the expected velocity correction are used. (See References 2.16 through 2.21). The results of these efforts represent a significant contribution toward extending the solution of the optimum control problem to a more meaningful formulation with respect to the midcourse guidance problem. However, the extensions are generally possible only through the artifice of certain restrictions which are often subtle but yet quite significant. Nonetheless, the results of these efforts promote an understanding of the optimum control problem as it applies to space flight technology. The most significant extension of the optimum control problem is referred to as "The Theory of Minimum Effort Control" which is discussed in section 2.2.3.

The essential difference between the linear-quadratic loss problem considered in Section 2.2.2.3 and minimum effort control is the nature of the loss function and the imposition of a terminal constraint. In the basic problem considered (see Reference 2.18), the optimum control is specified by minimizing the modulus of the expected value of velocity corrections with a constraint imposed upon the terminal state covariance matrix. The loss function for this case is a function of the control vector u only, i.e., in continuous form,

$$L(u) = \int_0^T \sqrt{E[u^T(t)u(t)]}\, dt$$

where the prime denotes transpose. The optimum control is defined as that which minimizes L(u) for a specified value of the covariance matrix of some linear function of the terminal state. It is shown that if the system dynamics and the observations are linear in the system state, and if the optimum control is a linear function of the observations, then the statistical optimization problem can be transformed into a deterministic optimization problem. The deterministic problem can be solved by the method of the maximum principle. This approach to the problem represents a rather provacative method of solution, however, the following comments are in order.

First, it is seen that the loss function is not a direct measure of the expected total velocity correction, instead, the loss function is the integral of the standard deviation, i.e., square root of the variance, of the commanded accelerations. In the case of a single control component or in the absence of random phenomena, the loss function becomes a direct measure of the expected total velocity correction which is of direct concern. In the more general case, the loss function bounds the expected total velocity correction. It is also seen that the loss function is not an explicit function of the state vector $\delta(t)$. (That is, the criterion of optimality is explicitly independent of the system state behavior intermediate to the terminal state). Thus, it appears that the form of the loss as a function of the control vector is motivated by mathematical considerations rather than physical ones. The adequacy or validity of the loss function in the general case is not clearly known.

Second, the "optimum" control is restricted to a linear function of the observations; therefore, the "optimum" control is the optimum <u>linear</u> control and can be sub-optimum in the class of all possible controls. On the other hand, the results of Reference 2.17 show that for linear system dynamics and Gaussian random phenomena, the optimum linear control is a linear function of the optimum estimate of the terminal state, which is itself a linear function of the observations; moreover, these results are established for a more general loss function than that considered in minimum effort control. That is, in Reference 2.17 the terminal state is explicitly included in the loss function which is not necessarily quadratic in control and state vectors. However, it cannot be concluded that the optimum control is linear in the observations for the general case. Rather, it is established that the "optimum" control of the class of controls which are linear in the observations is a linear function of the optimum estimation of the terminal state to be controlled. Whereas this is a significant result, care must be taken not to generalize erroneously. It should be noted that the results of Reference 2.17 are more general in terms of the loss function, as contrasted with linear-quadratic loss; however, the results are more restrictive than those for the case of a linear-quadratic loss function. In the latter case, linearity between the optimum control and the estimate of state was established without regard to the statistical distributions involved.

Third, the solution for the optimum control is dependent upon the behavior of the optimum estimate of the terminal state. In the formulation of the problem, it is assumed that the covariance matrix of the state estimation error reduces to zero at the terminal time. This is equivalent to assuming that at the terminal time all uncertainty is removed concerning the state. This assumption might be realistic for many cases since the relative errors can be quite small. However, this is not rigorous unless a sufficient set of perfect observations are available or a large number of uncorrelated observations are effectively filtered prior to the terminal time. Usually, the state uncertainty cannot be reduced to "zero" although it can be made arbitrarily small in a finite time under certain restrictions, e.g., a large number of observations with uncorrelated errors. It appears that the formulation of the problem does not require the assumption of "zero" uncertainty at the terminal time; however, the solution becomes more difficult to obtain. At present, the effect of terminal state uncertainty is not clearly known. It should be noted that the terminal state uncertainty is dependent upon the time correlation between observation errors and this effect is not included in the solution for the state estimation error, therefore, this effect upon the optimum control is not known.

Fourth, the formulation of the solution for the optimum control utilizes the optimum estimate (this estimate is a function of a' priori information concerning the state). However, even in the case when no a' priori information is available, the problem of optimum control still exists. In this case the solution must be a function of the a' posteriori information. This case is not included in the formulation of minimum effort control.

Fifth, in applying the maximum principle there exists the question of uniqueness of the solution. That is, in the general case, it is not known whether there exists a unique set of initial values for the adjoint variables which yield the specified terminal conditions. However, for the special case of a single terminal state component or a single linear combination of the state components being specified, the solution is shown to be unique. Uniqueness for the general case has not, as yet, been established.

Finally, several significant extensions to the basic results (given in Ref. 2.17 and 2.18) for minimum effort control have been made (See Refs. 2.20 and 2.21). In these extensions the loss function is defined as the expected total velocity correction rather than the integral of the standard deviation of the commanded acceleration. This modification results in a more meaningful loss function from the standpoint of design considerations for reaction control devices . However, the problem considered is restricted to that in which a constraint is applied to a single linear function of the terminal state. Also, the optimum control is restricted to be a linear function of the observations. It is further assumed that the optimum control is a linear function of the predicted terminal value function which is to be controlled. In Reference 2.17 this assumption is shown to be valid for linear system dynamics and Gaussian random phenomena. For other cases, the optimum <u>linear</u> control is not known to be the optimum of all controls. It should be pointed out that a nonlinear control for the same problem is considered in Reference 2.16, however, it is assumed that the nonlinear control is a function of the optimum estimate of the terminal state. (The results of Reference 2.17 indicate that this assumption is valid for linear dynamics and Gaussian distributions where the loss function includes only the terminal state).

2.2.4.4 Rudiments of a Unified Approach

For the purposes of space flight navigation and guidance, the ultimate objectives of a unified approach to statistical optimization can be summarized as follows:

(1) Determine the structure of the optimum solution in a sufficiently general form to show the interrelationships of the significant aspects of the problem.

(2) Evaluate the performance of the optimum solution with respect to the significant aspects of the problem in order to identify the critical aspects.

(3) Perform comparative evaluations of the performance of optimum and sub-optimum solutions such that the best sub-optimum solution can be selected when necessary.

The attainment of these objectives, however, requires that the significant aspects of space flight problems be considered. Some of the most important of these aspects for the present study are:

(1) The nature of the dynamics of the system (i.e., linear or non-linear) and/or adequacy of linear models.

(2) The characteristics of the random phenomena involved (i.e., type of distributions, Gaussian or otherwise), and the information required of the parameters which specify the distributions.

(3) The criteria of optimality (i.e., type of loss function used) and the minimization of the expected loss.

(4) Limitations in mechanization of procedures (i.e., finite data processing capabilities, word length, storage and time) and in control execution errors.

(5) In-flight system constraints as to number and type of available observations and/or the number and magnitude of the corrective actions.

The final result of the unified approach is a system configuration definition which is optimum in the "overall" sense and which embodies the best solution, optimum or sub-optimum, in accordance with the constraints of reality. Optimum in the "overall" sense implies that criteria of optimality and/or constraints include additional parameters than those usually considered in the mathematical formulation of the optimum problems. For example, in the use of a linear control, it is tacitly assumed that such a control is "optimum" where mechanization simplicity is considered as a criterion of optimality or perhaps as a constraint. However, the validity of this assumption cannot be established unless knowledge concerning the performance of "optimum" linear controls is available. That is, the overall optimum system configuration would embody a non-linear control if superior performance could be achieved within an allowable and/or tolerable increase in mechanization complexity. Indeed, it is not unusual to find that an unacceptable degree of complexity yields more than commensurate performance or results; for example, in methods of finite differences, quadrature, iterative solutions and parameter estimation, this situation normally exists.

The development of a unified approach to statistical optimization as outlined above could be a formidable endeavor if all of the efforts were to be accomplished within the study. Fortunately, significant contributions have been made in two general areas. First, although efforts to date generally consider space flight navigation and guidance problems on a restrictive basis, these efforts represent a basis for the formulation of the more general problem. The results of these efforts establish a minimum requirement for the unified approach and a set of indicative results which are to be expected, at least qualitatively. Second, the problems of statistical optimization as applied to space flight technology comprise a subset of the general class of problems which are considered in statistical decision theory. The general principles and methods of decision theory will undoubtedly provide the essential framework of a unified approach to statistical optimization for space flight problems. Thus, the efforts in developing the unified approach will be directed toward codifying the directly applicable principles and methods of decision theory into a special discipline suitable to the required treatment of space flight navigation and guidance problems. To this end some of the basic aspects of decision theory are discussed below as they apply to the desired unified approach. It should be pointed out that efforts to date often employ certain principles of decision theory on an implicit basis; however, unless these principles are consistently employed on an explicit basis, their full usefulness is not realized or exploited. This subject was treated in detail in a previous monograph concerning the problem of estimation (Ref. 2.23); note is made that the material of this reference covered a special form of the more general problem being considered here.

The basic problem which is considered in decision theory can be described as follows. There exists a situation wherein a decision must be made and/or an action must be taken in order to achieve some desired objective; however, the situation is not exactly known, i.e., there exists some uncertainty concerning the state of the situation. Further, the situation is characterized by some loss in making a decision and/or taking an action. The general problem in decision theory is to determine a strategy or policy which achieves the desired objective with the minimum loss. Thus, the basic problem in decision theory could be succinctly defined as that of determining an optimum strategy in a situation of uncertainty. It should be noted that the optimum strategy can consist of either the optimum estimate of the state or the optimum control, or both, as determined by the particular problem being considered. It should also be noted that the uncertainty in the state of the situation characterizes the problem as one in statistical decision theory; otherwise, the problem is one of deterministic optimization. Generally, there exists the possibility of acquiring information concerning the state of the situation and thus reducing the uncertainty concerning the situation. This capability is usually provided by acquiring observations from an information source which itself possesses some uncertainty. Whereas the uncertainty concerning the situation cannot be completely removed, it can be decreased by utilization of certain available information. It thus becomes apparent that two types of uncertainty exist in this problem which are of interest: (1) an uncertainty due to ignorance of the true state of the situation and (2) an uncertainty due to randomness in an information source.

In terms of space flight navigation and guidance problems, the true state corresponds to position and velocity deviations from a nominal and/or desired trajectory (or some linear function thereof) and the information source corresponds to a set of measurement deviations (observed minus computed residuals). The corresponding uncertainties are usually those due to orbit injection errors and measurement errors. With respect to the space flight problem, the optimum strategy is then the "best" possible action to be taken under the condition of uncertainty in the actual deviations from the desired trajectory.

It is important to note that the basic problem in decision theory is characterized by two distinct uncertainties. These uncertainties, in turn, are specified by two distinct probability spaces or statistical distributions which are usually independent. That is, the true state of the situation can be the outcome of a random phenomenon which is described by a probability distribution. In general, the appropriate action to be taken depends upon the true state of the situation or the unknown parameters which determine the probability space of the observations or the population to be sampled; where the latter probability space is a function of the randomness in the observations. The important point to be made is that the optimum strategy must, in general, consider all possible states, i.e., the optimum strategy must be optimum with respect to the probability space of the state.

A more formal description of the basic problem in decision theory can be formulated with the aid of the following notation. Let

(a) θ denote a parameter set which specifies the true state of a situation

(b) Ω denote the parameter space which is the space of all possible values of θ

(c) Y denote a set of random samples or observations which is a function of θ

(d) U denote a set of actions

(e) Δ denote the action space which is the space of all possible actions

(f) $L(U, \theta)$ denote a measure of "loss" associated with each possible U and θ, i.e., the loss is a function of U and θ

Before preceding, the following comments concerning the general situation should be noted.

First, Θ is not explicitly known, i.e., Θ is generally unknown; however, a' priori information is often available concerning the probability of occurence of Θ. This information is usually available in terms of a probability distribution function for Θ defined over the parameter space Ω.

Second, the set of observations Y usually increases the knowledge of Θ, i.e., the primary purpose of the observation process is to gain information concerning the true state of the situation. The information gain in the observation process is determined by the dependence of Y on Θ and the uncertainty in Y due to randomness.

Third, the concept of information as a measure of uncertainty is a fundamental consideration in this type of problem. However, two types of information are present: The a' priori information available concerning Θ, and the a' posteriori information obtained through the observation process. The information available concerning the true situation is a function of both types of information.

Fourth, it is axiomatic that the action U to be taken in a situation is a function of the a' priori and a' posteriori information concerning Θ, i.e., the action does not ignore the true state of the situation. Thus, U is generally a function of Y and the a' priori information concerning Θ. Conversely, U is not an explicit function of Θ, since Θ is generally not known.

Fifth, since the loss $L(U, \Theta)$ is a function of random variables, it is a random variable also. That is, the space of the loss is determined by the space of Θ and Y. Further, the probability distribution function of $L(U, \Theta)$ is determined by those of Θ and Y and the explicit functional relationships of $U(Y)$ and $L(U, \Theta)$. Since $L(U, \Theta)$ is a random variable it possesses an average or expected value which is defined as the "risk".

Sixth, since the loss $L(U, \Theta)$ is a function of two random variables, its total expectation can be written in terms of a conditional expectation; therefore, the risk can be written in terms of a "conditional" risk, given Θ. That is,

$$R(U,\Omega) = E[L(U,\theta)]$$

$$= E_\theta\, E[L(U,\theta)/\theta]$$

$$R(U,\Omega) = E_\theta\,[R(U/\theta)]$$

where

$$R(U/\theta) = E[L(U,\theta)/\theta]$$

The term $R(U, \Omega)$ denotes the "total" risk. This risk is not an explicit function of Θ; however, it is a function of the probability distribution of Θ over the parameter space Ω. The term $R(U/\Theta)$ is the conditional risk (that is, the expected value of the loss, given a particular value of Θ.)

A basic precept of decision theory is that a loss function can be defined such that the risk should be a minimum. Of course, the risk is a function of the particular loss function $L(U, \Theta)$ and the particular action U as a function of Y. Thus, the problem is to determine the action U as a function of Y which minimizes the risk associated with the loss $L(U, \Theta)$. This action, denoted by $\hat{U}(Y)$, minimizes the risk and is defined as the optimum strategy with respect to $L(U, \Theta)$. Such optimum strategies can, therefore, be defined as minimum risk strategies. There are two types of minimum risk strategies which have been formulated in decision theory. These strategies are referred to as (1) Bayes strategies and (2) Mini-max strategies. A Bayes strategy, denoted by U_B, minimizes $R(U, \Omega)$ over the parameter space Ω, i.e.,

$$MIN[R(U,\Omega)] = R(U_B, \Omega)$$

A mini-max strategy, U_M, minimizes the maximum risk, i.e.,

$$MIN\text{-}MAX[R(U,\Omega)] = MAX[R(U_M, \Omega)]$$

A mini-max strategy can also be defined for the conditional risk, i.e.,

$$MIN\text{-}MAX[R(U/\theta)] = MAX[R(U_M/\theta)]$$

It should be noted that a Bayes strategy is optimum over the parameter space and utilizes the a' priori information represented by the probability distribution of Θ over Ω. On the other hand, a mini-max strategy minimizes a least favorable situation in terms of either the distribution of Θ over Ω, or Θ itself. The latter strategy is applicable where a Bayes strategy cannot be formulated.

A Bayes strategy is generally implied in the usual formulations of space flight problems by the virtue of the available information. However, while the Bayes strategy is generally applicable, a mini-max strategy can be considered in situations where a' priori information for the Bayes strategy is incomplete or questionable. For a more exhaustive treatment of minimum risk strategies, see References 2.24, 2.25 and 2.26.

It should be apparent that the problem outlined includes the problems of optimum estimation and/or control. Indeed, the two problems possess the same form and are distinguishable only by the particular loss function employed. Generally, if $L(U, \Theta) = 0$ for $U(Y) = \Theta$ and $L(U, \Theta) > 0$ for $U(Y) \neq \Theta$, then the problem is generally that of estimating the true state of the situation and the loss is a function of the estimation error. The problem of estimation has been discussed in terms of decision theory in a previous monograph (see Section 2.3 of Reference 2.23). The principles which form methods of solution for the estimation problem also apply to the control problem. The essential difference is the definition of the loss function $L(U, \Theta)$. In general, however, a Bayes strategy is determined by minimizing the Bayes function, or a' posteriori risk, as defined in Section 2.3.5.2.1 of Reference 2.23. A Bayes strategy was derived for the optimum estimation problem in Reference 2.23. A Bayes strategy for the optimum control problem is discussed in the following section of this monograph.

It should be noted that the problem which has been defined includes the sequential problem or time sequence problem which is most common in space flight technology. This aspect of the problem does not present a major difficulty in determining minimum risk strategies. The essential difference with respect to the other problems being considered lies in the definition of the parameter (Θ), the action (U) and the sample (Y) sets; i.e., these sets can be defined as ordered sets of vectors (or subsets) which correspond to the state, control and observation vectors at particular times. The optimum strategy is, thus, the ordered set U of action subsets or vectors. Of course, the loss function $L(U, \Theta)$ is defined over the sets U and Θ. For instance, the "quadratic" loss function (see Section 2.2.2) can be written as

$$L(U,\theta) = \theta^T Q \theta + U^T \gamma U$$

where Θ and δ are vectors with subvectors which denote the state and control at discrete times, i.e.,

$$\theta^T = (\delta_1^T, \delta_2^T, \ldots, \delta_n^T, \ldots, \delta_N^T)$$

$$U^T = (u_0^T, u_1^T, \ldots, u_{n-1}^T, \ldots, u_{N-1}^T)$$

Thus, if the matrices Q and γ as partitioned diagonal matrices, the loss $L(U, \Theta)$ can be written in the usual quadratic form.

$$L(U,\theta) = \sum_{i=1}^{N} [\delta_i^T Q_i \delta_i + u_{i-1}^T \gamma_i u_{i-1}]$$

where

$$Q = \begin{vmatrix} Q_1 & O & \cdots & \cdots & O \\ O & Q_2 & \cdots & \cdots & O \\ \vdots & \vdots & Q_n & & \vdots \\ O & O & \cdots & \cdots & Q_N \end{vmatrix}$$

$$\gamma = \begin{vmatrix} \gamma_1 & O & \cdots & \cdots & O \\ O & \gamma_2 & \cdots & \cdots & O \\ \vdots & \vdots & \gamma_n & & \vdots \\ O & O & \cdots & \cdots & \gamma_N \end{vmatrix}$$

Thus, it is seen that the problem being formulated is not restricted to a "terminal" action.

An important goal in the establishment of minimum risk strategies is the determination of a set which contains all necessary information to define the optimum strategy. This set, termed a "sufficient statistic", was defined in a previous monograph (see Section 2.3 of Reference 2.23) for the estimation problem. The use of sufficient statistics in optimum control is discussed in Reference 2.17. The general result, as presented, is that minimum risk strategies are exclusive functions of sufficient statistics; therefore, only sufficient statistics need be considered in determining minimum risk strategies. This proof is a significant part of the sequential problem in which it is generally desired to express the "local" action in terms of the most "compact" set of information. In general, the sufficient statistic for a particular problem will depend upon the nature of the dynamics, the statistical distributions and the loss function involved in the problem. In all cases, however, the sufficient statistic determines a basic requirement of the optimum strategy for minimum risk which, in turn, determines a basic property of the general structure of the optimum solution.

2.2.4.5 Bayes Strategy in Optimum Control

Bayes strategies are applicable to a variety of problems of space flight navigation and guidance in the areas of estimation and control. The development of a Bayes strategy for the estimation problem was considered in detail in a previous monograph (Reference 2.22). In this section, the development of a Bayes strategy for the control problem is considered. However, it should be noted that this discussion is not an exhaustive treatment of the subject. Rather, the discussion is expository in nature with the primary purpose being to demonstrate the nature of the approach illustrating the formulation of the problems and the form of the results. Equivalence of solutions with those determined by other methods will be indicated and a general structure will be given for a Bayes strategy as it applies to space flight navigation and guidance problems. The essential feature of a Bayes strategy is the minimization of the Bayes function or the a' posteriori risk as defined in Section 2.3.5.2.1 of Reference 2.23. The basic steps are as follows: First, the Bayes risk is defined as

$$R(U,\Omega) = E_{\theta}\left\{E_{U}[L(U,\theta)/\theta]\right\}$$

$$= \int_{\Omega}\int_{Y} L(U,\theta)\, f(Y/\theta)\, f(\theta)\, dY\, d\theta$$

$$R(U,\Omega) = \int_{Y}\left[\int_{\Omega} L(U,\theta)\, f(Y,\theta)\, f(\theta)\, d\theta\right] dY$$

where f( ) denotes the probability density function of its argument and f(/) denotes the conditional probability density function of its arguments. Now, since all of the functions used in the definition of R $[U, \Omega]$ are positive, the optimum strategy (the Bayes strategy) minimizes the inner integral for all Y. However, the inner integral can be reduced further by use of the identity f(Y /Θ)f(Θ) = f(Θ/Y)f(Y). i.e.,

$$R(U,\Omega) = \int_Y \left[\int_\Omega L(U,\theta)\, f(\theta/Y)\, d\theta \right] f(Y)\, dY$$

$$= \int_Y B\,[U(Y)]\, f(Y)\, dY$$

where $B[U(Y)]$ is defined as the Bayes function. (This function is recognized as being the conditional expected loss, given a particular set of observations Y).

$$B[U(Y)] = \int_\Omega L(U,\theta)\, f(\theta/Y)\, d\theta$$

Again, the optimal (Bayes) strategy minimizes the Bayes function $B[U(Y)]$.

In the more general form of the optimum control problem, the loss is a function of sequential state and control vectors. However, this dependence can be included in the parameter and action sets Θ and U by defining these sets as vectors which contain the state vectors and control vectors as subvectors, i.e.,

$$\theta = \begin{vmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_n \\ \vdots \\ \delta_N \end{vmatrix} \qquad U = \begin{vmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \\ \vdots \\ u_N \end{vmatrix}$$

where δ_n and u_n are state and control vectors, respectively, defined as

$$\delta_n = \delta(t_n) \quad \text{and} \quad u_n = u(t_{n-1})$$

and where N is the total number of control points. Note that u_n denotes a control vector subsequent to δ_n . This minor difference in notation should be contrasted with earlier discussions of quadratic loss. Consider the case of a "generalized" quadratic loss function in Θ and U , to wit,

$$L(U,\theta) = \theta^T Q \theta + U^T \gamma U$$

where Q and γ are symmetrical square matrices whose orders are determined by the number of control points N and the number of components in the vectors δ_n and u_n . It should be noted that the loss function $L(U, \theta)$ is more general than the usual quadratic loss function J_n which is defined as (see Section 2.2.2)

$$J_N = \sum_{i=1}^{N} [\delta_i^T Q_i \delta_i + u_{i-1}^T \gamma_i u_{i-1}]$$

The loss J_N is a special case of $L(U, \Theta)$ where Q and γ are partitioned diagonal matrices containing the matrices Q_i and γ_i.

Now, the parameter set Θ is a function of the action set U. In particular, the state vector δ_n is a function of the previous state δ_{n-1} and the control u_n. For a linear system, the relationship of δ_n, and u_n is usually expressed as

$$\delta(t_n) = \phi_{n,n-1}\,\delta(t_{n-1}) + \Gamma_{n,n-1}\,u(t_{n-1})$$

However, the system can also be described by the following equivalent relationships.

$$\begin{aligned}
\delta_1 &= \phi_{10}\,\delta_0 + \Gamma_{11} u_1 \\
\delta_2 &= \phi_{20}\,\delta_0 + \Gamma_{21} u_1 + \Gamma_{22} u_2 \\
&\vdots \\
\delta_n &= \phi_{n0}\,\delta_0 + \sum_{i=1}^{n} \Gamma_{ni} u_i \\
&\vdots \\
\delta_N &= \phi_{N0}\,\delta_0 + \sum_{i=1}^{N} \Gamma_{Ni} u_i
\end{aligned}$$

Note that the system state δ_n is expressed in terms of the initial state δ_0 and the contribution of all previous control vectors u_i for i = 1, 2, . . . , n. This system of equations can now be written in terms of the sets δ and θ as

$$\theta = \phi\,\delta_0 + \Gamma U$$

where δ_0 is the initial state vector, Θ and U are the parameter and action sets, respectively, and ϕ and Γ are matrices defined as follows

$$\phi = \left| \begin{array}{c} \phi_{10} \\ \hline \phi_{20} \\ \hline \vdots \\ \hline \phi_{n0} \\ \hline \vdots \\ \hline \phi_{N0} \end{array} \right|$$

$$\Gamma = \left| \begin{array}{cccccc}
\Gamma_{11} & 0 & 0 & \cdots & 0 & \cdots\ 0 \\
\Gamma_{21} & \Gamma_{22} & 0 & \cdots & 0 & \cdots\ 0 \\
\Gamma_{31} & \Gamma_{32} & \Gamma_{33} & \cdots & 0 & \cdots\ 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
\Gamma_{n1} & \Gamma_{n2} & \Gamma_{n3} & \cdots & \Gamma_{n,n} & \cdots\ 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
\Gamma_{N1} & \Gamma_{N2} & \Gamma_{N3} & \cdots & \Gamma_{N,n} & \cdots\ \Gamma_{N,N}
\end{array} \right|$$

The matrices ϕ and Γ are thus partitioned matrices which contain the sub-matrices ϕ_{io} and Γ_{ij}, respectively. It is important to note that Γ is a lower triangular matrix.

The loss function (for the present purposes, loss is quadratic) can now be expressed in terms of the action set U and the initial state vector δ_o, i.e., since the parameter set Θ is a function of U and δ_o it follows that

$$L[U(Y)\ ;\ \theta(\delta_o, U)] = L^*[U(Y), \delta_o]$$

where

$$L^*\left[U(Y),\ \delta_o\right] = (\phi\,\delta_o + \Gamma U)^T Q(\phi\,\delta_o + \Gamma U) + U^T \gamma U$$

$$= \delta_o^T \phi^T Q \phi\, \delta_o + 2U^T \Gamma^T Q \phi\, \delta_o + U^T \Gamma^T Q \Gamma U + U^T \gamma U$$

$$L^*\left[U(Y), \delta_o\right] = \delta_o^T \phi^T Q \phi\, \delta_o + 2U^T \Gamma^T Q \phi\, \delta_o + U^T(\Gamma^T Q \Gamma + \gamma)\,U$$

$$L^*(U, \delta_o) = L^*[U(Y), \delta_o]$$

Substituting $L^*(U, \delta_o) = L^*[U(Y), \delta_o]$ into the Bayes function and taking the first partial derivative with respect to the action set, it is found that

$$\frac{\partial}{\partial U} B[U(Y)] = \frac{\partial}{\partial U} E[L^*(U, \delta_o)/Y]$$

$$= E\left\{\left[\frac{\partial}{\partial U} L^*(U, \delta_o)\right]/Y\right\}$$

$$\frac{\partial}{\partial U} B[U(Y)] = 2E\left\{[\Gamma^T Q \phi\, \delta_o + (\Gamma^T Q \Gamma + \gamma)\,U]/Y\right\}$$

Thus, setting the first partial of $B[U(Y)]$ equal to zero determines the Bayes strategy U_B, i.e.,

$$(\Gamma^T Q \Gamma + \gamma)U_B = -E[(\Gamma^T Q \phi X_o)/Y] = -\Gamma^T Q \phi E(\delta_o/Y)$$

$$(\Gamma^T Q \Gamma + \gamma)U_B = -\Gamma^T Q \phi\, \hat{\delta}_o$$

where

$$\hat{\delta}_o = E(\delta_o/Y)$$

Thus, the Bayes strategy becomes

$$U_B = -(\Gamma^T Q \Gamma + \gamma)^{-1} \Gamma^T Q \phi \hat{\delta}_o \equiv -K \hat{\delta}_o$$

and the corresponding loss is

$$\begin{aligned} L[U(Y); \theta(\delta, U)] &= \delta_o^T [\phi^T Q \phi - 2K^T Q \phi + K^T(\Gamma^T Q \Gamma + \gamma) K] \delta_o \\ &\equiv \delta_o^T A \delta_o \end{aligned}$$

The "optimum" schedule for these corrections can now be determined by *minimizing* the *maximum* eigenvalue of the *matrix* A. This *process* has its origin in the fact that

$$\lambda_{min} \leq \frac{L[U, \theta]}{\delta_o^T \delta_o} = \frac{\delta_o^T A \delta_o}{\delta_o^T \delta_o} \leq \lambda_{max}$$

where the λ are the eigenvalues of A. Note that Dynamic Programming has not been employed at any point.

Several comments are in order concerning the Bayes strategy for the optimum control (quadratic loss). First, note that the form of U_B is the same as the form for the optimum estimate in the linear case and Gaussian distributions (see Sections 2.3.6.3 of Reference 2.23). Second, the optimum control is a linear function of the conditional expectation of δ_o, given the observations, which is a Bayes strategy for the optimum estimate of δ_o under rather general conditions (see Section 2.3.5.2 of Reference 2.23). Third, the Bayes strategy (U_B) is the "total" optimum control since it contains as subvectors, the "local" optimum control vectors (u_n^*); i.e., U_B is a single expression for the set of optimum control vectors (u_n^*) for n = 1,2, ..., N. Fourth, the Bayes strategy (U_B) is a more general solution than that for the special loss function J_N (discussed earlier) which is the sum of quadratic forms of "local" state and control vectors. Fifth, the Bayes strategy (U_B), in total form, has the same form as the local optimum control which is usually derived by the method of Dynamic Programming (see Section 2.2.2.2.1).

The Bayes strategy (U_B) for the optimum control is given as a total optimum control; i.e., the optimum action set (U_B) contains all the optimum sequential controls. It is, however, desirable to determine the local optimum controls u_n^* from the optimum action set (U_B). This objective can be accomplished as follows. The Bayes strategy can be written in terms of the following equations:

$$(\Gamma^T Q \Gamma + \gamma) U_B = -\Gamma^T Q \phi \hat{\delta}_0$$

$$\gamma U_B = -[\Gamma^T Q \phi \hat{\delta}_0 + \Gamma^T Q \Gamma U_B] = -\Gamma^T Q [\phi \hat{\delta}_0 + \Gamma U_B]$$

$$\gamma U_B = -\Gamma^T Q [Z + D U_B]$$

where

$$Z = \phi \hat{\delta}_0 + (\Gamma - D) U_B$$

and where D is a partitioned diagonal matrix containing the matrices Γ_{ii} for $i = 1, 2, \ldots N$. That is

$$D = \begin{bmatrix} \Gamma_{11} & & 0 \\ & \Gamma_{22} & \\ 0 & & \ddots \ \Gamma_{NN} \end{bmatrix}$$

The vector Z contains the subvectors Z_n defined as

$$Z_n \equiv \phi_{n0} \hat{\delta}_0 + \sum_{i=1}^{n-1} \Gamma_{ni} u_i$$

which are in fact the conditional expectation of the state at t_n, given the controls $u_1, u_2, \ldots u_{n-1}$, for no control at t_{n-1}, i.e., $u_n = 0$. Alternately, Z_n is the conditional expectation of the state just prior to the control (u_n). It follows that

$$Z_n = \phi_{n,n-1} \hat{\delta}_{n-1}$$

and

$$\hat{\delta}_n = Z_n + \Gamma_{n,n} u_n$$

Consider the case of $L(U, \theta) = J_N$ for which Q and γ are partitioned diagonal matrices containing the matrices Q_i and γ_i. For this case, the matrix product $\Gamma^T Q$ is an upper traingular matrix and the final optimum control vector $u_N^* = u^*(t_{N-1})$ can be written as

$$\gamma_N u_N^* = -\Gamma_{N,N}^T Q_N [Z_N + D_N u_N^*] = -\Gamma_{N,N}^T Q_N [\phi_{N,N-1} \hat{\delta}_{N-1} + D_N u_N^*]$$

$$= -\Gamma_{N,N}^T Q_N \phi_{N,N-1} \hat{\delta}_{N-1} - \Gamma_{N,N}^T Q_N D_N u_N^*$$

But, since $D_N = \Gamma_{N,N}$, It follows that

$$(\gamma_N + \Gamma_{N,N}^T Q_N \Gamma_{N,N}) u_N^* = \Gamma_{N,N}^T Q_N \phi_{N,N-1} \hat{\delta}_{N-1}$$

Thus, u_N^* is a linear function of $\hat{\delta}_{N-1}$, i.e.,

$$u_N^* = -(\gamma_N + \Gamma_{N,N}^T Q_N \Gamma_{N,N})^{-1} \Gamma_{N,N}^T Q_N \phi_{N,N-1} \hat{\delta}_{N-1} \equiv K_N \hat{\delta}_{N-1}$$

where

$$K_N = -(\gamma_N + \Gamma_{N,N}^T Q_N \Gamma_{N,N})^{-1} \Gamma_{N,N} Q_N \phi_{N,N-1}$$

It is seen that the optimum control $u_N^* = u^*(t_{N-1})$ is identical to that determined by applying the method of Dynamic Programming to solve the optimum control problem (see section 2.2.2.2.1). It is possible to continue the process to determine the previous optimum control vectors. The control u^*_{N-1} is derived from the following expression.

$$\begin{aligned}\gamma_{N-1} u_{N-1}^* &= -[D_{N-1}^T Q_{N-1}(z_{N-1} + D_{N-1} u_{N-1}^*) + \Gamma_{N,N-1}^T Q_N (z_N + D_N u_N^*)] \\ &= -D_{N-1}^T Q_{N-1} z_{N-1} - D_{N-1}^T Q_{N-1} D_{N-1} u_{N-1}^* - \Gamma_{N,N-1} Q_N (z_N + D_N u_N^*)\end{aligned}$$

Thus,

$$(\gamma_{N-1} + D_{N-1}^T Q_{N-1} D_{N-1}) u_{N-1}^* = -D_{N-1}^T Q_{N-1} z_{N-1} - \Gamma_{N,N-1}^T Q_N (z_N + D_N u_N^*)$$

Now, since

$$u_N^* = K_N \hat{\delta}_{N-1}$$

$$z_N = \phi_{N,N-1} \hat{\delta}_{N-1}$$

$$\hat{\delta}_{N-1} = z_{N-1} + \Gamma_{N-1,N-1} u_{N-1}^*$$

it follows that

$$u_{N-1}^* = K_{N-1} \hat{\delta}_{N-2}$$

where substitution and algebraic manipulation yield the relationship

$$K_{N-1} = -[\gamma_{N-1} + \Gamma_{N-1,N-1}^T (Q_{N-1} + P_1) \Gamma_{N-1,N-1}]^{-1} \Gamma_{N-1,N-1}^T (Q_{N-1} + P_1) \phi_{N-1,N-2}$$

and where

$$P_1 = \phi^T_{N,N-1} \, Q_N \, \phi_{N,N-1} + \phi^T_{N,N-1} \, Q_N \, \Gamma_{N,N} \, K_N$$

Again, it is seen that the optimum control $u^*_{N-1} = u^*(t_{N-2})$ is identical to that determined by applying the method of dynamic programming to solve the optimum control problem (see section 2.2.2.2.1).

It is possible to extend this process to determine the general control u_n^* in terms of a linear function of δ_{n-1}, i.e., $u^*_n = K_n \, \delta_{n-1}$. This capability follows from the fact that the optimum solution can be written in terms of an upper triangular matrix, i.e.

$$(\gamma + \Gamma^T Q D) \, U_B = -\Gamma^T Q z$$

The matrix Γ^T is upper triangular and the matrices γ, Q and D are diagonal; thus, the matrix $[\gamma + \Gamma^T Q D]$ is upper triangular. This relation can be inverted recursively for the control vector u^*_n from the following equations.

$$\gamma_n \, u_n^* = -\sum_{i=n}^{N} \Gamma_{in}^T \, Q_i \, (z_i + \Gamma_{i,i} \, u_i^*)$$

where

$$z_{n+1} = \phi_{n+1,n} \, \hat{\delta}_n$$

$$\hat{\delta}_n = \phi_{n,n-1} \, \hat{\delta}_{n-1} + \Gamma_{n,n} \, u_n^*$$

$$\hat{\delta}_n = z_n + \Gamma_{n,n} \, u_n^*$$

The procedure starts at n = N and proceeds backward as indicated previously. The important point to be made is that the Bayes strategy leads to the <u>same</u> recursive form of solution which is obtained by using the method of dynamic programming; however, the Bayes strategy does not require the "principle of optimality" which is the basis of dynamic programming, nor is it as restricted, as were the previous earlier discussions, to the case where Q and γ are diagonal (this fact can be appreciated since the upper triangular nature of the solution matrix can be assured for any symmetric weighting by the simple expedient of introducing an "<u>equivalent</u>" triangular array

$$Q = \begin{bmatrix} Q_{11} & & \\ & Q_{22} \ddots & 0 \\ & 0 & Q_{nn} \end{bmatrix} + \begin{bmatrix} 0 & 2Q_{12} & 2Q_{13} & \cdots & 2Q_{1n} \\ & 0 & \ddots & & \vdots \\ & & 0 & \ddots & \\ & 0 & & \ddots & 2Q_{n-1,n} \\ & & & & 0 \end{bmatrix}$$

$$\gamma' = \begin{bmatrix} \gamma_{11} & & & \\ & \gamma_{22} & & 0 \\ & & \ddots & \\ 0 & & & \gamma_{nn} \end{bmatrix} + \begin{bmatrix} 0 & 2\gamma_{12} & \cdots & 2\gamma_{1n} \\ & 0 & \ddots & \vdots \\ & 0 & \ddots & 2\gamma_{n-1,n} \\ & & & 0 \end{bmatrix}$$

While the algorithms thus produced are different, the procedure involved in producing them is identical). Indeed, since the last two controls as obtained by the two methods were shown to be identical, the two solutions must be completely identical, assuming the optimum solution is unique.

The structure of a Bayes strategy can now be given for the general case. The minimization of the Bayes function is equivalent to the requirement that the conditional expectation of the first variation of the loss function given the data should vanish. This fact is seen as follows.

$$B[U(Y)] = E[L(U,\theta)/Y]$$

$$\frac{\partial}{\partial U} B[U(Y)] = \frac{\partial}{\partial U} E[L(U,\theta)/Y]$$

$$= E[\frac{\partial}{\partial U} L(U,\theta) Y]$$

Setting the first variation of $B[U(Y)]$ equal to zero, it is found that

$$E[\frac{\partial}{\partial U} L(U,\theta)/Y] = 0$$

Now, consider the case of a general loss function defined by

$$L(U,\theta) = F_1(\theta) + F_2(U)$$

For this case, the Bayes strategy is defined by

$$\frac{\partial}{\partial U} F_2(U) = -E\left\{\frac{\partial}{\partial U} F_1[\theta(U)]/Y\right\}$$

This expression forms the basis of a general structure of the optimum solution.

From the foregoing, it becomes apparent that a Bayes strategy is generally applicable to problems of space flight navigation and guidance; however, it is extremely important to note that the foregoing implies that the complete observation set Y is available to determine each element of the optimum action set U_B or each optimum control u_n^*. That is, the optimum action set U_B is a function of the conditional exception of θ, given Y. It is characteristic of space flight navigation and guidance problem, however, that only a subset of Y is available to determine each control u_n, i.e., at time t_n only the subset of Y, say Y_n which has been acquired prior to t_n is available to determine the control u_n. Thus, the problem becomes that of determing the optimum action set U subject to the condition that each element of U is a function of the corresponding element of Y. In particular, let U and Y be the following sets.

$$U(u_1, u_2, \ldots, u_n, \ldots, u_N)$$

$$Y(y_1, y_2, \ldots, y_n, \ldots, y_N)$$

where the following subsets of Y are defined

$$Y_1 = (y_1)$$

$$Y_2 = (y_1, y_2)$$

$$\vdots$$

$$Y_n = (y_1, y_2, \ldots, y_n)$$

That is, Y_n is the subset of Y which contains the observation vectors for $i = 1, 2, \ldots, n$.

In actual control problems, the control vector at t_n can be a function only of the subset Y_n where, of course, appropriate definitions of the observation subsets are made. Thus, it is required to determine the optimum action set U as follows.

$$U[u_1(Y_1), u_2(Y_2), \ldots, u_n(Y_n), \ldots u_N(Y_N)]$$

To accomplish this objective, consider the class of loss functions which can be written as

$$L[U(Y), \theta] = \sum_{i=1}^{N} L_i[U(Y_i), \theta]$$

It is important to note that this class of loss functions contains those of interest in space flight navigation and guidance problems, among them being the quadratic forms just analyzed. Now, a Bayes strategy can be determined in the following manner. First, the risk $R[U(Y)]$ is

$$R[U(Y)] = \iint_{\Omega\, Y} L[U(Y), \theta]\, f(Y/\theta)\, f(\theta)\, dY d\theta$$

$$=\int\!\!\int_{\Omega\, Y} \sum_{i=1}^{N} L_i\,[U(Y_i),\theta]\; f(Y/\theta)\; f(\theta)\, dY d\theta$$

$$=\int_{\Omega} \sum_{i=1}^{N} \int_{Y} L_i\,[U(Y_i),\theta]\; f(Y/\theta)\, f(\theta)\, dY d\theta$$

Now, it is noted that $L_i[U(Y_i),\theta]$ is independent of the observation set $(Y-Y_i)$, where

$$Y-Y_i = (y_{i+1}, y_{i+2}, \ldots, y_N)$$

Thus, each term in $R[U(Y)]$ can be integrated over a Y set for which $L_i[U(Y_i),\theta]$ is constant, i.e., That is,

$$\int_{Y} L_i\,[U(Y_i),\theta]\;\; f(Y/\theta)\, f(\theta)\, dY$$

can be integrated as

$$\int_{Y_i} L_i\,[U(Y_i),\theta] \int_{Y-Y_i} f(Y/\theta)\, f(\theta)\, dY$$

However, since $F(Y/\theta)\, f(\theta) = f(Y;\theta)$, the integral over $Y-Y_i$ is the marginal density of Y_i and θ, i.e., $f(Y_i,\theta)$. Now, since $f(Y_i,\theta) = f(Y_i/\theta)f(\theta)$ it follows that

$$\int_{Y} L_i\,[U(Y_i),\theta]\;\; f(Y/\theta)\;\; f(\theta)\, dY = \int_{Y_i} L_i\,[U(Y_i),\theta]\, f(Y_i/\theta)\, f(\theta)\, dY_i$$

Therefore,

$$R[U(Y)] = \int_{\Omega} \sum_{i=1}^{N} \int_{Y_i} L_i\,[U(Y_i),\theta]\;\; f(\theta/Y_i)\;\; f(Y_i)\, dY_i\, d\theta$$

Now, substituting $\qquad f(Y_i/\theta)\;\; f(\theta) = f(\theta/Y_i)\, f(Y_i)$

the Bayes risk can be written as

$$R[U(Y)] = \int_{\Omega} \sum_{i=1}^{N} \int_{Y_i} L_i\,[U(Y_i),\theta]\, f(\theta/Y_i)\;\; f(Y_i)\, dY_i\, d\theta$$

or

$$R[U(Y)] = \sum_{i=1}^{N} \int_{Y_i} \left\{ \int_{\Omega} L_i\,[U(Y_i),\theta]\;\; f(\theta/Y_i)\, d\theta \right\} f(Y_i)\, dY_i$$

$$\equiv \sum_{i=1}^{N} \Lambda_i\,[U(Y_i)]$$

where the $\mathcal{R}_i$ are the Bayes risks for the i steps of the process. i.e.,

$$\mathcal{R}_i\,[U(Y_i)] = \int_{Y_i}\left\{\int_{\Omega} L_i[U(Y_i),\,\theta]\, f(\theta/Y_i)\,d\theta\right\} f(Y_i)\,dY_i$$

$$= \int_{Y_i} b_i\,[U(Y_i)]\, f(Y_i)\,dY_i$$

and where the b_i are the Bayes functions for the steps. i.e.,

$$b_i\,[U(Y_i)] = \int_{\Omega} L_i[U(Y_i),\,\theta]\; f(\theta/Y_i)\,d\theta$$

The final step in the present development is to express $R\,[U(Y)]$ in terms of a Bayes function which is the sum of $b_i\,[U(Y_i)]$.

$$R[U(Y)] = \sum_{i=1}^{N}\int\left\{\int L_i[U(Y_i),\,\theta]\; f(\theta/Y_i)\;d\theta\right\} f(Y)\,dY$$

$$= \int_{Y}\sum_{i=1}^{N}\left\{\int_{\Omega} L_i\,[U(Y_i),\,\theta]\; f(\theta/Y_i)\,d\theta\right\} f(Y)\,dY$$

$$= \int_{Y}\left\{\int_{\Omega}\sum_{i=1}^{N} L_i[U(Y_i),\,\theta]\;\; f(\theta/Y_i)\,d\theta\right\} f(Y)\,dY$$

Finally

$$R\,[U(Y)] = \int_{Y} B\,[U(Y)]\, f(Y)\; UY$$

Thus.

$$B\,[U(Y)] = \sum_{i=1}^{N} b_i\,[U(Y_i)] = \sum_{i=1}^{N}\int_{\Omega} L_i\,[U(Y_i),\,\theta]\; f(\theta/Y_i)\,d\theta$$

$$= \sum_{i=1}^{N} E\left\{L_i\,[U(Y_i),\theta]/Y_i\right\}$$

The Bayes strategy, U_B, is the action set which minimizes the Bayes function or a' posteriori risk, $B[U(Y)]$.

It is seen that for the case of sequential observations, the problem of optimization reduces to a form which is similar to that for the previously considered case where the observations were unordered; however, there exists an essential difference. This difference is that in the case of ordered observations, the conditional expectations which form the optimum solution are taken with respect to the ordered observation sets Y_i rather than with respect to the total observation set Y.

Now as before, consider the class of loss functions which can be written as follows.

$$L_i[U(Y_i), \theta(U_i)] = F_i[U(Y_i)] + G_i[\theta(U_i)]$$

where U_i is the subset of the action set U which contains the control vectors ; i.e.,

$$\begin{aligned} U_1 &= (u_1) \\ U_2 &= (u_1, u_2) \\ &\vdots \\ U_i &= (u_1, u_2, \ldots, u_i) \end{aligned}$$

Setting the first partial of $B[U(Y)]$ equal to zero, it is found that

$$\sum_{i=1}^{N} \frac{\partial}{\partial U} F_i[U(Y_i)] = -\sum_{i=1}^{N} E\left\{ \frac{\partial}{\partial U} G_i[\theta(U_i)]/Y_i \right\}$$

The solution to this expression determines the Bayes strategy, U_B, for the class of loss functions $L[U(Y), \theta]$ which can be written as follows.

$$L[U(Y), \theta] = \sum_{i=1}^{N} \left\{ F_i[U(Y_i)] + G_i[\theta(U_i)] \right\}$$

This class of loss functions contains the loss functions of general interest in space flight; therefore, the solution of the Bayes strategy can be considered as a general structure of the optimum solution for statistical optimization problems involving sequential observations. But more important, this general structure makes it possible to study the interrelationships between significant aspects of the problem.

At this point, it is informative to consider the case previously formulated where $L(U, \theta) = J_N$ but where this time the observations will be ordered. The essential difference in the two cases arises from the fact that the conditional expectations are taken with respect to the observation sets Y_i. Thus, the optimum control $u^*(t_n)$ is expressed as a linear function of the conditional expectation of the state, given the set of observations Y_n. That is, all of the available information is used to estimate the state in order to define the control.

A particular case of interest is the problem of optimum control of the terminal state. This problem can be specified by the loss J_N where only Q_N is non-zero, i.e.,

$$J_N = \delta_N^T Q_N \delta_N + \sum_{i=1}^{N} u_i^T \gamma_i u_i$$

The Bayes strategy U_B for this problem is a special case of the previous solution for which Q is defined as follows.

$$Q = \begin{vmatrix} 0\,0 \ldots\ldots 0\,0 \\ 0\,0 \ldots\ldots 0\,0 \\ \vdots \quad \ddots \quad \vdots \\ 0\,0 \ldots\ldots 0\,0 \\ 0\,0 \ldots\ldots 0\,Q_{NN} \end{vmatrix}$$

For this case, the matrix product $\Gamma^T Q$ contains a single matrix column, i.e.,

$$\Gamma^T Q = \begin{vmatrix} 0 & 0 & \ldots\ldots & \Gamma_{N1}^T Q_{NN} \\ 0 & 0 & \ldots\ldots & \Gamma_{N2}^T Q_{NN} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots\ldots & \Gamma_{Nn}^T Q_{NN} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots\ldots & \Gamma_{NN}^T Q_{NN} \end{vmatrix}$$

the corresponding Bayes strategy is then given by

$$(\gamma + \Gamma^T Q \Gamma) U_B = \begin{vmatrix} \Gamma_{N1}^T Q_{NN} & \phi_{N0} & \hat{\delta}_0(1) \\ & \vdots & \\ \Gamma_{N2}^T Q_{NN} & \phi_{N0} & \hat{\delta}_0(2) \\ & \vdots & \\ \Gamma_{Nn}^T Q_{NN} & \phi_{N0} & \hat{\delta}_0(n) \\ & \vdots & \\ \Gamma_{NN}^T Q_{NN} & \phi_{N0} & \hat{\delta}_0(N) \end{vmatrix}$$

where $\hat{\delta}_0(n) = E(\delta_0/Y_n)$. Thus, the Bayes strategy (U_B) is an explicit function of the estimated terminal state as defined by the propagated initial state. Note the important fact that the additional data are providing successively improved estimates of δ_0. Thus, errors in the controls applied at previous times due to èstimation errors are becoming known. This fact allows for an adjustment to be made in the motion to null the effect of these errors and to prevent their continued contribution to the loss function. Of course nothing can be done for their effects prior to the time at which they were estimated to an improved level.

Of course, a Bayes strategy is not limited to "quadratic" loss functions. As another example, consider this particular case where the terminal state is controlled but where the total expected velocity correction rather than its square is used as a criterion of optimality. This problem, as was noted, is also included in the general structure. For this case, F_i and G_i are defined as follows:

$$F_i = \sqrt{u_i^T u_i}$$

$$G_i = \delta_i^T Q_i \delta_i$$

where $Q_i = 0$ for $i = 1, 2, \ldots, N-1$ and arbitrary for $i = N$. The loss function, then, becomes

$$L(U, \theta) = \sum_{i=1}^{N} \sqrt{u_i^T u_i} + \delta_N^T Q_N \delta_N$$

Now, for linear system dynamics

$$\delta_N = \phi_{NO} \delta_0 + \sum_{i=1}^{N} \Gamma_{Ni} u_i$$

$$\delta_N = \phi_{NO} \delta_0 + \Gamma U$$

where

$$\Gamma = | \Gamma_{N1}, \Gamma_{N2}, \ldots, \Gamma_{Nn}, \ldots, \Gamma_{NN} |$$

For this case, the Bayes strategy is defined by

$$\frac{\partial}{\partial U} \sum_{i=1}^{N} F_i = -E\left[\frac{\partial}{\partial U} (\delta_N^T Q_N \delta_N) / Y \right]$$

$$= -2E\left[\Gamma^T Q_N \phi_{NO} \delta_0 + \Gamma^T Q_N \Gamma U) / Y \right]$$

$$1(U) = -2\left[E(\Gamma^T Q_N \phi_{NO} \delta_0) / Y + \Gamma^T Q_N \Gamma U \right]$$

where $1(U)$ is a vector of unit subvectors, 1_n, each of which defines the direction of the optimum control u_n^*. Each of these unit vectors is then defined by the following equation.

$$1_n = -2 \Gamma_{Nn}^T Q_N \left[\phi_{NO} \hat{\delta}_0(n) + \sum_{i=1}^{N} \Gamma_{Ni} u_i \right]$$

where $\hat{\delta}_0(n) = E(\delta_0 / Y_n)$. The corresponding magnitudes for the controls can be determined once information pertaining to the constraints imposed is provided (see section 2.2.3.2.9).

These examples illustrate the use of a Bayes strategy in statistical optimization problems which are of particular interest. Further, they show that the determination of explicit solutions depends upon the specific loss function, the system dynamics, the statistical distributions, etc. The most important observation, however, is that the general structure of the solution provides the basis for a unified approach to the general problem, and serves as motivation for further investigations designed to develop an effective means of determining solutions for particular problems. These further analyses must also develop consistent methods for optimizing the navigation and guidance procedures with respect to the sighting schedules, the types of observations employed, the description of the system dynamics, etc.

2.2.4.6 Some Concluding Comments

The present effort does not permit the full development of the unified approach discussed herein; however, several concluding comments are in order concerning the material which is presented.

First, it should be apparent that the basic problem of decision theory is sufficiently general to include the problems of space flight technology on a non-restrictive basis.

Second, the methods of solution derived from the principles of decision theory provide an adequate basis for the formulation of a unified approach to statistical optimization for space flight problems.

Third, the two most important principles in the analysis are those of minimum risk strategies and sufficient statistics. These principles underlie the general structure of the optimum solution.

Fourth, the Bayes strategy provides a general structure for optimum solutions and is applicable to most situations of interest. However, the MINI-MAX strategies should also be considered.

Fifth, the principles of decision theory have been extensively applied to the problem of estimation and have adequately solved this problem. However, the problem of optimum control, which is a more significant and difficult problem, has not been extensively reported from this point of view.

Sixth, the most important result of a unified approach is that the problems can be considered in their most general form.

Lastly, the objectives of the unified approach can be fulfilled through the principles of decision theory. However, an adequate effort must be devoted to this development.

3.0 RECOMMENDED PROCEDURES

The subject of midcourse guidance, particularly in regard to the stochastic formulation, is still in a state of infancy. Thus, at this time, there are solutions available to only a small set of problems; further, many restrictions and assumptions are implicit in these results which tend to limit the freedom with which they can be applied. (Many of these limitations have been noted in the text). Therefore, there is neither a clearly definable superiority of approach nor a unified theory of midcourse guidance for the stochastic problem. Rather, there is an impressive list of weaknesses in the present work and considerable amount of motivation for completing the development of the theory. (These details have been enumerated in Section 2.2.4).

For the reasons outlined, recommendations must be made reservedly until such time as the unified framework for evaluating various midcourse policies can be constructed and a valid comparison performed. However, with tongue-in-cheek a recommendation will nonetheless be made to satisfy the present needs until such a theory is available.

At this time, there appear to be only two basically different formulations of the stochastic optimum control problem which have been developed and which are applicable to linear midcourse guidance. These two theories (discussed in the text) are constructed around quadratic cost (e.g., References 2.1 - 2.15) and minimum effort (References 2.16 - 2.22) criterion. Of these two, the latter is generally the more efficient (according to the references) from the standpoint of propellant expended since the cost function more closely models the dependencies between the various corrective actions. The former is conceptually simpler and appears to require fewer computations; as a result, the former is probably more suited for present applications to self-contained G&N systems which are severely restricted as to the number of operations which can be performed.

However, the midcourse energy requirements for most of the missions investigated to date are small (for chemical systems, these requirements have generally corresponded to less than 5% of the mass of the vehicle for one way interplanetary voyages - including approach guidance). Thus, savings of 25% as apparently is possible with the mechanization of Minimum Effort Control produce almost negligible changes in the requirements imposed on the vehicle. For this reason and for the reasons of simplicity and additional numerical experience in the evaluation and mechanization of the quadratic loss approach, it is recommended (by the authors) for application to the midcourse guidance problem during the conceptual design of the system. During subsequent efforts it is recommended that the relative merits of both approaches be contrasted to define, in a quantitative manner, the importance of the energy reductions and the corresponding implications for the system (hardware, software). Only in this manner can an intelligent choice of one of the existing approaches to midcourse guidance be made. These recommendations should be considered in the light of the comments made in the text and should be disregarded as a more unified approach to the problem evolves and as treatments of the stochastic control problem with constraints are developed.

Quadratic cost midcourse guidance has the feature that the control policy can be readily predetermined using a detailed simulation of the guidance process. This feature is afforded by the fact that the gain structure of the stochastic problem (i.e., the true midcourse problem) is idential to that of the deterministic problem. This simplification provides the opportunity to optimize the problem with relative ease for the case where no constraints are applied without concerning the analysis with the statistics of probable error sources. Further, if constraints are imposed, the effects of the inclusion can be introduced either in preflight simulations or in a real time mechanization. These objectives are accomplished as follows:

(1) The weighting parameters (i.e., Q_i, γ_i,) and a series of times (at which corrections are to be considered) be specified.

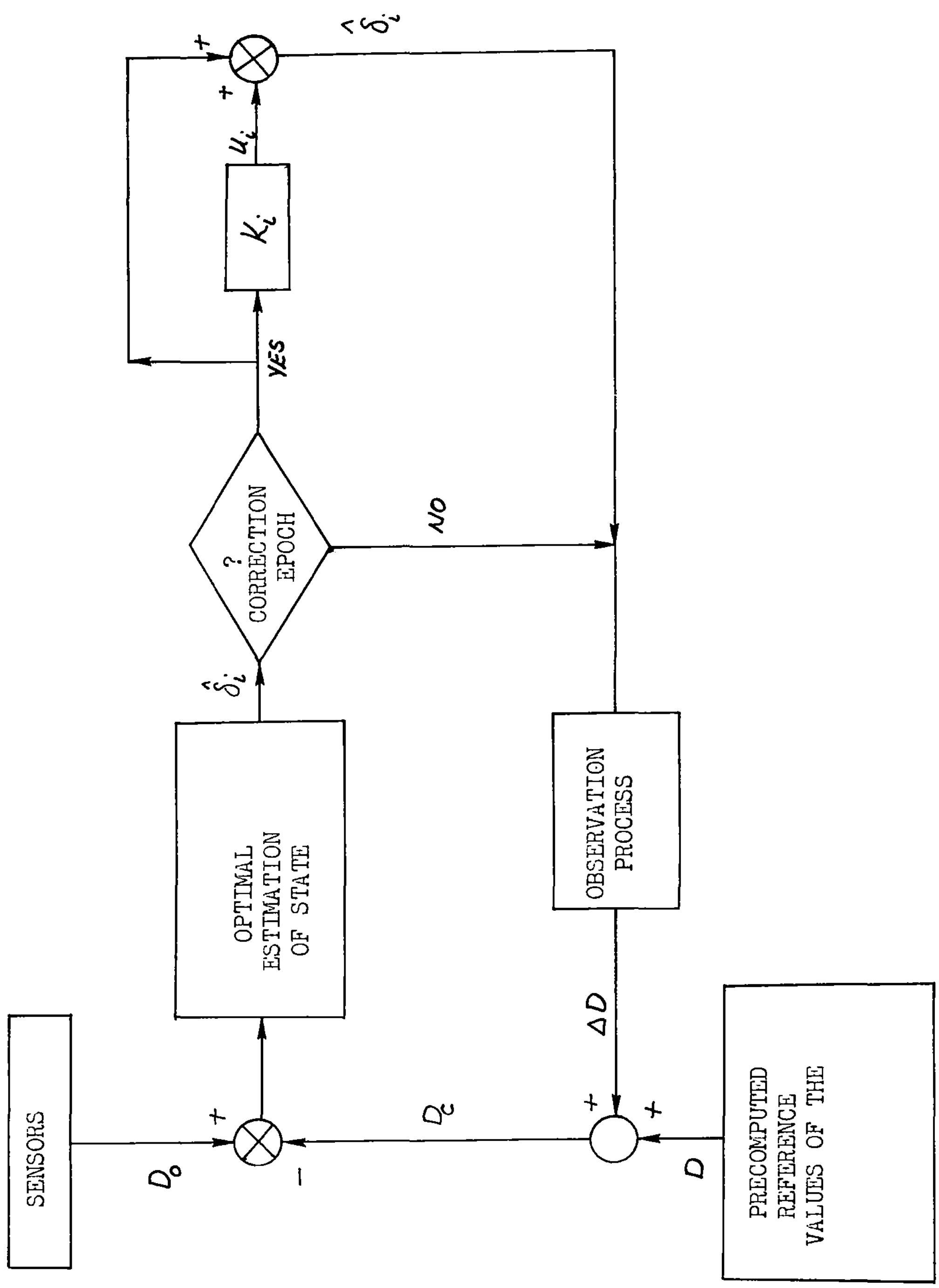
(2) The covariance matrices for the estimation of the state must be computed (These matrices are independent of true samples).

(3) The gains for the nth to the first correction epoch be computed.

(4) The table of costs as a function of the correction epochs be constructed and optimized by Dynamic Programming to define the proper sequence of correction. At least one correction must be made in an interval where the estimation error in the parameters being controlled is small.

(5) The weighting parameters be varied within any limits desired to define their effect on the total cost and the corrective strategy. (As shown in the text, the variation of the matrix Q_N will allow terminal constraints to be satisfied under the assumption that the state is ever known to the required precision.)

Once these tasks have been performed, the gains can be employed in a particular mechanization of the midcourse guidance problem as follows:



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1.12 (see 1.11)

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