

GUIDANCE, FLIGHT MECHANICS AND TRAJECTORY OPTIMIZATION

Volume I - Coordinate Systems and Time Measure

By M. B. Tamburro, A. S. Abbott, and G. E. Townsend

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Issued by Originator as Report No. SID 65-1200-1

Prepared under Contract No. NAS 8-11495 by
NORTH AMERICAN AVIATION, INC.
Downey, Calif.

for George C. Marshall Space Flight Center
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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FOREWORD

This report was prepared under contract NAS 8-11495 and is one of a series intended to illustrate analytical methods used in the fields of Guidance, Flight Mechanics, and Trajectory Optimization. Derivations, mechanizations and recommended procedures are given. Below is a complete list of the reports in the series.

Volume I	Coordinate Systems and Time Measure
Volume II	Observation Theory and Sensors
Volume III	The Two Body Problem
Volume IV	The Calculus of Variations and Modern Applications
Volume V	State Determination and/or Estimation
Volume VI	The N-Body Problem and Special Perturbation Techniques
Volume VII	The Pontryagin Maximum Principle
Volume VIII	Boost Guidance Equations
Volume IX	General Perturbations Theory
Volume X	Dynamic Programming
Volume XI	Guidance Equations for Orbital Operations
Volume XII	Relative Motion, Guidance Equations for Terminal Rendezvous
Volume XIII	Numerical Optimization Methods
Volume XIV	Entry Guidance Equations
Volume XV	Application of Optimization Techniques
Volume XVI	Mission Constraints and Trajectory Interfaces
Volume XVII	Guidance System Performance Analysis

The work was conducted under the direction of C. D. Baker, J. W. Winch, and D. P. Chandler, Aero-Astro Dynamics Laboratory, George C. Marshall Space Flight Center. The North American program was conducted under the direction of H. A. McCarty and G. E. Townsend.

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NOMENCLATURE LIST

a	Semi-major axis of ellipsoid Earth model, X direction (triaxial)
a_{ij}	Elements of a matrix (direction cosines)
a_s, a_m	Semi-major axis of Earth's (Moon's) orbit
A	Azimuth angle
b	Semi-major axis of ellipsoid Earth Model, Y direction (triaxial)
c	Semi-major axis of ellipsoid Earth Model, Z direction (triaxial)
$E.T.$	Ephemeris time
E	Elevation angle
e_s, e_m	Eccentricity of Earth's (Moon's) orbit
f	Flattening of the Earth
G	Universal gravitational constant
$G.S.T.$	Greenwich Sidereal Time
h	Altitude angle
H	Perpendicular distance from a point in space to surface of spheroid Earth model (altitude)
$H.A.$	Hour angle
i	Angle of inclination between orbital plane and some reference plane
i_m	Inclination of Moon's equatorial plane to the Earth's equatorial plane
i_o	Obliquity of Moon's orbital plane with respect to the true ecliptic
i_t	Obliquity of true ecliptic with respect to the fixed ecliptic at some epoch
i_e	Inclination of ecliptic to the mean equator of date
I_{xx}, I_{yy}, I_{zz}	Moment of inertia of Earth about X, Y and Z axes
J	Jacobian operator
$J.D.$	Julian date
$J.E.D.$	Julian ephemeris date

L.S.T.	Local sidereal time
L.M.T.	Local mean time = U.T. - λ
l, m, n	Direction cosine
M_m, M_s, M_e	Mass of Moon, Sun, Earth
M_s, M_m	Mean anomaly of Earth, Moon
N	Longitude of Earth's ascending node plus the angle to the lunar node
\bar{N}_G	Gradient of Earth model surface equation
n_e	Mean angular velocity of Earth in its orbit
\bar{n}_G	Unit outward normal vector to surface of Earth model
n_m	Mean angular velocity of Moon in its orbit
p	Frequency of Free Eulerian Nutation
q_r	The r th generalized coordinate in Lagrange's equation
\bar{r}	A vector from the origin of a coordinate system to a point in space
R	Magnitude of the distance from origin of coordinate system to a point on the Earth's surface
S.T.	Sidereal time
T	Rotational transformation or rotational kinetic energy of Earth
T	Time in mean solar days from 1950.0 (J.D. 2433282.423)
U.T.	Universal time
U_m	Work function for Moon-Earth system
U_s	Work function for Sun-Earth system
U	$U_m + U_s$
v	True anomaly
V	Periodic nutation potential

$\hat{x}, \hat{y}, \hat{z}$	Unit vector in X, Y, Z direction
X_G, Y_G, Z_G	X, Y, Z coordinates of ellipsoid Earth model surface definition
α	True right ascension angle
α_0	Right ascension angle uncorrected for precession
α'_0	Right ascension angle uncorrected for nutation or precession
β_{GE}	Ecliptic declination angle for Free Eulerian nutation
Δ	Distance from differential element of mass in Earth model to center of mass of a celestial body
δ	Declination angle
δ_0	Declination angle uncorrected for precession
δ'_0	Declination angle uncorrected for nutation
$\delta\phi$	Difference between astronomical latitude and geodetic latitude
$\delta\lambda$	Longitude angle between zenith and norm to Earth model
ξ, η, ξ	Cartesian variables for differential mass location
θ, ϕ, ψ	Euler angles of rotation
θ_m	Obliquity of Earth's mean celestial pole to the mean ecliptic pole due solely to solar-lunar precession
Θ	Periodic variation in the obliquity associated with nutation due to solar-lunar gravitation
λ	Longitude angle from reference
μ	Gravitational constant
ρ	Distance between centers of gravity of the Earth and a celestial body
τ	Period of Free Eulerian nutation
τ'	Projection of the vernal equinox of date on the orbital plane
τ'_M	Intersection of the Earth equatorial meridian passing through the true vernal equinox of date and the Moon's equatorial plane
γ	Vernal equinox of date

τ_M	Mean vernal equinox of date
τ	True vernal equinox of date
τ_{50}	Mean vernal equinox of epoch 1950.0
ϕ	Latitude in spheroidal coordinates
ϕ'	Latitude angle in spherical polar coordinates
Ψ	Periodic variation in the precession angle (nutations)
ψ_m	Precession about the mean ecliptic pole
$\bar{\omega}$	Angular velocity of Earth's rotation about its axis
ω_{zo}	The constant angular velocity of the spheroid Earth model about the body fixed polar axis
ω	Argument of perigee
$\bar{\omega}_m$	Angular velocity of Moon's rotation about its axis
$\tilde{\omega}_s$	Longitude of perihelion of Earth's orbit
$\tilde{\omega}_m$	Longitude of perihelion of Moon's orbit
Ω	Longitude of ascending node

1. STATEMENT OF THE PROBLEM

All of the problems to be discussed in this and subsequent monographs of the series require that the position and velocity of the particle being studied be describable in a simple and definitive manner. Thus, major attention must be placed on systems of coordinate measure. The complete description of these systems is the objective of this monograph. This emphasis will be exhibited in the orderly progression from the basic requirements of measure to the definition of coordinate frameworks utilized for trajectory problems and those employed for the description of body oriented axis systems. This discussion will take advantage of a NASA prepared document (Reference 1) on coordinate systems for the Apollo Project and of its notation. However, due to the broad spectrum of problems to be encountered in this series of monographs, adaptations and additions have at times been necessary.

Once these basic coordinate frames are described, attention will turn to the description of the nature of the motions of these fundamental reference systems due to the motion of the moon about the earth (non-spherical and non-homogeneous) and due to gravitational attractions on the mass anomalies produced by other bodies in the solar system. This analysis will take the form of the development of a math model to be employed, the derivation and simplification of the equations of motion and the solution of these equations to yield the nutation and precession corrections to the basic reference frame. This analysis, while not precise (due to the math model employed), is singularly helpful in identifying the nature of the motion and the primary sources of the disturbances. Equations describing the observed motions of these coordinate systems are also included for the purpose of completing the analysis.

The final discussion in the monograph is of time standards employed in the study of astronomy (sidereal time, universal time and ephemeris time). These standards, which are still employed as the basis of civil time, etc., (ephemeris time is a consequence of atomic clocks) are developed from the historical as well as the physical point of view for the purpose of relating the rotating coordinate systems described earlier to measurements of the celestial sphere. The section includes an example which demonstrates the relationships between these time standards and the process to be employed in a given reduction (frequent reference is, of course, made to data available in the American Ephemeris and Nautical Almanac).

2. STATE OF THE ART

2.1 Coordinate Systems

2.1.1 Introduction to Coordinate Systems

2.1.1.1 The Purpose of Coordinate Systems

In the process of conducting an engineering study in the realm of flight mechanics, trajectory optimization or guidance, the first decision to be made pertains to the selection of a coordinate system to uniquely describe the position of the vehicle being considered and in which the equations of motion in a Newtonian sense can be written. (Relativistic mechanics will not be considered since the velocities generally encountered in these problems are much less than that of light.) Thus, at this point, a simple statement of the purpose of such systems is in order.

A coordinate system is a framework constructed to allow the specification of an ordered triplet of real numbers associated with the position of a point in three-dimensional space. (The idea of using velocities as coordinates of a point without distinction between velocity and position coordinates constitutes the phase space approach to dynamics and is discussed in several of the monographs which follow this, the first of the series.)

2.1.1.2 Newtonian Mechanics and the Inertial Frame

If G is a frame of reference with point O fixed in G , and P a moving particle, its position relative to O can be described as a vector quantity $\vec{r} = \vec{OP}$. (The velocity and acceleration of P are defined to be $\vec{v} = \frac{d\vec{r}}{dt}$ and $\vec{a} = \frac{d\vec{v}}{dt}$.) On the basis of astronomical observations, Newton conjectured that in a basic frame (an inertial, non-rotating, non-accelerating axis system) of reference there existed a proportionality between the force acting on a body and its acceleration. This constant of proportionality determines the mass of the body. The equivalent of his fundamental laws are:

Every body persists in a state of rest or in uniform motion in a straight line except if it is compelled by force to change that state.

The time rate of change of linear momentum is equal to the force producing it and the change occurs in the direction the force is acting. (In vector notation this statement becomes $\vec{F} = \frac{d}{dt}(m\vec{v})$.)

The mutual actions of any two bodies are always equal and opposite in direction.

The constitution of a basic frame (in which Newton's laws are valid) is an elusive and possibly a metaphysical concept. It is possible, however, to establish such reference axes to a satisfactory degree of approximation due to the almost "fixed" pattern of the stars in a given celestial field. The extremely slow observed motions result from the large, though finite, distances between our solar system and a typical star. These distances,

in turn, are responsible for the fact that the direction from any point in the solar system to the star is approximately constant in inertial space (i.e., the angle between two direction lines from any point in the system to two such "fixed" stars remains constant). A set of inertial axes can, therefore, be established using these direction lines to define the non-rotating system and the mass center of the solar system as the non-acceleratin origin.

2.1.1.3 Coordinate System Requirements

Before describing the many systems of coordinate measurement in use, it is beneficial to first look at the four characteristics of every coordinate system. First, all of these systems must have an origin. In a three-dimensional space the origin is simply the location of the null set [0,0,0] of the coordinates. Second, there must be a reference plane which contains this point (any of an infinite set). This plane can be defined by any two vectors lying in it or by defining one of its two poles, (i.e., the positive or negative normal to the surface emanating from the origin). Third, an arbitrarily selected but identifiable vector in the reference plane must be selected as a principal direction. Finally, a technique of measuring the coordinates of a point in the system must be established. The methods used for coordinate measurement to be described here are the rectangular cartesian, the spherical polar and the spheroidal.

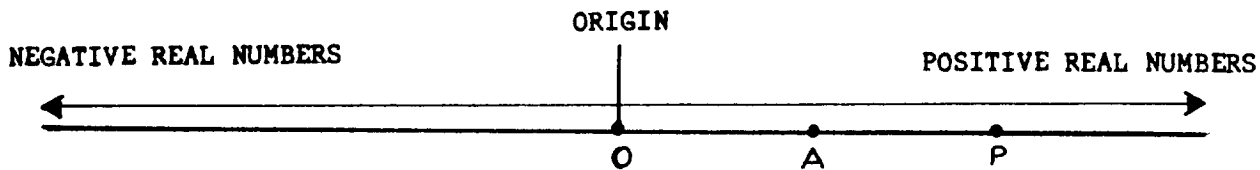
2.1.1.4 Coordinate Measurement Techniques

2.1.1.4.1 Rectangular Cartesian Coordinate Measurement

One of the principal ideas leading to the establishment of the coordinate system is the idea of identifying the complete set of real numbers with the set of points comprising a straight line; that is, to each real number there corresponds a single line and vice versa. This idea enables a one-dimensional coordinate system to be constructed where the coordinate [X] of a point P on the line is given by

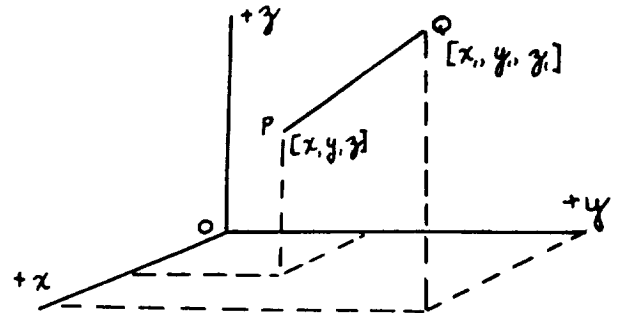
$$X = \frac{OP}{OA}$$

OP is the length of line from O to P and OA is a unit length



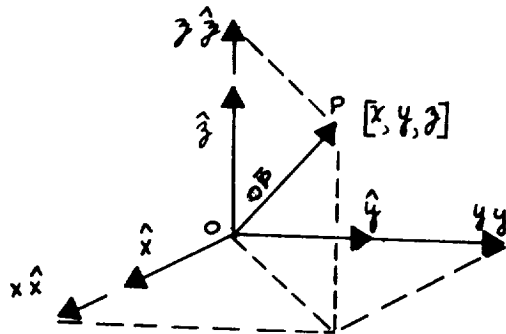
By taking three noncoplanar straight lines intersecting at an origin O an oblique cartesian coordinate system, associating each point in a three-dimensional space with an ordered triplet of real numbers, can be constructed. If the coordinate axes intersect at right angles to each other the system is said to be a rectangular cartesian coordinate system. A feature of the rectangular cartesian system is that the length of a line joining points $P[x, y, z]$ and $Q[x', y', z']$ is given by the expression

$$PQ = [(x-x')^2 + (y-y')^2 + (z-z')^2]^{\frac{1}{2}}$$



A three-dimensional space having a coordinate system with this property is said to be a Euclidian 3 - space. Since the ordered triplet of numbers $[x, y, z]$ satisfies all the requirements of vector spaces it will be referred to as a vector and denoted by a bar above the quantity (i.e., \overline{op}). This vector space is called a three-dimensional linear vector space since any vector in the space can be represented as a linear combination of three noncoplanar vectors. If these noncoplanar vectors are chosen to be of unit magnitude having only one non-zero coordinate (i.e., $[1, 0, 0]$, $[0, 1, 0]$ and $[0, 0, 1]$) they are called fundamental unit vectors. In this monograph the fundamental unit vectors are denoted by the symbols $\hat{x}, \hat{y}, \hat{z}$. Thus any vector $[x, y, z]$ or \overline{op} , can be expressed as the following linear combination

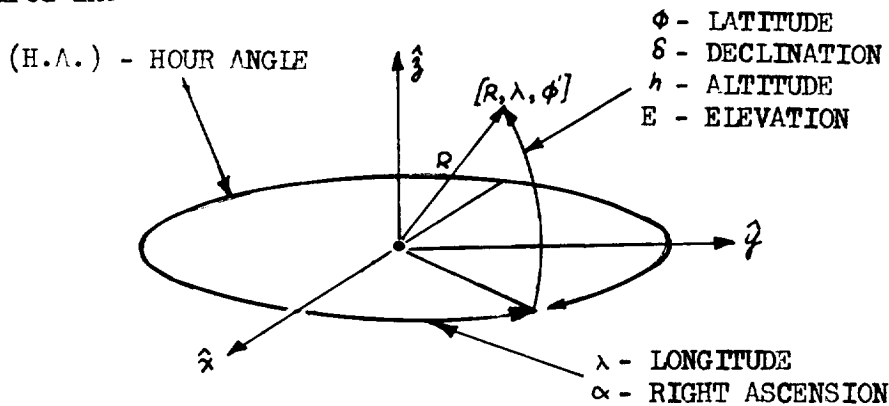
$$\overline{OP} = x\hat{x} + y\hat{y} + z\hat{z}$$



2.1.1.4.2 Spherical Polar Coordinate Measurement

The basis of spherical polar coordinate measurement is an ordered triplet of real numbers $[R, \lambda, \phi]$. The first number is associated with the distance between the point being measured and the origin. The other two numbers are associated with two angles specifying the direction to the point from the origin. When a reference line, of length R and imbedded in a rectangular cartesian system, is rotated through these angles about two selected axes the end of the line specifies the location of the point.

Consider a rectangular cartesian system determined by the unit vectors \hat{x} , \hat{y} , \hat{z} . If the reference line is chosen to be along the \hat{x} axis the first angle is determined by a rotation about the \hat{z} axis in a positive right-hand manner for the longitude and right ascension techniques and in a negative manner for the hour angle technique. The second angle is measured in a plane passing through the rotated reference line and the \hat{z} axis in a direction positive towards the \hat{z} axis. This angle is referred to as the latitude, declination, or altitude depending on the particular coordinate system it is being measured in.



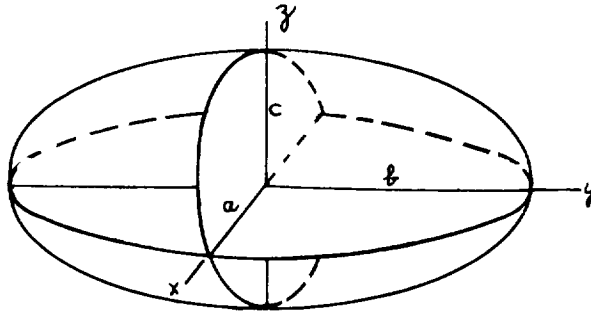
From the diagram, it is evident that the spherical coordinates of a point are unique and may not be added in the sense of rectangular coordinates. (That is, the sum of two vectors in spherical coordinates is not the result obtained by adding the two ordered triplets for the respective vectors).

2.1.1.4.3 Spheroidal Coordinate Measurement

The surface defined by a second degree equation in the rectangular cartesian coordinates $[x, y, z]$ is called a quadric surface. A quadric surface given by an equation of the form

$$\frac{x_g^2}{a^2} + \frac{y_g^2}{b^2} + \frac{z_g^2}{c^2} = 1$$

is called an ellipsoid, since the intersection of any plane passing through the origin with this surface is an ellipse. The subscript g indicates these are coordinates of a surface point.



The figure described by this equation resembles a sphere which has been flattened along two of its axes with the x, y, z axes being the semi-major, mean, and minor axes of the figure. The length of these axes are respectively, a, b , and c . If $a = b$, then the figure is called a spheroid. For $a = b > c$ the figure is called an oblate spheroid. Since the figure of a spheroid can be obtained by rotating an ellipse about one of its principal axes it is also called an ellipsoid of revolution. The figure of most concern to geodesy is the oblate spheroid which satisfies the equation

$$\frac{x_g^2 + y_g^2}{a^2} + \frac{z_g^2}{c^2} = 1, \quad a > c$$

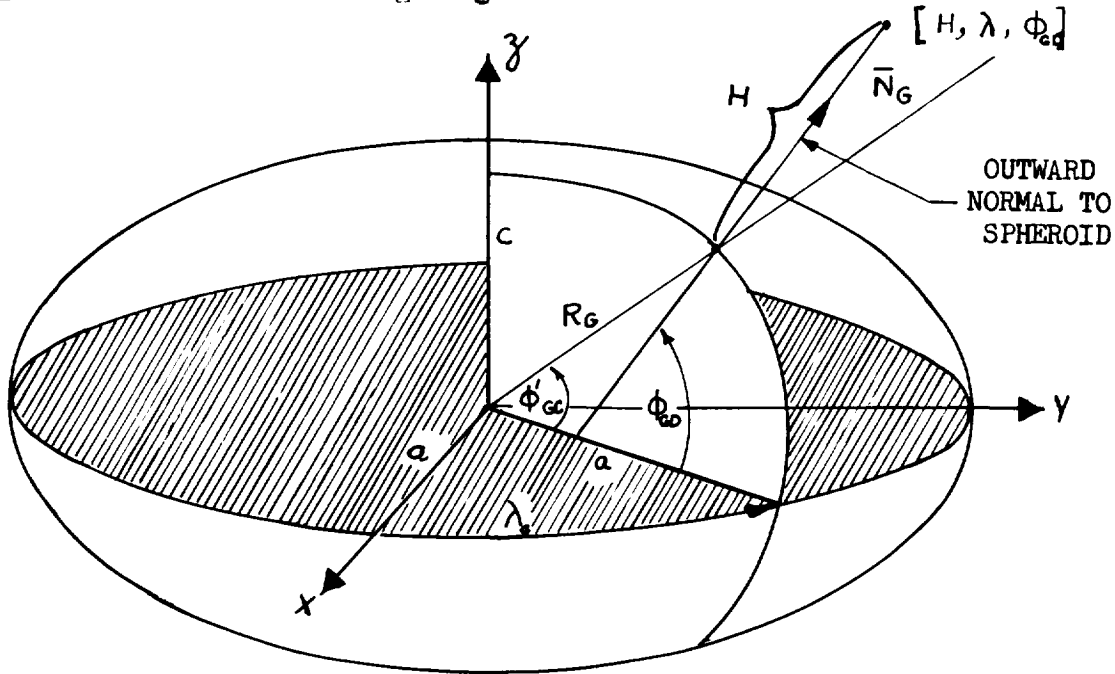
The parameters usually chosen to describe an oblate spheroid are the major axis length (or equatorial radius), a and the flattening f given by

$$f = 1 - \frac{c}{a}$$

Since, for an oblate spheroid, $c < a$ the flattening is always a positive number.

The basis of spheroidal coordinate measurements is the ordered triplet of real numbers $[H, \lambda, \phi]$. The first number H is associated with the perpendicular distance of the point being measured to the surface of the spheroid. The second number λ , called the longitude is the same second number used in the spherical polar measurement technique. The third number ϕ is associated with the angle measured in the plane of the z axis and the point being considered between the normal to the spheroid in the direction of

the point and the major axis. The method of spheroidal coordinate measurement is illustrated in the following diagram.



A point $[x_G, y_G, z_G]$ on the surface of the spheroid is given in spheroidal coordinates by $[c, \lambda, \phi_G]$, whereas in spherical polar coordinates it is $[R_G, \lambda_G, \phi'_G]$. It is a simple matter to express the rectangular cartesian coordinates in terms of the corresponding spherical coordinates by means of trigonometric functions

$$x_G = R_G \cos \phi'_G \cos \lambda_G$$

$$y_G = R_G \cos \phi'_G \sin \lambda_G$$

$$z_G = R_G \sin \phi'_G$$

However, to express these coordinates in terms of the spheroidal coordinates for any given a and c preliminary steps must be taken. It is first necessary to compute the rectangular cartesian components of the unit outward normal vector in terms of both spherical polar and spheroidal coordinates. An outward normal vector N_G is simply a vector lying in the direction of the maximum rate of change of the scalar function determining the surface. N_G is obtained by operating on the function using the gradient operator denoted by the symbol ∇ . That is

$$\begin{aligned}
\bar{N}_G &= \nabla \left[\frac{x_G^2 + y_G^2}{a^2} + \frac{z_G^2}{c^2} - 1 \right] \\
&= 2 \left[\frac{x_G}{a^2} \hat{x} + \frac{y_G}{a^2} \hat{y} + \frac{z_G}{c^2} \hat{z} \right] \\
&= 2 R_G \left[\frac{\cos \phi'_{GC} \cos \lambda_G}{a^2} \hat{x} + \frac{\cos \phi'_{GC} \sin \lambda_G}{a^2} \hat{y} + \frac{\sin \phi'_{GC}}{c^2} \hat{z} \right]
\end{aligned}$$

The unit outward normal vector \bar{N}_G in terms of spherical coordinates is

$$\bar{n}_G = \frac{\bar{N}_G}{|\bar{N}_G|} = \frac{\left[\frac{\cos \phi'_{GC} \cos \lambda_G}{a^2} \hat{x} + \frac{\cos \phi'_{GC} \sin \lambda_G}{a^2} \hat{y} + \frac{\sin \phi'_{GC}}{c^2} \hat{z} \right]}{\left[\frac{\cos^2 \phi'_{GC}}{a^4} + \frac{\sin^2 \phi'_{GC}}{c^4} \right]^{\frac{1}{2}}}$$

In terms of spheroidal coordinates the unit normal vector is

$$\bar{n}_G = (\cos \phi_0 \cos \lambda) \hat{x} + (\cos \phi_0 \sin \lambda) \hat{y} + (\sin \phi_0) \hat{z}$$

Now equating the components of these two expressions for the unit normal vector results in the relation

$$\tan \phi'_{GC} = \frac{c^2}{a^2} \tan \phi_{GD}$$

To establish the value of $\phi_0 - \phi'_{GC}$ as a function of the angle ϕ_{GD} the procedure is as follows:

$$\begin{aligned}
\tan(\phi_0 - \phi'_{GC}) &= \frac{\tan \phi_0 - \tan \phi'_{GC}}{1 + \tan \phi_0 \tan \phi'_{GC}} = \frac{\tan \phi_0 (1 - \frac{c^2}{a^2})}{1 + \frac{c^2}{a^2} \tan^2 \phi_{GD}} \\
&= \frac{\sin \phi_0 \cos \phi_0 (1 - \frac{c^2}{a^2})}{\cos^2 \phi_0 + \frac{c^2}{a^2} \sin^2 \phi_{GD}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\sin 2\phi_{GD}(2f - f^2)}{2 - 2(2f - f^2)\sin^2\phi_{GD}} \\
&= \frac{[(2f - f^2)/(2 - 2f + f^2)] \sin^2\phi_{GD}}{1 + [(2f - f^2)/(2 - 2f + f^2)] \cos^2\phi_{GD}} \\
&= \frac{l \sin 2\phi_{GD}}{1 + l \cos 2\phi_{GD}}
\end{aligned}$$

where

$$l = \frac{2f - f^2}{2 - 2f + f^2}$$

since

$$\begin{aligned}
\exp(i\chi) &= \cos \chi + i \sin \chi \\
\exp(-i\chi) &= \cos \chi - i \sin \chi
\end{aligned}$$

then

$$\exp(2i\chi) = \frac{1 + i \tan \chi}{1 - i \tan \chi}$$

or

$$2i\chi = \ln \frac{1 + iy}{1 - iy}$$

where

$$y = \tan \chi$$

Hence

$$x = \tan^{-1} y = \frac{1}{2i} \ln \left\{ \frac{1+iy}{1-iy} \right\}$$

Therefore

$$\begin{aligned} \phi_{60} - \phi'_{6c} &= \frac{1}{2i} \ln \left\{ \frac{1 + i \left[(l \sin 2\phi_{60}) / (1 + l \cos 2\phi_{60}) \right]}{1 - i \left[(l \sin 2\phi_{60}) / (1 + l \cos 2\phi_{60}) \right]} \right\} \\ &= \frac{1}{2i} \ln \left\{ \frac{1 + l \exp(2i\phi_{60})}{1 + l \exp(-2i\phi_{60})} \right\} \end{aligned}$$

or expanding this expression in series

$$\phi_{60} - \phi'_{6c} = l \sin 2\phi_{60} - \frac{l^2}{2} \sin 4\phi_{60} + \frac{l^3}{3} \sin 6\phi_{60} - \dots$$

An expression for the radial distance R_G to the spheroid can be derived as follows:

$$\begin{aligned} R_G &= [x_G^2 + y_G^2 + z_G^2]^{\frac{1}{2}} \\ &= \left[\frac{\cos^2 \phi'_{6c}}{a^2} + \frac{\sin^2 \phi'_{6c}}{c^2} \right]^{-\frac{1}{2}} \\ &= a(1-f) \left[1 - (2f-f^2) \cos \phi'_{6c} \right]^{-\frac{1}{2}} \\ &= a \left[1 - f^2 \sin^2 \phi'_{6c} - \frac{3}{2} f^2 \sin^2 \phi'_{6c} + \frac{3}{2} f^2 \sin^4 \phi'_{6c} + \dots \right] \end{aligned}$$

The average (or mean sea-level) figure of the earth is best represented as an oblate spheroid with the minor axis being the axis of revolution (polar axis). This model is not exact, however, it is adequate for most trajectory studies. For this reason the best values of the equatorial radius a and the flattening f are desired. These data along with the polar radii are presented in the following table.

Table.1. Equatorial Radius a , and Flattening f

	Baker	Kaula	Statistical Estimate from Available Data
Equatorial radius, a (km)	6378.150 ± 0.050	6378.163 ± 0.021	6378.210 ± 0.045
$\frac{1}{f}$	298.30 ± 0.05	298.24 ± 0.04	298.27 ± 0.03
Polar radius, c (km)	6356.768 ± 0.050	6356.777 ± 0.021	6356.826 ± 0.045
Confidence Level	?	?	95%

In addition, Kaula's values of a and f have been utilized to construct tables of $\phi_{GD} - \phi'_{GC}$ and R_G/a (Tables 2 and 3)

2.1.1.5 Classification of Coordinate Systems

For the purposes of this monograph, all coordinate systems (regardless of their method of coordinate measurement) will be classified as being either observational or dynamical in nature. Observational coordinate systems will be considered to be fixed with respect to either an observer or a central body and thus include most of the astronomical frames. (A central body is a body having a very large mass as compared to a body moving under its sole influence.) On the other hand the dynamical coordinate system will be considered to be fixed with respect to the orbital plane of the vehicle or some identifiable feature of the vehicle's structure. A further classification of coordinate systems will be made based on whether the system is rotating or fixed with respect to the "fixed" stars.

In order to accurately describe the coordinates of a number of bodies in the many coordinate systems described in this monograph a superscript and subscript is given to each coordinate. The superscript gives the number of the body being measured. For the purposes of describing an arbitrary body number, no superscript is given. The subscript indicates the coordinate system in which the body is being measured. These superscripts and subscripts are easily read by consulting the given listing.

Geodetic Latitude ϕ (Degrees)	$\phi - \phi'_G$ (Seconds)					
G-4	0.	0.24223325E 02	0.48417434E 02	0.72553143E 02	0.96601336E 02	
5-9	0.12053300E 03	0.14431927E 03	0.16793143E 03	0.19134097E 03	0.21451965E 03	
10-14	0.23743945E 03	0.26007270E 03	0.28239203E 03	0.30437046E 03	0.32598141E 03	
15-19	0.34719872E 03	0.36799669E 03	0.38835013E 03	0.40823436E 03	0.42762526E 03	
20-24	0.44649928E 03	0.46483351E 03	0.48260564E 03	0.49979403E 03	0.51637776E 03	
25-29	0.53233659E 03	0.54765105E 03	0.56230243E 03	0.57627278E 03	0.58954498E 03	
30-34	0.60210275E 03	0.61393065E 03	0.62501412E 03	0.63533948E 03	0.64489396E 03	
35-39	0.65366571E 03	0.66164384E 03	0.66881837E 03	0.67518034E 03	0.68072173E 03	
40-44	0.68543553E 03	0.68931570E 03	0.69235726E 03	0.69455621E 03	0.69590956E 03	
45-49	0.69641538E 03	0.69607275E 03	0.69488179E 03	0.69284366E 03	0.68996053E 03	
50-54	0.68623566E 03	0.68167326E 03	0.67627863E 03	0.67005807E 03	0.66301891E 03	
55-59	0.65516946E 03	0.64651906E 03	0.63707801E 03	0.62685763E 03	0.61587014E 03	
60-64	0.60412878E 03	0.59164769E 03	0.57844191E 03	0.56452743E 03	0.54992109E 03	
65-69	0.53464059E 03	0.51870450E 03	0.50213218E 03	0.48494381E 03	0.46716033E 03	
70-74	0.44880340E 03	0.42989545E 03	0.41045957E 03	0.39051953E 03	0.37009971E 03	
75-79	0.34922511E 03	0.32792131E 03	0.30621442E 03	0.28413107E 03	0.26169834E 03	
80-84	0.23894379E 03	0.21589535E 03	0.19258135E 03	0.16903043E 03	0.14527155E 03	
85-89	0.12133393E 03	0.97247005E 02	0.73040415E 02	0.48743938E 02	0.24387476E 02	
90	0.					

Table 2

Angular difference between geodetic and geocentric

latitude ($\phi - \phi'_G$) as a function of geodetic latitude
for the Kaula Earth model ($1/f = 298.24$) measured in
seconds of arc

Geodetic Latitude ϕ (Degrees)	$\frac{R_G}{a}$							
0-4	0.99999997E 00	0.99999896E 00	0.99999593E 00	0.99999086E 00	0.99998379E 00			
5-9	0.99997471E 00	0.99996364E 00	0.99995058E 00	0.99993556E 00	0.99991859E 00			
10-14	0.99989969E 00	0.99987888E 00	0.99985619E 00	0.99983165E 00	0.99980528E 00			
15-19	0.99977712E 00	0.99974719E 00	0.99971555E 00	0.99968222E 00	0.99964723E 00			
20-24	0.99961064E 00	0.99957249E 00	0.99953283E 00	0.99949170E 00	0.99944915E 00			
25-29	0.99940524E 00	0.99936000E 00	0.99931349E 00	0.99926578E 00	0.99921693E 00			
30-34	0.99916699E 00	0.99911602E 00	0.99906407E 00	0.99901124E 00	0.99895754E 00			
35-39	0.99890307E 00	0.99884790E 00	0.99879207E 00	0.99873568E 00	0.99867877E 00			
40-44	0.99862141E 00	0.99856370E 00	0.99850567E 00	0.99844742E 00	0.99838901E 00			
45-49	0.99833051E 00	0.99827200E 00	0.99821353E 00	0.99815520E 00	0.99809706E 00			
50-54	0.99803918E 00	0.99798165E 00	0.99792453E 00	0.99786787E 00	0.99781178E 00			
55-59	0.99775629E 00	0.99770150E 00	0.99764746E 00	0.99759424E 00	0.99754190E 00			
60-64	0.99749052E 00	0.99744014E 00	0.99739084E 00	0.99734266E 00	0.99729569E 00			
65-69	0.99724998E 00	0.99720557E 00	0.99716253E 00	0.99712090E 00	0.99708075E 00			
70-74	0.99704211E 00	0.99700505E 00	0.99696960E 00	0.99693581E 00	0.99690370E 00			
75-79	0.99687335E 00	0.99684476E 00	0.99681800E 00	0.99679308E 00	0.99677004E 00			
80-84	0.99674890E 00	0.99672969E 00	0.99671245E 00	0.99669718E 00	0.99668391E 00			
85-89	0.99667266E 00	0.99666342E 00	0.99665623E 00	0.99665108E 00	0.99664799E 00			
90	0.99664699E 00							

Table 3 Variation of Earth's radial distance with geodetic latitude
for the Kaula Earth model ($1/f = 298.24$)

<u>Superscripts</u>	<u>1st Subscript</u>	<u>2nd Subscript</u>
L Launch site	G Geo	A astronomical
T Observer (topos)	O orbital	C centric
V Vehicle	S seleno	D detic
		E equatorial
		ecliptic
		G graphic
M Moon	T topo	H horizontal
		L lunar equatorial
S Sun	V vehicle	P principal
		S stable

Using the above notation, the coordinate ${}^V\chi_{SG}$ for instance is interpreted to be the x coordinate of the vehicle as measured in the selenographic coordinate system. In general, the first subscript can be identified with the center of origin of coordinates and the second subscript associated with the reference plane. Although all combinations of the two are not used, this notation suggests an abundance of possible coordinate systems. Some of these systems will be discussed and illustrated later (sections 1.1.2.2 and 1.1.2.3).

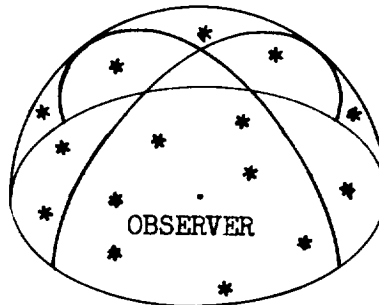
2.1.1.6 Selection Basis

The bases upon which a coordinate system is selected are the requirements of the particular problem under investigation, and interface considerations. For problems involving the contributions of many technologies it is generally wise to adopt a set of standard coordinate systems.

2.1.2 Observational Coordinate Systems

2.1.2.1 The Celestial Sphere

The first class of coordinate systems to be considered is that of the observer or the observer's central body. (These are the systems commonly associated with astronomical and radar observations.) In each of these systems, the observer and the observed body are in motion. Further, the motion can be described only with the aid of an "inertial" coordinate system in which Newton's equations can be evaluated. Thus, since an observer has no concept of direction except in relation to other bodies, objects, etc.; and since the less the motion of the reference direction, the more accurately the observer can correlate data acquired at different times, the concept of the celestial sphere has developed. The celestial sphere is an imaginary surface of infinite radius on which the positions of the stars are projected (see sketch).



Since the celestial sphere is considered to have an infinite radius, every point in the solar system can be regarded as being a center and all lines (or planes) parallel to each other will intersect the sphere at common points (or great circles). This fact not only allows positions to be quoted relative to the "fixed" stars but it also drastically simplifies the problem of reducing data acquired from several observers, each of which was employing a reference system of finite proportions.

The difference in position of celestial bodies projected on the celestial sphere due to a change of origin is known as parallax. Since the dimensions of the Earth's orbit are known, this effect can be measured by comparing the apparent positions of a celestial body at different times during the Earth's transit about the Sun. For origins located within the solar system parallax is negligible for all but the nearest stars. The celestial sphere therefore provides a convenient framework for fixing the relative positions of the heavenly bodies.

2.1.2.2 Origin of Coordinates

The first characteristic of a coordinate system is an origin of coordinates; that is, the point at which the null, or zero set is located. The origins of observational coordinate systems and their designations are:

- 1) The center of the Earth - geocentric, (from the Greek geo-earth and Gr. kentron - center)
- 2) The center of the Moon - selenocentric (Gr. selene-moon)
- 3) The center of the Sun - heliocentric (Gr. helios - sun)
- 4) The center of Mass - **barycentric**,
(Gr. barys-heavy)
- 5) The observer - topocentric (Gr. topos-place)

2.1.2.3 Reference Planes

Associated with every reference plane are its poles, or the points at which a normal to the reference plane pierces the celestial sphere. The position of the poles on the celestial sphere does not change with a change in origin of coordinates since the celestial sphere was assumed to have an infinite radius. There are four reference planes, known as fundamental planes, that are the basis of most astronomical measurements. They are:

- 1) The plane of the Earth's equator

This is the plane perpendicular to the instantaneous axis of rotation of the Earth. The great circle in which it intersects the celestial sphere is called the Earth's celestial equator (or celestial equator). The poles of the Earth's equator are known as celestial poles.

2) The plane of the ecliptic

The plane in which the Earth moves around the Sun (withstanding perpendicular disturbances caused by the sun's oblateness, the Moon and the planets) is designated as the ecliptic. Its great circle on the celestial sphere is simply called the ecliptic.

3) The plane of the geodetic horizon

This is the plane tangent to the surface of the oblate spheroid Earth model, at the observer. If the actual Earth coincided with this model, it would also be the plane normal to the direction of gravity at the observer.

4) The plane of the astronomical horizon

This is the plane perpendicular to the direction of the local gravity vector or the direction of a plumb line at the observer. Due to certain anomalies, such as surface terrain effects, it does not exactly coincide with the plane of the geodetic horizon. The great circle in which the astronomical horizon intersects the celestial sphere is called the celestial horizon. The pole of the plane of the horizon in the upward direction is known as the zenith, and in the downward direction, the nadir.

In addition, the following non-fundamental reference planes will also be considered:

5) The plane of the Moon's equator

This is the plane perpendicular to the axis of rotation of the Moon. The great circle in which it intersects the celestial sphere is called the Moon's celestial equator.

6) The plane of the Earth-Moon systems

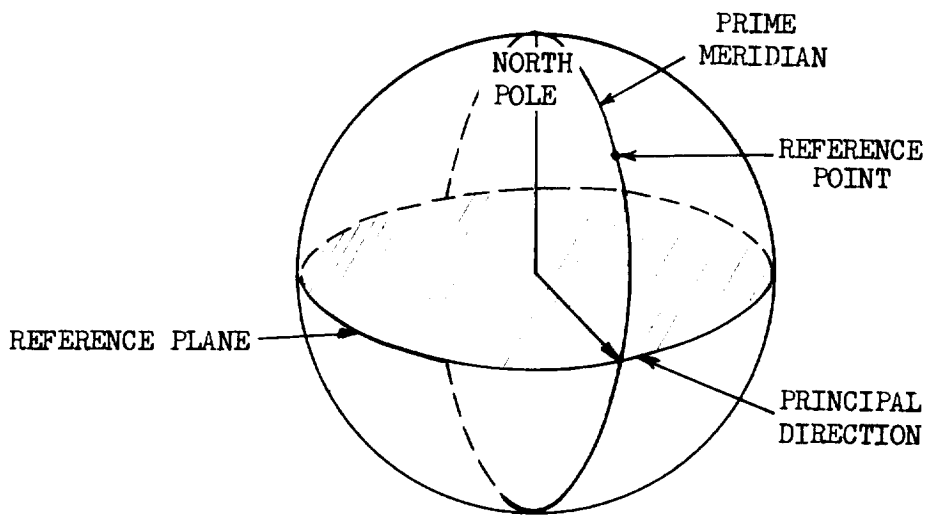
This is the plane described by a line passing through the center of the Earth and the center of the Moon, as the Moon revolves about the earth.

7) The plane of the Galaxy

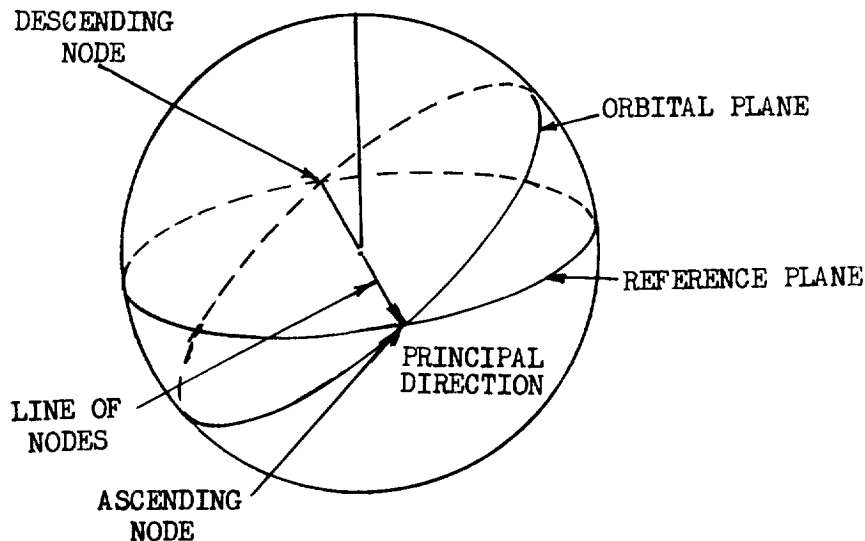
This is the plane perpendicular to the axis of rotation of the Milky Way Galaxy. The great circles intersecting the celestial sphere are known as galactic circles. Its poles are called galactic poles.

2.1.2.4 Principal Directions

Having established an origin and reference plane for an axis system, it is necessary to prescribe a direction to complete a unique description of an axis system. If the chosen direction lies in the reference plane, it is called a principal direction. All principal directions are determined by the line of intersection between a great circle on the celestial sphere and the reference plane. If the great circle passes through the poles of the reference plane, it is called a meridian of the reference plane.



If the great circle is the celestial circle of another reference plane, the line of intersection, (determining the principal direction) is called the line of nodes of the two reference planes. Most often, one of these reference planes is an orbital plane. If this is the case the node passed as the orbiting body goes toward the northern hemisphere of the reference plane is called the ascending node and the other the descending node.

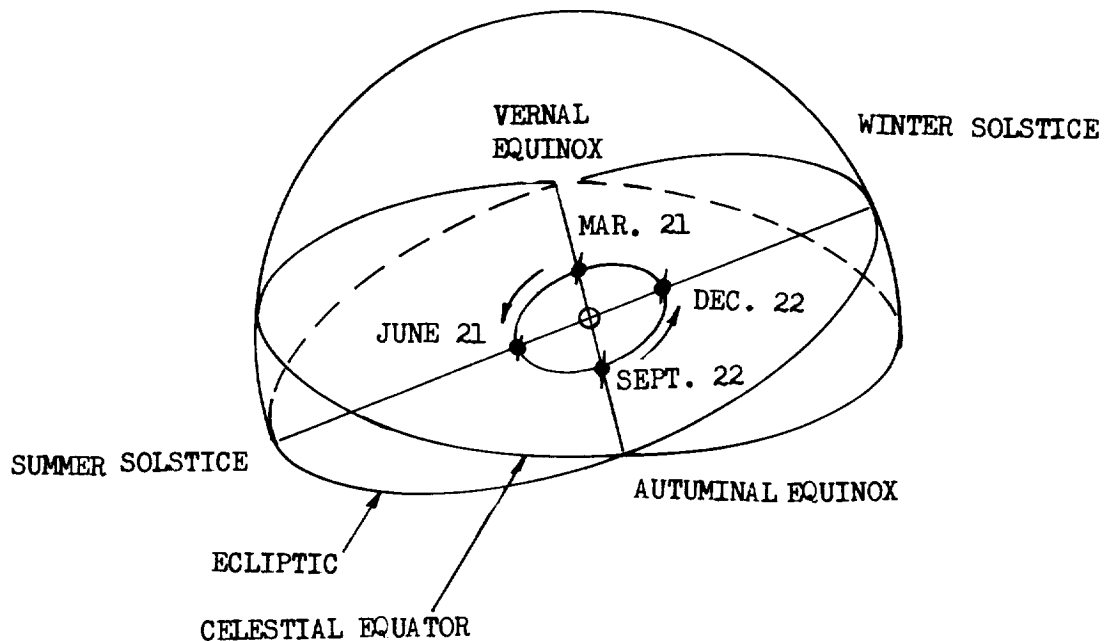


The three most common principal directions defined in terms of a meridian are those determined by the Greenwich or prime meridian, the lunar prime meridian and the local meridian. For a geocentric system, a principal direction is determined by the plane of the Earth's equator and the meridian passing through the Royal Observatory at Greenwich, England, positive in the direction nearest to Greenwich.

For a selenocentric system a principal direction is determined by the plane of the Moon's equator and the lunar meridian passing through the mean center point (MCP) of the Moon, positive in the direction of the mean center point.

The local meridian is the great circle normal to the fundamental plane passing through the position of the observer. It fixes a principal direction which is in the direction nearest to the observer.

The most useful principal direction in astronomical work is that established by the intersection of two reference planes, that of the Earth's equator and the ecliptic. The principal direction is taken in the direction of the ascending node, that is, the point on the celestial sphere where the apparent orbit of the sun crosses the celestial equator going north. Since this occurs during the spring season of the Northern Hemisphere at the time when sunlight and darkness intervals are equal it is called the vernal equinox. (from the Latin vernalis - spring, aliquis - equal, and nox - night). The point on the celestial sphere where the plane of the ecliptic is nearest to the north celestial pole is called the summer solstice and the point nearest the south celestial pole the winter solstice.



Unfortunately, from the standpoint of computing the positions, or ephemerides of the celestial bodies with the passage of time, both the poles of the ecliptic and the celestial poles are continuously in motion. Hence, the equinox is continuously moving on the celestial sphere. This motion will be discussed in detail in section 2.2 of this monograph; however, the general nature of the motion will be discussed in subsequent paragraphs.

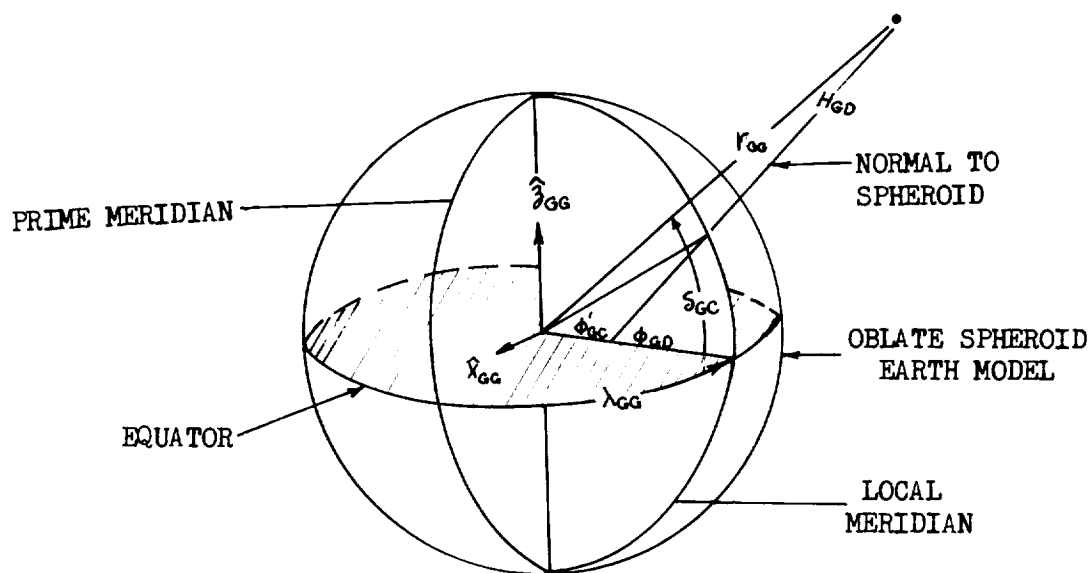
The motion of the pole of the ecliptic is a result of planetary attractions upon the Earth as a whole and is called planetary precession. It consists of a slow rotation of the ecliptic about a slowly moving diameter. The effect of this is to decrease the angle between the pole of the ecliptic and the celestial pole (the obliquity of the ecliptic) by about 47" a century and displace the equinox about 12" per century. The motion of the celestial pole is due primarily to the attraction of the Moon and Sun on the equatorial bulge of the Earth. The theory of this motion is given in Section 1.1.3 (The Effects of Precession and Nutation). However, since the discussion of the coordinate frames relates so closely with this material, some of the more salient facts will be summarized for the purpose of describing the motion of the principal direction determined by the vernal equinox. The first component of this motion is called Luni-Solar precession and consists of the mean or continuous motion of the celestial pole about the ecliptic pole. As the mean celestial pole moves, or precesses, about the pole of the ecliptic its direction line describes a conical surface (with semi-vertex angle of approximately 23.5°) about the pole of the ecliptic. The period of revolution is about 25,800 years.

Superimposed on this motion is a small amplitude (about 9") irregular motion called nutation which carries the true celestial pole about the mean pole in a period of 18.6 years. The position of the vernal equinox, disregarding the effects of nutation, is called the mean equinox (γ_m). The true equinox includes all effects due to precession and nutation. Since the mean equinox of date is not a fixed inertial direction, a principal direction corresponding to the mean equinox of 1950.0 has been adopted as an inertial principal direction. The notation 1950.0 is used to describe the beginning of the Besselian year of 1950 which actually occurs 22.15 hrs. E.T., 31 December 1949 (J.D. 2,433, 282 . 423).

2.1.2.5 Rotating Observational Systems

This section is concerned with the definition of coordinate systems in which observations can be readily made from the Earth or Moon. The intent is to present an orderly discussion and graphic portrayal of these systems and to extend the basic notation and format advantages realized in Reference 1. To this end the following subsections have been prepared.

2.1.2.5.1 Geographic Coordinate Systems (GG) (Geocentric (GC) and Geodetic (GD) Latitudes).



<u>Type</u>	Rotating, earth referenced, observational
<u>Origin</u>	Center of the earth (geocentrics)
<u>Reference Plane</u>	Earth's equator
<u>Principal Direction</u>	Intersection of the prime meridian with the earth's equator.
<u>Rectangular Cartesian Coordinates</u>	<p>x_{GC} axis is in the principal directions</p> <p>y_{GC} axis is normal to the x_{GC}, z_{GC} plane in the right hand sense</p> <p>z_{GC} axis lies in the direction of the celestial pole</p>

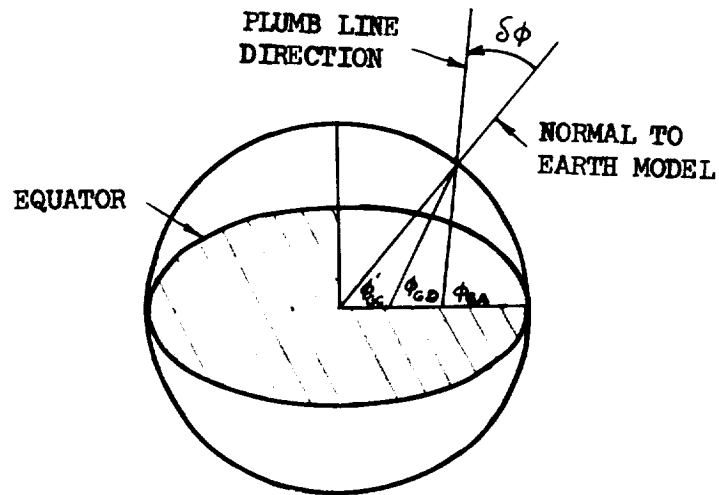
Spherical Polar Coordinates

- r_{ee} - radial distance from origin to the point being measured
- λ_{ee} - longitude measured positive eastward from the prime meridian to the meridian containing the point of interest (local meridian)
- δ_{ee} - geocentric declination (angle between the radius vector to the measured point and the earth's equatorial plane), measured positive north of the equatorial plane
- ϕ'_{ee} - geocentric latitude (angle between the earth's equatorial plane and the radius vector to the point of intersection of the earth model and the normal to the spheroid to the point being measured) measured positive north of the equatorial plane

Spheroidal Coordinates

- H_{eo} - perpendicular distance of the point being measured to the surface of the earth spheroid model
- λ_{eo} - same as longitude measured in spherical coordinates
- ϕ_{eo} - **geodetic** latitude (angle between the normal to earth spheroid model passing through the point of interest and the earth's equatorial plane)

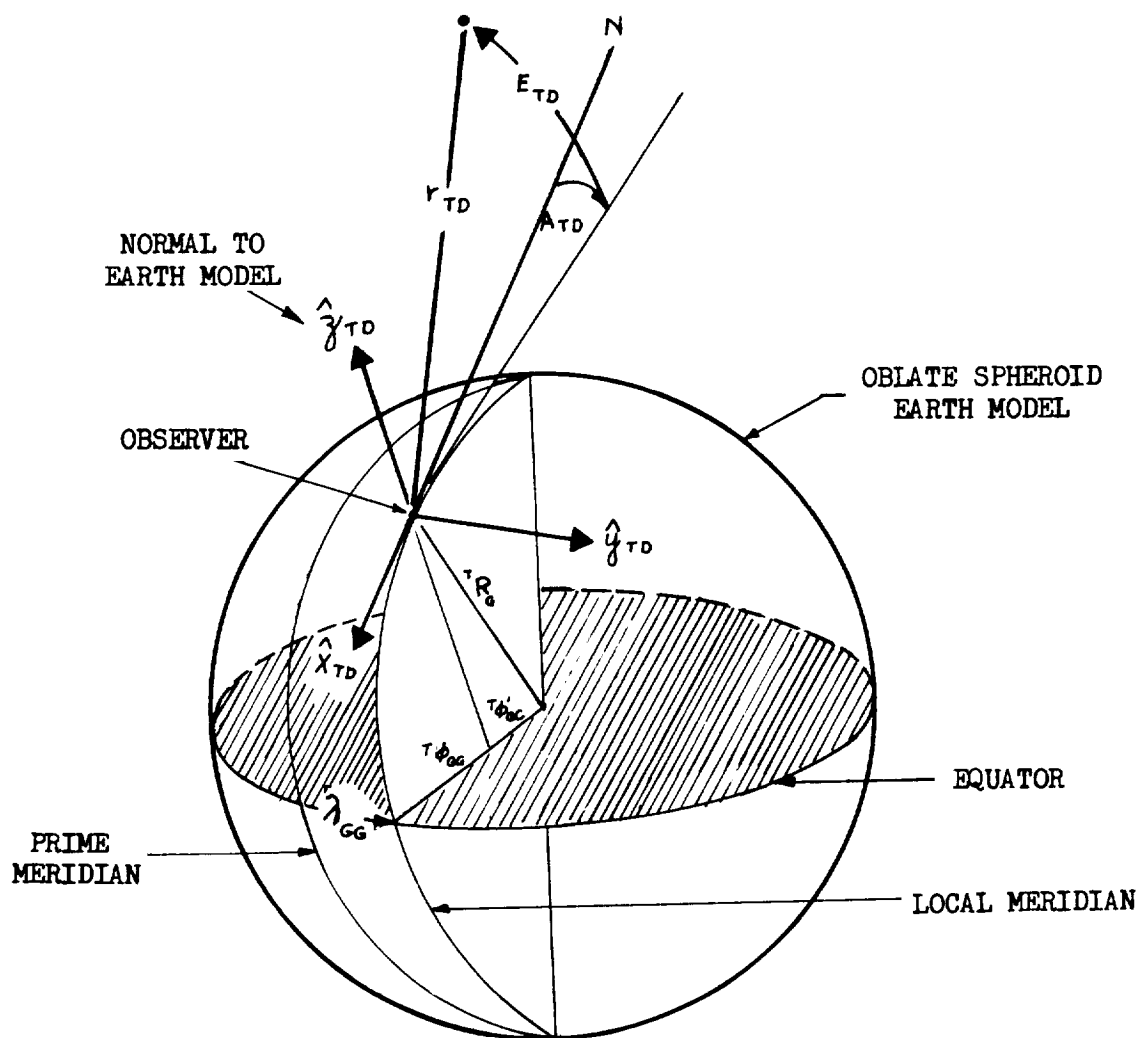
2.1.2.5.2 Geographic Coordinate System (Astronomical Latitude (GA))



This coordinate system is identical to the geographic system with the exception that the latitude of a reference point is now given by the angle between the local gravity vector (or plumb line) and the plane of the equator. If the earth were exactly spheroidal in shape and the mass distributed in uniform concentric layers the direction of the gravity vector would be normal to the spheroid. However, due to deviations resulting from surface terrain effects and other causes the astronomical latitude ϕ_{GA} differs from the geodetic latitude ϕ_{GD} by a small quantity $\delta\phi$ called station error, where

$$\delta\phi = \phi_{GD} - \phi_{GA}$$

2.1.2.5.3 Radar AZ-EL or Topodetic Axes (TD).



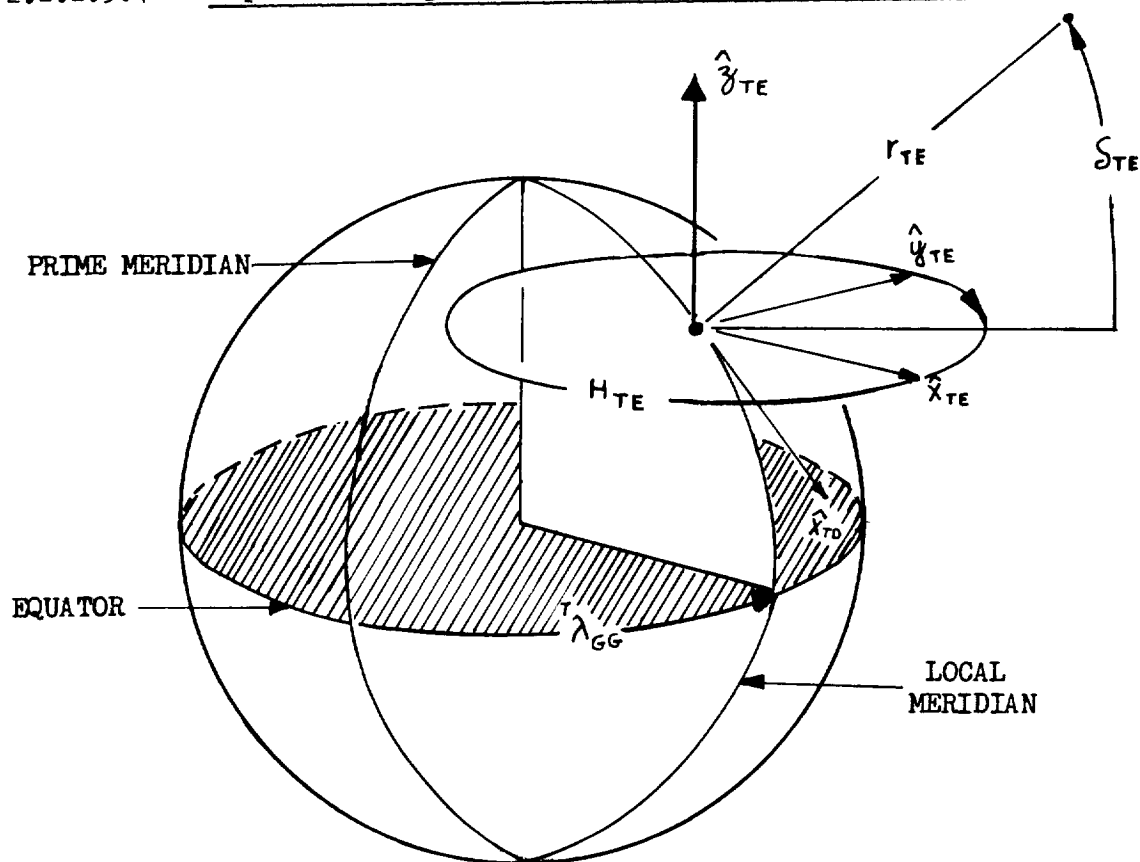
<u>Type</u>	Rotating, earth-referenced, observational
<u>Origin</u>	Observer (topocentric)
<u>Reference Plane</u>	Tangent plane to the spheroid earth model at the observer
<u>Principal Direction</u>	Local south direction on the tangent plane to the earth model
<u>Rectangular Cartesian Coordinates</u>	x_{TD} axis is in the principal direction y_{TD} axis is normal to the x_{TD}, z_{TD} axes in the right hand sense z_{TD} axis is in the direction of the normal to the earth spheroid model

Spherical Polar Coordinates

r_{to} - radial distance from the observer to the point being measured

A_{to} - azimuth angle measured from the local north direction on the tangent plane to the projection of the point being measured on the tangent plane, positive to the east

E_{to} - elevation angle measured between the tangent plane to the earth model and the line from the observer to the point, positive towards the outward normal to the earth.

2.1.2.5.4 Topocentric Equatorial (TE) or Radar (H.A.-Dec.)

<u>Type</u>	Rotating, earth-referenced, observational
<u>Origin</u>	Observer (Topocentric)
<u>Reference Plane</u>	Earth's equatorial plane
<u>Principal Direction</u>	Intersection of the local meridian through the observer with the earth's equatorial plane in a outward direction from the observer

Rectangular Cartesian Coordinates

χ_{TE} axis is in the principal direction
 y_{TE} axis is normal to the χ_{TE} , z_{TE} axes in
right hand sense
 z_{TE} axis is in the direction of the north
celestial pole

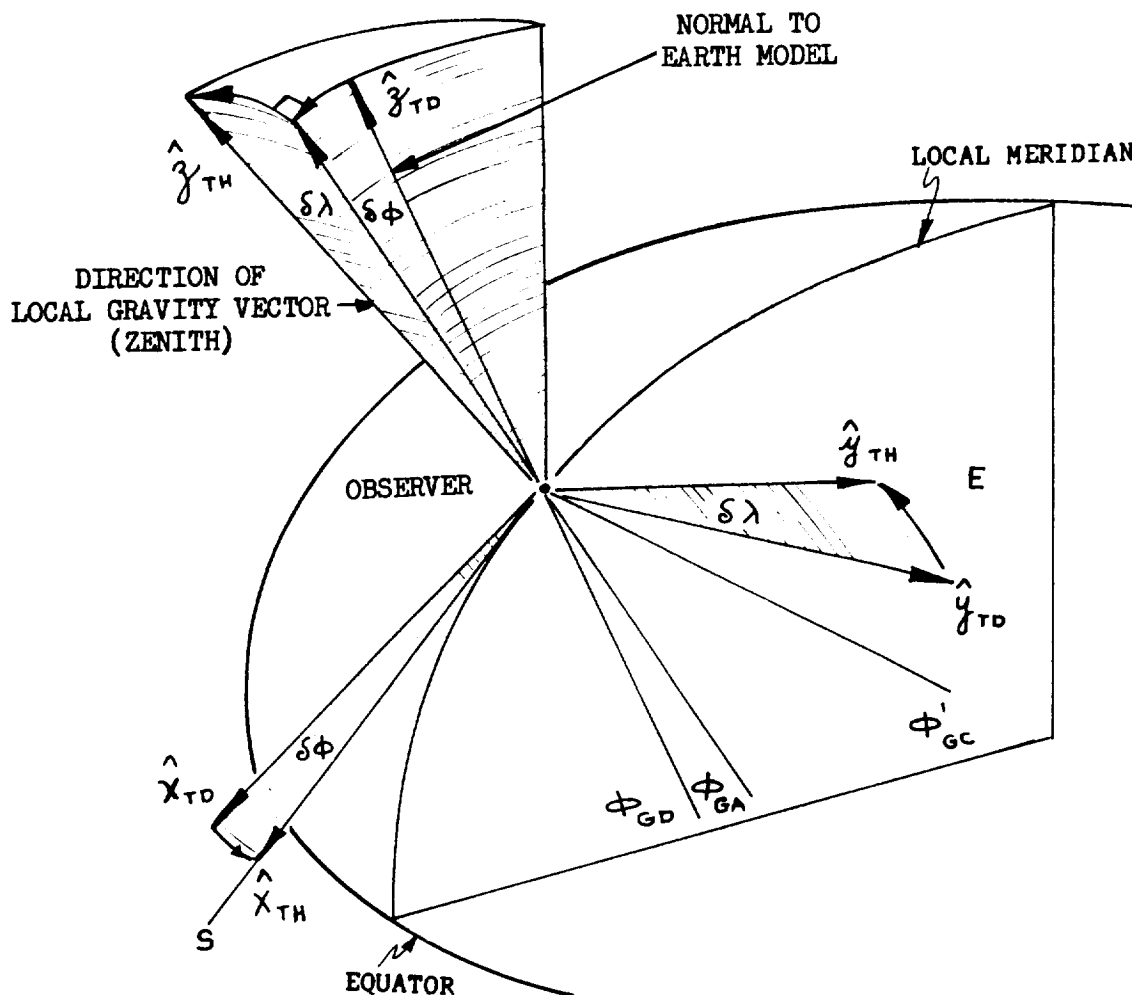
Spherical Polar Coordinates

r_{TE} - radial distance from the observer to the
point being measured

H_{TE} - topocentric hour angle measured from the
 χ_{TE} axis, in a clockwise manner when
looking from the positive z_{TE} axis to the
projection of the point on the observer's
equatorial plane

δ_{TE} - topocentric equatorial declination angle
measured between the observer's equatorial
plane and the line from the observer to
the point being measured

2.1.2.5.5 Topocentric Horizontal (TH) or Local Astronomical Axes.



<u>Type</u>	Rotating, earth-referenced, observational
<u>Origin</u>	Observer, (topocentric)
<u>Reference Plane</u>	Plane of the astronomical horizon (the plane whose normal is in the direction of the local gravity vector)
<u>Principal Direction</u>	Local south direction on the plane of the horizon determined by the intersection of the horizon and the plane passing through the earth's axis of rotation containing the observer

Rectangular Cartesian Coordinates

x_{TH} axis is in the principal direction
 y_{TH} axis is normal to the x_{TH}, z_{TH}
in the right hand sense
 z_{TH} axis is in the outward direction of the
local gravity vector (plumb line
direction)

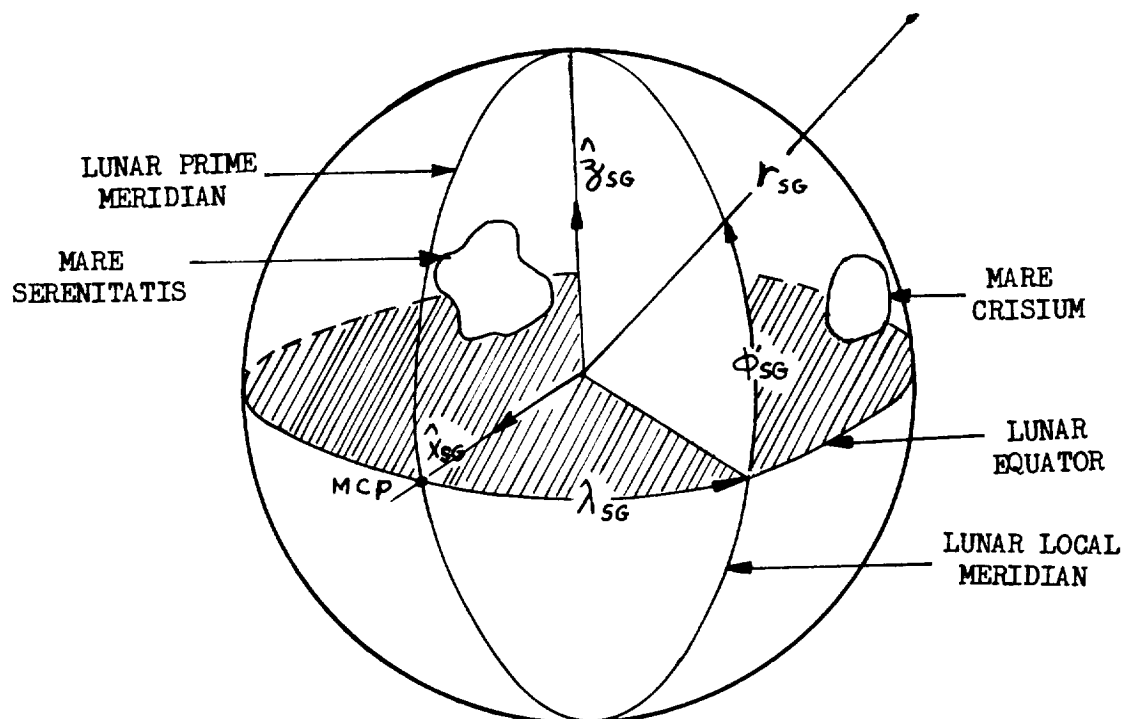
Spherical Polar Coordinates (not shown)

r_{TH} - radial distance from the observer to the
point being measured

A_{TH} - astronomical azimuth measured from the local
North direction in the plane of the
astronomical horizon to the projection of the
point on this plane

h_{TH} - astronomical altitude measured from the
plane of the astronomical horizon to the
line from the origin to the point being
measured, positive toward the zenith

2.1.2.5.6 Selenographic Coordinate System (SG).



Type Rotating, moon referenced, observational

Origin Center of the moon (Selenocentric)

Reference Plane Lunar equatorial plane

Principal Directions Mean center point of the Apparent Disc

Rectangular Cartesian Coordinates

x_{SG} axis lies in the principal direction
 y_{SG} axis is normal to the x_{SG}, z_{SG} plane in
 in the right hand sense
 z_{SG} axis lies in the direction of the lunar
 celestial pole

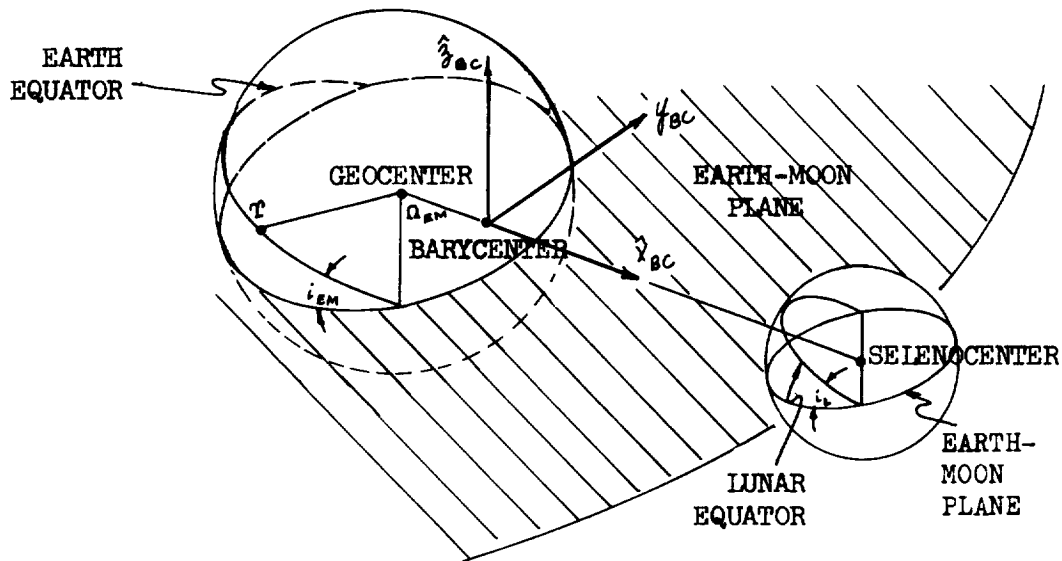
Spherical Polar Coordinates

r_{SG} - radial distance from the center of the
 moon to the point being measured

λ_{SG} - selenographic longitude measured positive
 from the lunar prime meridian (passing
 through the MCP) to the point being
 measured in the direction of Mare Crisium.

ϕ_{SG} - selenographic latitude

2.1.2.5.7 Earth-Moon Barycentric Axes (BC)

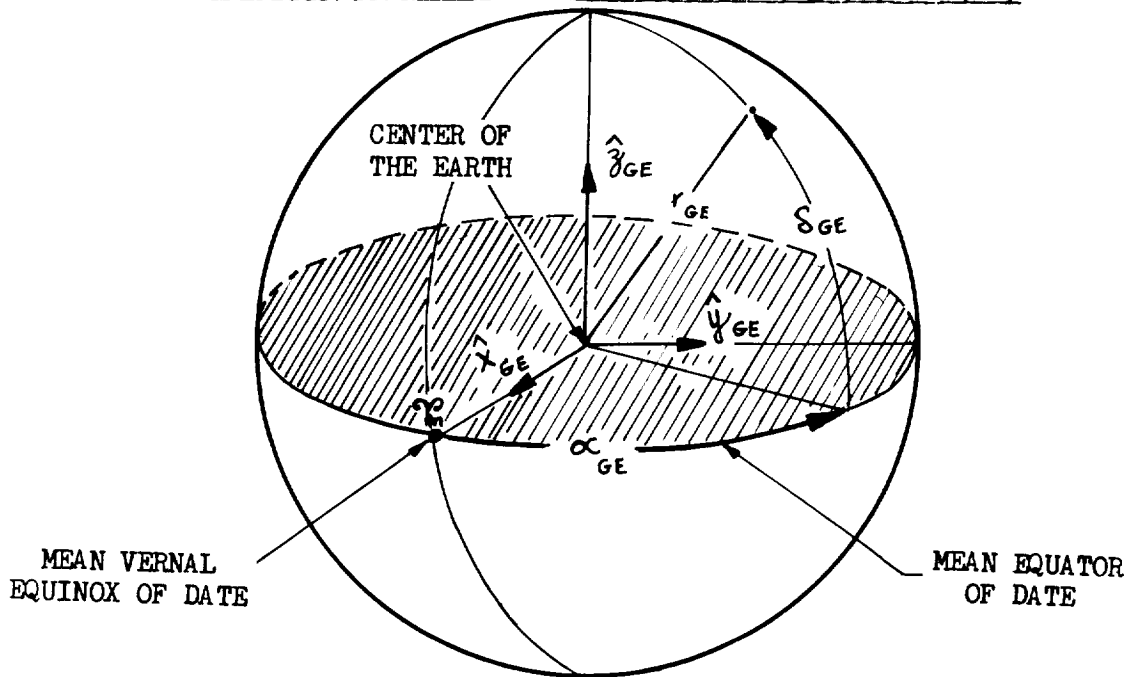


<u>Type</u>	Rotating, lunar trajectory applications
<u>Origin</u>	Center of mass of the Earth-Moon system (barycentric)
<u>Reference Plane</u>	Earth-Moon plane
<u>Principal Directions</u>	Direction of the line from the center of the Earth to the center of the Moon (Earth-Moon line)
<u>Rectangular Cartesian Coordinates</u>	X_{BC} axis is in the principal direction Y_{BC} axis is normal to the X_{BC} , Z_{BC} axes in the right-hand sense Z_{BC} axis is in the direction normal to the Earth-Moon plane positive towards the north celestial pole

2.1.2.6 Quasi-Inertial and Inertial Observational Systems

The coordinate frames presented on the previous pages are all tied in some fashion to the Earth's crust (or that of the Moon) and all move with it. However, for studies of motion, it is generally desirable from the standpoint of the complexity of the solution to write the differential equations describing the trajectory in a frame for which the coriolis accelerations are negligibly small. The subsections which follow present some candidate systems defined with reference to equatorial planes, etc. of the epoch of date. These frames are moving very slowly (thus the terminology quasi-inertial) and in a predictable fashion (section 2.2). Therefore, should the rotational accelerations involved still be too large to be neglected for a particular application, an arbitrary epoch in the past (generally 1950.0) can be selected for the definition of the basic reference frame and corrections for nutation and precession can be made when communicating between the frame of the reference epoch and that of date.

2.1.2.6.1 Earth Centered Quasi Inertial or Geoequatorial (GE).



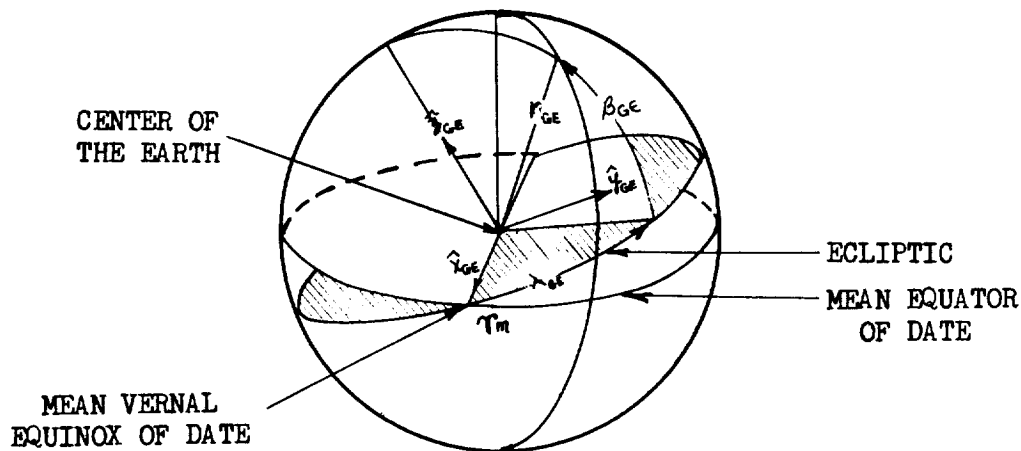
<u>Type</u>	Quasi, inertial, observational
<u>Origin</u>	Center of the earth (geocentric)
<u>Reference Plane</u>	Mean earth equatorial plane of date
<u>Principal Direction</u>	Mean vernal equinox of date
<u>Rectangular Cartesian Coordinates</u>	

x_{GE} axes is in the principal direction
 y_{GE} axes is normal to the x, z plane in the right hand sense
 z_{GE} axes is in the direction of the mean north celestial pole of date

Spherical Polar Coordinates

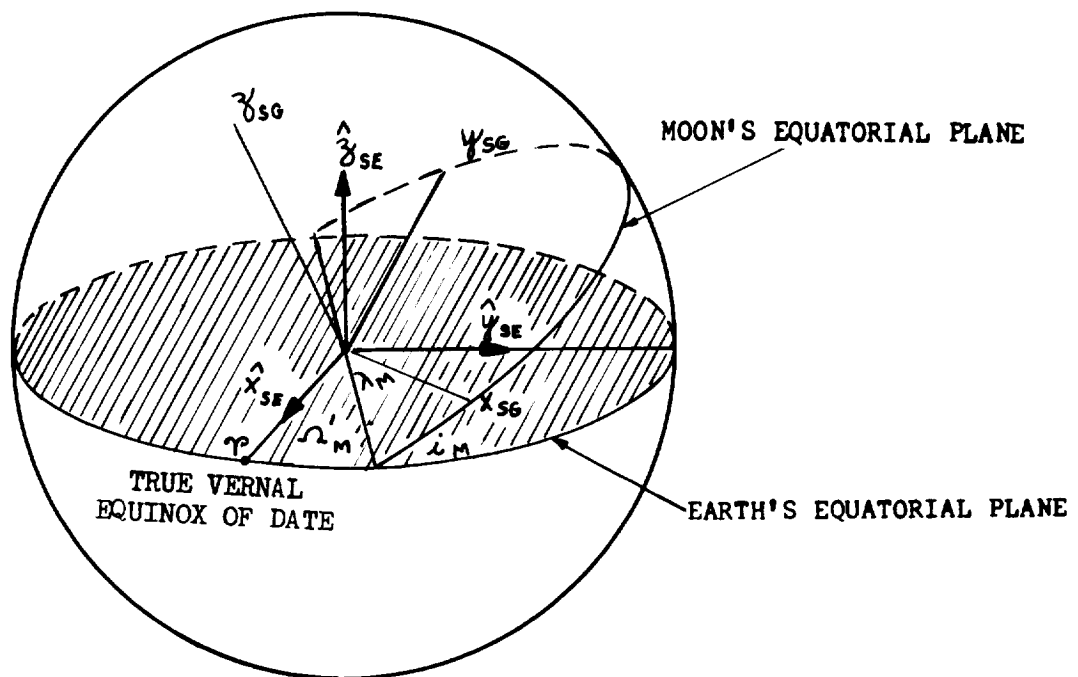
r_{GE} - radial distance from the center of the earth to the point being measured
 α_{GE} - right ascension measured from the x_{GE} axis to the projection of the line from the origin to the point on the mean equatorial plane of date, positive in a counterclockwise manner when viewed from the positive z_{GE} axis.
 δ_{GE} - declination, the angle between the mean equatorial plane of date and the line from the center of the earth to the point being measured.

2.1.2.6.2 Geoecliptic (G ϵ).



<u>Type</u>	Quasi - Inertial, observational
<u>Origin</u>	Center of the earth (geocentric)
<u>Reference Plane</u>	Ecliptic
<u>Principal Direction</u>	Mean vernal equinox of date
<u>Rectangular Cartesian Coordinates</u>	$x_{G\epsilon}$ axis is in the principal direction $y_{G\epsilon}$ axis is normal to the $x_{G\epsilon}, z_{G\epsilon}$ plane in the right hand sense $z_{G\epsilon}$ axis is in the direction of the north pole of the ecliptic
<u>Spherical Polar Coordinates</u>	$R_{G\epsilon}$ - radial distance from the center of the earth to the point being measured $\lambda_{G\epsilon}$ - ecliptic longitude measured from the axis to the projection of the line from the origin to the point being measured on the plane of the ecliptic $\beta_{G\epsilon}$ - ecliptic declination measured between the plane of the ecliptic and the line from the origin to the point

2.1.2.6.3 Selenocentric Equatorial (SE)



Type: Quasi-inertial, observational

Origin: Center of the Moon (selenocentric)

Reference Plane: Earth's true equatorial plane of date

Principal Direction: True vernal equinox of date

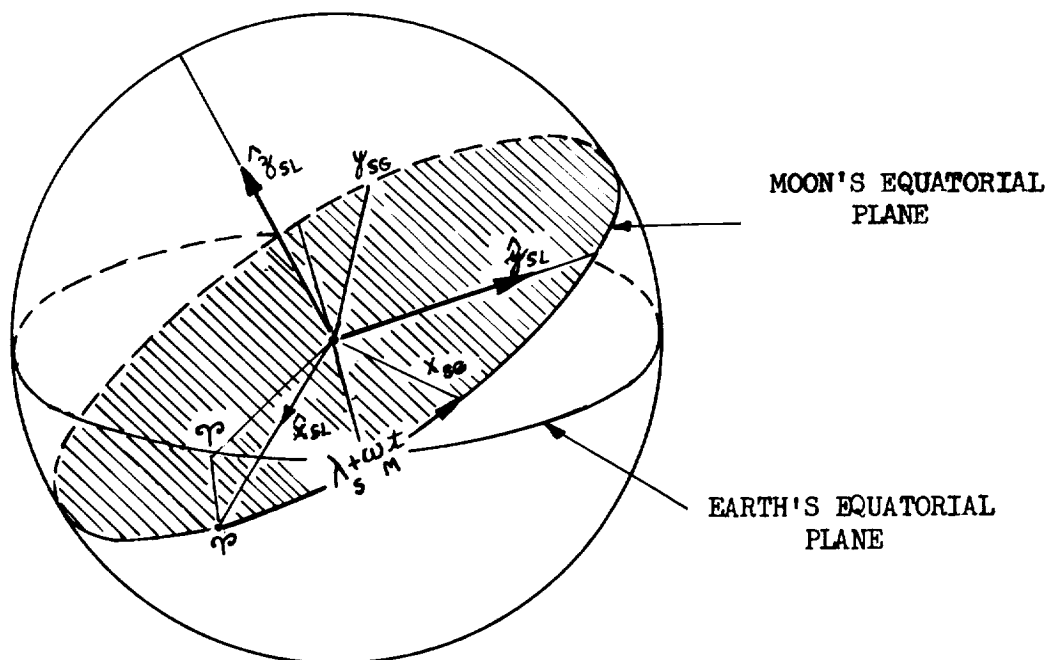
Rectangular Cartesian Coordinates:

x_{SE} axis is in the principal direction

y_{SE} axis is normal to the x_{SE} , z_{SE} axes in the right-hand sense

z_{SE} axis is in the direction of the true north celestial pole of date

2.1.2.6.4 Selenocentric Lunar Equatorial (SE)



<u>Type:</u>	Quasi-inertial, observational
<u>Origin:</u>	Center of the Moon (selenocentric)
<u>Reference Plane:</u>	Moon's equatorial plane
<u>Principal Direction:</u>	Intersection of the Earth equatorial meridian passing through the true vernal equinox of date and the Moon's equatorial plane

Rectangular Cartesian Coordinates:

x_{sl} axis is in the principal direction

y_{sl} axis is normal to the x_{sl} , z_{sl} axes in the right-hand sense

z_{sl} axis is in the direction of the Moon's axis of rotation

2.1.2.6.5 Earth-Centered Inertial of 1950.0 (GE 1950.0)

(diagram same as geoequatorial - except equatorial plane and mean vernal equinox are of epoch 1950.0)

Type: Inertial, observational
Origin: Center of the Earth (geocentric)
Reference Plane: Mean Earth equatorial plane of epoch 1950.0
Principal Direction: Mean vernal equinox of epoch 1950.0
Rectangular Cartesian Coordinates:

X_{GE50} axis is in the principal direction

Y_{GE50} axis is normal to the X_{GE50} Z_{GE50} plane in the right-hand sense

Z_{GE50} axis is in the direction of the north celestial of epoch 1950.0

2.1.2.6.6 Galactic and Heliocentric Coordinate Systems

The use of galactic coordinate systems is at present limited to radio astronomy studies. When used in conjunction with this branch of physics, the Earth is chosen as the origin of coordinates, the plane of the Milky Way galaxy is the reference plane, and either the direction toward the center of the galaxy or the intersection of the plane of the galaxy with the celestial equator is chosen as the principal direction. If the former principal direction is used, the axis normal to this direction and the direction of the galactic pole is in the direction of galactic rotation.

The reference plane for heliocentric or Sun-centered coordinate systems is the plane normal to the Sun's axis of rotation. Except for a slight angle of inclination, this plane coincides with the plane of the ecliptic. Any number of principal directions can be chosen to determine the heliocentric axes.

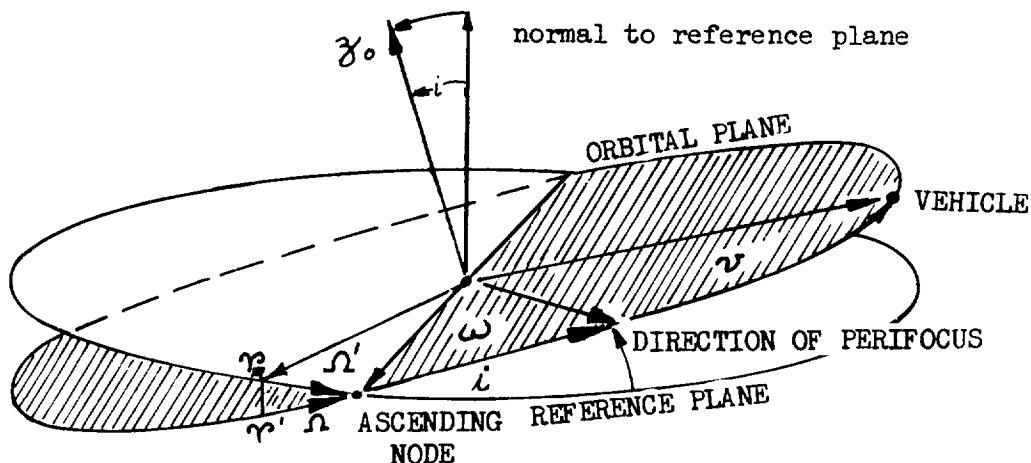
2.1.3 Dynamical Coordinate Systems

2.1.3.1 Orbital Plane Systems

It will be shown in the monograph on Theory of Motion - Two Body that in a dynamical system consisting of a vehicle with negligible mass and a spherical central body with symmetric mass distribution that the vehicle moves in a plane (known as the orbital plane) which passes through the center of the central attracting body. Rectangular cartesian coordinate systems with two arbitrarily defined axes imbedded in this plane are, thus, frequently useful in analyzing this motion; although in the true case, disturbances such as those due to an oblate central body produce a motion out of this plane. However, for small disturbances it is still convenient to use coordinates based on this plane for many orbit computations. Generally, these orbital plane systems are right handed and are oriented such that their Z axis is perpendicular to the orbital plane (i.e., along the angular momentum). The X axis (the principal direction) is selected to be a readily definable vector in the plane of motion (toward periapse, toward $\bar{r}(\dot{t})$, toward the node, etc.).

The orientation of the orbital plane is given with respect to some reference plane associated with the central body and containing an arbitrarily selected principal direction (with the Earth as a central body - the equatorial plane is commonly taken as the reference plane; and the vernal equinox of date (Υ), the principal direction). This orientation is given in terms of the angle of inclination (i) between the orbital and reference planes within the range $0^\circ \leq i \leq 180^\circ$ and the longitude of the ascending node measured from the principal direction.

In those cases in which the desired coordinate frame is selected in such a fashion that the principal direction locates the perifocus of the orbit, a third angle ω , called the argument of perigee and measured in the orbital plane from the ascending node to the perifocus must be specified. The angle ν (referred to as the true anomaly) measured in the orbital plane from the direction of the perifocus now locates the vehicle.

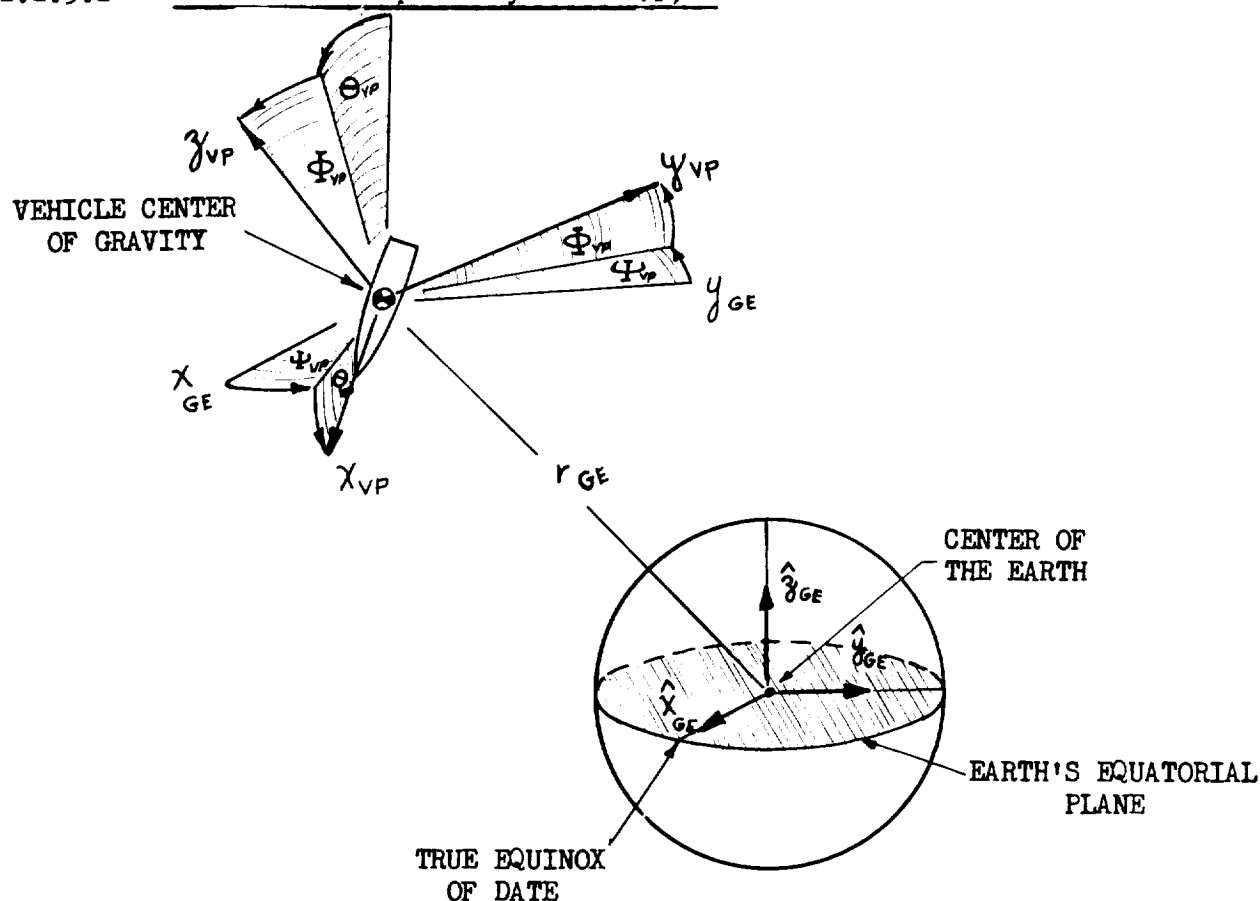


A summary of the principal directions of the orbital plane coordinate systems and their coordinate measurement notation is given below:

Principal Direction (X-axis)	Rectangular Cartesian Coordinates	Spherical Coordinates
Projection of the vernal equinox of date on the orbital plane, γ'	$x_{or'}, y_{or'}, z_{or'}$	$R_{or'}, \lambda_{or'}, \phi_o$ ($\lambda_{or'} = \Omega_o + \omega + \nu$)
Ascending node	x_{on}, y_{on}, z_{on}	$R_{on}, \lambda_{on}, \phi_o$ ($\lambda_{on} = \omega + \nu$)
Perifocus	x_{op}, y_{op}, z_{op}	R_{op}, ν, ϕ_o
Vehicle (rotating frame)	$x_{ov}, 0, z_{ov}$	$R_{ov}, 0, \phi_o$
Vehicle (inertial frame of t_o)	x_{oi}, y_{oi}, z_{oi}	$R_{oi}, \lambda_{oi}, \phi_o$

In each of these orbital plane coordinate systems, the angle ϕ_o is measured normal to the orbital plane (that is, in the plane containing the Z_o axis and the vehicle) and is positive toward the Z_o axis. In the absence of disturbing forces $\phi_o = 0$. All orbital plane systems have their origin at the central spherical body and their Z axis aligned normal to the orbital plane.

2.1.3.2 Vehicle Principal Body Axes (VP)



<u>Type:</u>	Rotating, vehicle referenced
<u>Origin:</u>	Vehicle center of mass
<u>Reference Plane:</u>	Plane normal to the longitudinal axis of symmetry
<u>Principal Direction:</u>	One of the principal axes of inertia in the reference plane
<u>Rectangular Cartesian Coordinates:</u>	

x_{VP} axis is the vehicle longitudinal axis of symmetry

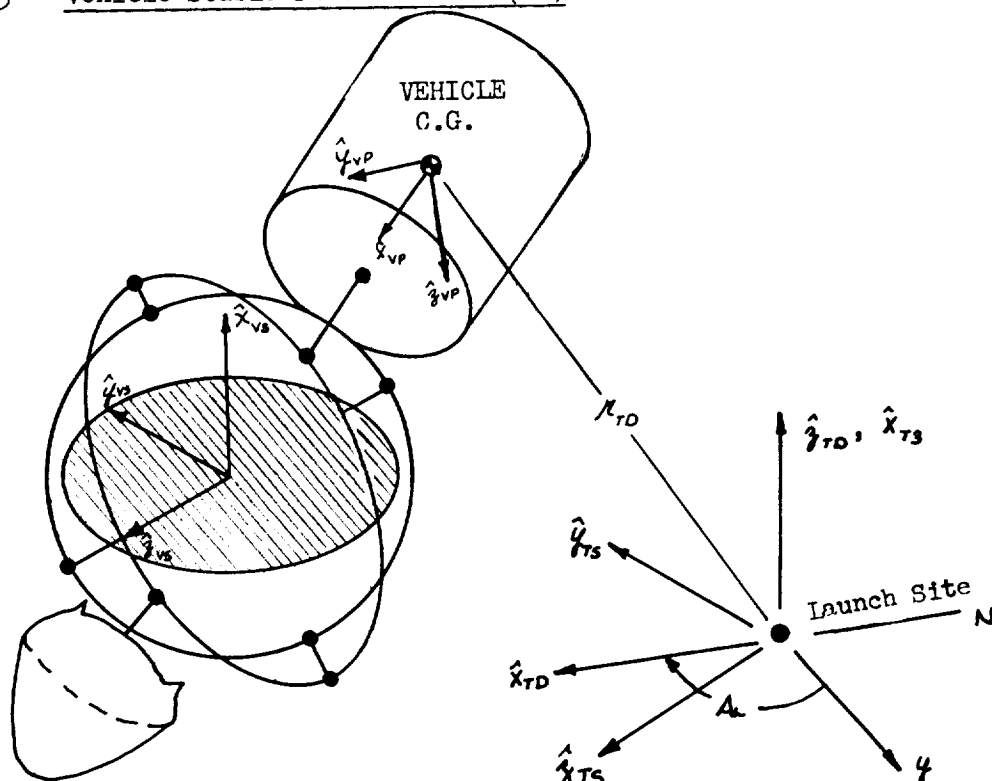
y_{VP} axis is a principal axis of inertia

z_{VP} axis is a principal axis of inertia normal to the x_{VP} , y_{VP} plane

Vehicle Principal Axes (VP) Orientation With Relation to
the Geoequatorial Coordinate System (GE)

The χ_{vp} , ψ_{vp} , ϕ_{vp} vehicle principal axes directions are derived from the geo-equatorial axes by successive rotations about the z_{GE} axis, the intermediate y_{vp} axis, and the final x_{vp} axis by the angles Ψ_{vp} , θ_{vp} , and ϕ_{vp} , respectively. For a vehicle aligned normal to the Earth spheroidal model at launch, the orientation angles Ψ_{vp} and θ_{vp} correspond to the right ascension of the launch site ($^L\alpha_{GE}$) and the south geodetic latitude of the launch site.

2.1.3.3 Vehicle Stable Platform Axes (VS)



<u>Type</u>	Non-rotating, translates with vehicle, guidance applications
<u>Origin</u>	Intersection of the primary axes of the accelerometers in the vehicle guidance package
<u>Rectangular Cartesian Axes</u>	<p>x_{VS} axis is parallel to the Earth spheroid outward normal at the launch site at the guidance reference release time</p> <p>y_{VS} axis is normal to the x_{VS}, z_{VS} plane in the right-hand sense</p> <p>z_{VS} axis is parallel to the aiming azimuth at the guidance reference time positive downrange</p>

Vehicle Principal Axes (VP) Orientations With Relation to the Vehicle Stable Axes (VS)

The x_{VP} , y_{VP} , z_{VP} vehicle principal axes directions are derived from the vehicle stable axes by successive rotations about the x_{VS} axis, the intermediate y_{VP} axis and the final z_{VP} axis by the angles ϕ_v , θ_v , and ψ_v

Vehicle Stable Axes at Launch Point (TS)

The vehicle stable axes at the launch point is a non-rotating axis system with the origin fixed to and moving with the launch point as the Earth rotates. Its axes are oriented in the same direction as the vehicle stable axes.

2.1.4 Coordinate Transformations

2.1.4.1 Introduction to Transformations

A transformation is a group of simultaneous equations relating a set of independent variables to a set of dependent variables. If $[u, v, w]$ is the set of independent variables and $[x, y, z]$ the set of dependent variables, then the equations

$$x = f(u, v, w)$$

$$y = g(u, v, w)$$

$$z = h(u, v, w)$$

where f , g , and h are given functions, are said to determine the transformation T or transformations of coordinates. This transformation which can also be written as:

$$[x, y, z] = T[u, v, w]$$

is to be regarded as a law of correspondence whereby to each set of numbers $[u, v, w]$ there corresponds a set of numbers $[x, y, z]$. To be considered as a transformation, the number of independent variables does not, in general, have to equal the number of dependent variables; however, in transformations resulting from translations or rotations of three dimensional coordinate systems, the number of variables in each set are the same; namely, three. The equations relating spherical polar and rectangular cartesian coordinates are also of this type. It will be assumed that, except at certain singular points, the functions f , g , and h are continuous, differentiable functions; therefore, there exists an inverse transformation T^{-1} of the form

$$u = F(x, y, z)$$

$$v = G(x, y, z)$$

$$w = H(x, y, z)$$

if the functions f , g , and h have continuous partial derivatives and if the Jacobian determinant

$$J = \begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & \frac{\partial g}{\partial w} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} & \frac{\partial h}{\partial w} \end{vmatrix}$$

does not vanish at any point where the inverse transformation is defined. It is furthermore assumed that the transformation and its inverse are single-valued; that is, there exists a one-to-one correspondence between all sets of the real-valued triplets $[u, v, w]$ and $[x, y, z]$.

2.1.4.2 Transformation Between Spherical Polar and Rectangular Cartesian Coordinates

A common transformation of coordinates is the one between spherical polar $[R, \lambda, \phi]$ and rectangular cartesian coordinates $[x, y, z]$. This group of equations can be written by means of the trigonometric functions as

$$x = R \cos \phi \cos \lambda$$

$$y = R \cos \phi \sin \lambda$$

$$z = R \sin \phi$$

And, the inverse transformation can be written in terms of the inverse trigonometric functions as

$$R = (x^2 + y^2 + z^2)^{1/2}$$

$$\lambda = \sin^{-1} \frac{y}{(x^2 + y^2)^{1/2}} \begin{cases} -90^\circ \leq \lambda \leq +90^\circ & \text{if } x > 0 \\ 90^\circ \leq \lambda \leq 270^\circ & \text{if } x < 0 \end{cases}$$

$$\phi = \sin^{-1} \frac{z}{(x^2 + y^2 + z^2)^{1/2}} \quad -90^\circ \leq \phi \leq +90^\circ$$

Since the Jacobian J of the transformations is found to be

$$J = R^2 \cos \phi$$

the inverse transformation exists for $R \neq 0$ and $\phi \neq 90^\circ$.

2.1.4.3 Linear Transformations

A transformation of coordinates T is said to be linear if:

$$T\{a[x_1, y_1, z_1] + b[x_2, y_2, z_2]\} = aT[x_1, y_1, z_1] + bT[x_2, y_2, z_2]$$

where a and b are constants. The general form of the equations determining a linear transformation between $[x, y, z]$ and $[x', y', z']$ is given by

$$x' = a_{10} + a_{11}x + a_{12}y + a_{13}z$$

$$y' = a_{20} + a_{21}x + a_{22}y + a_{23}z$$

$$z' = a_{30} + a_{31}x + a_{32}y + a_{33}z$$

where the a 's are constants. This system of linear equations can be written in matrix notation as:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a_{10} \\ a_{20} \\ a_{30} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

or in abbreviated matrix notation as

$$\underline{X}' = \underline{X}_0 + [A] \underline{X}$$

where

$$\underline{X}' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \quad \underline{X}_0 = \begin{bmatrix} a_{10} \\ a_{20} \\ a_{30} \end{bmatrix} \quad \underline{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

If the $[x', y', z']$ are rectangular cartesian coordinates, this transformation can be visualized geometrically as two transformations: a transformation due to a translation of the origin of the axis system, and a transformation due to a rotation of the axis system.

The elements a_{ij} of the matrix array $[A]$ are called the direction cosines of the linear transformation. This follows from the fact that the unit vectors in each system are orthogonal to each other. These direction cosines are formed by taking all possible dot products between the unit vectors in each system, that is:

$$\begin{array}{lll} \hat{x}' \cdot \hat{x} = a_{11} & \hat{x}' \cdot \hat{y} = a_{12} & \hat{x}' \cdot \hat{z} = a_{13} \\ \hat{y}' \cdot \hat{x} = a_{21} & \hat{y}' \cdot \hat{y} = a_{22} & \hat{y}' \cdot \hat{z} = a_{23} \\ \hat{z}' \cdot \hat{x} = a_{31} & \hat{z}' \cdot \hat{y} = a_{32} & \hat{z}' \cdot \hat{z} = a_{33} \end{array}$$

Since the Jacobian of the total transformation is simply the determinant of the $[A]$ matrix, the inverse transformation

$$\underline{X} = [A]^{-1} (\underline{X}' - \underline{X}_0)$$

exists if

$$\det [A] \neq 0$$

2.1.4.4 Properties of the Rotation Matrix [A]

Since the matrix [A] can be thought of as representing a transformation by rotation of a rectangular cartesian axis system, it is called the rotation matrix. By taking the dot product of two unit vectors in the transformed axes, the orthogonality conditions of the direction cosines can be derived. They are:

$$\hat{x}' \cdot \hat{x}' = a_{11}^2 + a_{12}^2 + a_{13}^2 = 1$$

$$\hat{y}' \cdot \hat{y}' = a_{21}^2 + a_{22}^2 + a_{23}^2 = 1$$

$$\hat{z}' \cdot \hat{z}' = a_{31}^2 + a_{32}^2 + a_{33}^2 = 1$$

$$\hat{x}' \cdot \hat{y}' = a_{11} a_{21} + a_{12} a_{22} + a_{13} a_{23} = 0$$

$$\hat{x}' \cdot \hat{z}' = a_{11} a_{31} + a_{12} a_{32} + a_{13} a_{33} = 0$$

$$\hat{y}' \cdot \hat{z}' = a_{21} a_{31} + a_{22} a_{32} + a_{23} a_{33} = 0$$

All linear transformations having these properties are said to be orthogonal. If successive orthogonal transformations [A], [B] are applied to a system, the resulting transformation [C] is also orthogonal. Symbolically, this product is represented as:

$$[C] = [A] [B]$$

where [A] [B] \neq [B] [A]. For orthogonal transformations, the transposed matrix [A]^T obtained by interchanging the rows and columns of [A] is identical to the inverse matrix [A]⁻¹. This fact can be proven by comparing the transpose matrix with the transformation matrix formed by reversing the order and sense of the rotations involved.

The rotation matrix defining a transformation by positive rotation about the X axis has the form

$$[T_{x, \Phi}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \Phi & \sin \Phi \\ 0 & -\sin \Phi & \cos \Phi \end{bmatrix}$$

Similarly, for positive rotations about the x and z axes by the angles θ and ψ , respectively, there corresponds the rotation matrices $[T_{y,\theta}]$ and $[T_{z,\psi}]$, where:

$$[T_{y,\theta}] = \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix}$$

$$[T_{z,\psi}] = \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using this notation, successive rotations about the z, y and x axes through the angles ψ , θ and φ respectively yields the transformation.

$$[T] = [T_{x,\varphi}] [T_{y,\theta}] [T_{z,\psi}]$$

and any vector is transformed as

$$\vec{X}' = [T] \vec{X}$$

FROM TO	GEOGRAPHIC (GG)	GEOGRAPHIC (GG)	GEOEQUATORIAL (EARTH-CENTERED QUASI-INERTIAL), (GE)	GEOECLIPTIC (GE)
	GEOGRAPHIC (GG)		$[T_{\mathcal{G}}(\lambda_E + \omega t)]$	$[T_{\mathcal{G}}(\lambda_E + \omega t)][T_{\mathcal{Z}, -i_E}]$
GEOEQUATORIAL (EARTH-CENTERED QUASI-INERTIAL), (GE)		$[T_{\mathcal{Z}, -i_E}(\lambda_E + \omega t)]$		$[T_{\mathcal{Z}, -i_E}]$
GEOECLIPTIC (GE)		$[T_{\mathcal{Z}, i_E}][T_{\mathcal{G}, -(\lambda_E + \omega t)}]$	$[T_{\mathcal{Z}, i_E}]$	

2.1.4.5 Geocentric Coordinate System Transformations

Note λ_E - right ascension of the prime meridian at time zero, ω - angular velocity of the earth, i_E - obliquity of the ecliptic

TO	FROM	RADAR AZ-EL (TD)	TOPOCENTRIC HORIZONTAL (TH)	TOPOCENTRIC EQUATORIAL (TE)
RADAR AZ-EL (TD)			$[T_{\gamma, -\delta\phi}][T_{x, \delta\lambda}]$	$[T_{\gamma, 90^\circ - \phi_{eo}}]$
TOPOCENTRIC HORIZONTAL (TH)		$[T_{x, \delta\lambda}][T_{\gamma, \delta\phi}]$		$[T_{x, \delta\lambda}][T_{\gamma, 90^\circ - \phi_{eo} + \delta\phi}]$
TOPOCENTRIC EQUATORIAL (TE)		$[T_{\gamma, -(90 - \phi_{eo})}]$	$[T_{\gamma, -(90^\circ - \phi_{eo})}]$	

2.1.4.6 Topocentric Coordinate System Transformations

Note ϕ_{eo} - geodetic latitude of observe, $\delta\phi$ - difference between geodetic and astronomical latitudes, $\delta\lambda$

FROM TO	SELENOGRAPHIC (SG)	SELENOCENTRIC EQUATORIAL (SE)	SELENOCENTRIC LUNAR EQUATORIAL (SL)
SELENOGRAPHIC (SG)		$[T_{g,\lambda_m}][T_{x,i_m}][T_{g,\Omega_m}]$	$[T_{g,(\lambda_s+\omega_m t)}]$
SELENOCENTRIC EQUATORIAL (SE)	$[T_{g,\Omega_m}][T_{x,i_m}][T_{g,\lambda_m}]$		$[T_{g,\Omega_m}][T_{x,i_m}][T_{g,(\lambda_s+\omega_m t)}]$
SELENOCENTRIC LUNAR EQUATORIAL (SL)	$[T_{g,(\lambda_s+\omega_m t)}]$	$[T_{g,(\lambda_s-\lambda_m+\omega_m t)}][T_{x,i_m}][T_{g,\Omega_m}]$	

2.1.4.7 Selenocentric Coordinate System Transformations

Note λ_s - quasi-inertial longitude of the lunar prime meridian, at time zero ω_m - angular velocity of the Moon,

λ_m - longitude of the lunar prime meridian measured from the Earth equatorial line of nodes,

i_m - inclination of the Moon's equatorial plane, Ω_m - angle between true vernal equinox and Earth-Moon equatorial line of nodes.

FROM TO	GEOEQUATORIAL (GE)	ORBITAL AXES PRINCIPAL DIRECTION- ASCENDING NODE (ON)	ORBITAL AXES PRINCIPAL DIRECTION- PERIFOCUS (OP)	ORBITAL AXES PRINCIPAL DIRECTION- VEHICLE (OV)
GEOEQUATORIAL (GE)		$[T_{g,-\Omega}][T_{x,-i}]$	$[T_{g,-\Omega}][T_{x,-i}][T_{g,-\omega}]$	$[T_{g,-\Omega}][T_{x,-i}][T_{g,-(\omega+\nu)}]$
ORBITAL AXES PRINCIPAL DIRECTION- ASCENDING NODE (ON)	$[T_{x,i}][T_{g,\Omega}]$		$[T_{g,-\omega}]$	$[T_{g,-(\omega+\nu)}]$
ORBITAL AXES PRINCIPAL DIRECTION- PERIFOCUS (OP)	$[T_{g,\omega}][T_{x,i}][T_{g,\Omega}]$	$[T_{g,\omega}]$		$[T_{g,-\nu}]$
ORBITAL AXES PRINCIPAL DIRECTION- VEHICLE (OV)	$[T_{g,(\omega+\nu)}][T_{x,i}][T_{g,\Omega}]$	$[T_{g,(\omega+\nu)}]$	$[T_{g,\nu}]$	

2.1.4.8 Geoequatorial and Orbital Axes Transformations

2.1.4.9 Transformation by Translation

The transformations between any two coordinate systems having different origins is simplified by considering an intermediate system orientated such that its axes are aligned with the axes of one of the systems and its origin located at the other. The transformation between this intermediate system and the system located at the other origin then consists of a simple translation and is written in the form

$$\bar{X}_o = \bar{X}_p + \bar{X}_{op}$$

In this section intermediate axes systems will be employed to describe the transformations between the geocentric, topecentric and vehicle-centered systems.

2.1.4.9.1 Transformation Between Geocentric and Topocentric Systems

These transformations are accomplished by first considering an intermediate system known as the local geocentric equatorial axes (GE') with origin at the center of the earth and axes aligned with the topocentric equatorial axes (TE). The transformation between a vector measured in the topocentric equatorial axes and one measured in the local geocentric equatorial axes is given by

$$\bar{X}_{GE'} = \bar{X}_{TE}^T + \bar{X}_{GE'}$$

where

$$\bar{X}_{GE'}^T = \begin{bmatrix} r_G \cos \phi'_{GC} \\ 0 \\ r_G \sin \phi'_{GC} \end{bmatrix}$$

By means of this transformation and Tables 2.1.4.5, 6, and 10 all geocentric and topocentric system relationships can be easily written.

2.1.4.9.2 Transformation Between Topocentric and Vehicle-Centered Systems

The relationship between topocentric and vehicle-centered systems is easily written once the transformation between the vehicle stable axes at the launch point (VS') and the vehicle stable axes (VS) is established by

$$\bar{X}_{TS} = \bar{X}_{VS}^V + \bar{X}_{TS}$$

where

$$\bar{X}_{VS}^T = \begin{bmatrix} v_{x_{TS}} \\ v_{y_{TS}} \\ v_{z_{TS}} \end{bmatrix}$$

Tables 2.1.4.6 and 10 can then be used to write the transformation between any topocentric and vehicle-centered system.

2.1.4.9.3 Transformation Between Vehicle-Centered Systems and Geocentric Systems

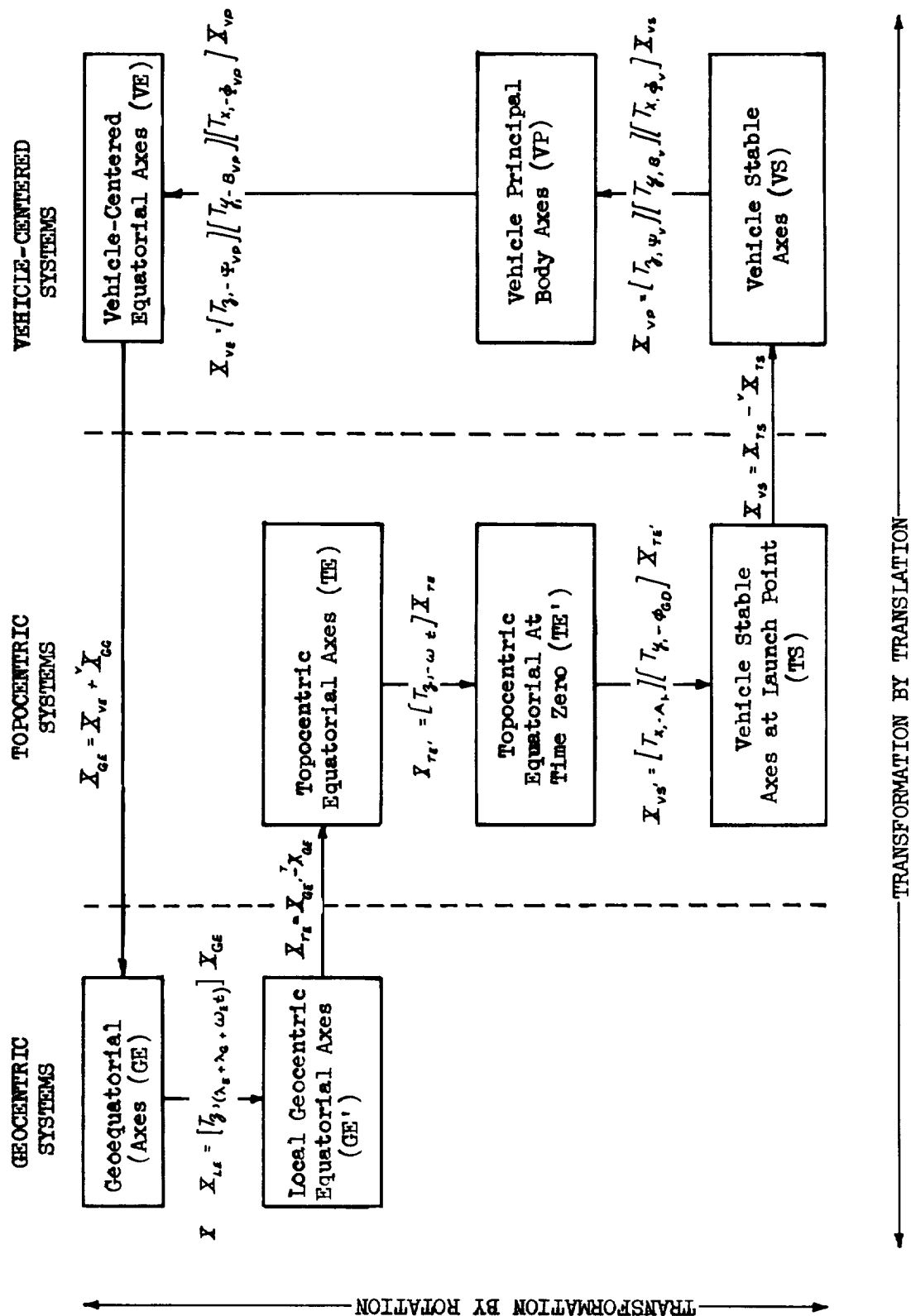
These transformations are established by considering the intermediate axis system known as the Vehicle-Centered Equatorial System (VE) with origin at the vehicle center of mass and axes aligned with the Geoequatorial (GE) system. The transformation between these coordinate systems is given by

$$\mathbf{X}_{GE} = \mathbf{X}_{VE} + \mathbf{X}_{GE}^V$$

where

$$\mathbf{X}_{GE}^V = \begin{bmatrix} \mathbf{R}_{GE}^V \cos \delta_{GE}^V \cos \alpha_{GE}^V \\ \mathbf{R}_{GE}^V \cos \delta_{GE}^V \sin \alpha_{GE}^V \\ \mathbf{R}_{GE}^V \sin \delta_{GE}^V \end{bmatrix}$$

All other transformations between geocentric and vehicle-centered systems can be written by consulting the tables 2.1.4.5, 8, and 10.



2.1.4.10 Intermediate Axis System Relationships used in Relating Geocentric, Topocentric and Vehicle-centered Coordinate Systems

2.2 THE EFFECTS OF PRECESSION AND NUTATION

2.2.1 Description of the Various Types of Precession and Nutation

Over 2000 years ago it was discovered that the vernal equinox moved from east to west by $50''$. 2453 every year. This motion is called precession and is caused by the gravitational attraction of other celestial bodies acting on the equatorial bulge of the earth. If the earth were perfectly spherical and radially homogeneous, it would not experience any deviation from its mean equatorial pole. However, since the earth has an equatorial bulge, it experiences torques from the gravitational attraction of the sun and the moon. Due to the fact that the lunar orbital plane is approximately 5° oblique to the mean ecliptic, both the lunar and solar torques tend to align the equator with the ecliptic. The earth responds to this torque much like a spinning top responds to a torque. It precesses about the mean ecliptic pole. This precession is called luni-solar precession. Since the moon is so much closer to the earth than the sun, its contribution to luni-solar precession is approximately twice as much as that from the sun. The equatorial pole has an obliquity of about 23.5° so at the rate of precession mentioned earlier, the equatorial pole would very nearly trace a right circular cone every 25,800 years.

Just as the sun and moon cause the equatorial pole to precess, so do the planets of our solar system cause the ecliptic pole to precess; however, the magnitude of this planetary precession is very small and will be considered negligible in this discussion.

"Total general precession" is the sum of planetary and luni-solar precession and gives the changes in the mean vernal equinox of date from some epoch. Total general precession amounts to $50''$. 2453/year and can be considered uniform for practical use. This is the rate of westward rotation of the mean vernal equinox of date.

As the equatorial pole precesses about the ecliptic pole, it also experiences further disturbances known as nutations. Free Eulerian Nutations are those which would occur if the earth were simply set in rotation and left to itself without any disturbing forces. This motion is analogous to the torque free motion of a body of revolution in which the moment of inertia about the spin axis is not equal to that about a perpendicular axis. The body precesses about the angular momentum vector just as the earth nutates about its mean equatorial pole. Forced nutations are those which are caused by the changing positions in space of the sun, earth, and moon, which in turn cause variations in their respective gravitational attractions to the earth.

The most significant nutation is the 19 Year Lunar Nutation. This nutation is caused by the precession of the moon's orbit. As mentioned earlier, the moon's orbital plane is about 5° oblique to the mean ecliptic. The line of nodes associated with these planes precesses with a period of about 18.6 years. The result is to change the direction of the small fluctuations in potential experienced by the earth-moon system.

Other forced nutations include the Semi-annual Solar Nutation and the Fortnightly Lunar Nutation. These phenomena are the result of the decreasing torque that the sun and moon apply to the earth as they approach the passing of the equatorial plane. Due to symmetry, the net torque, as one of these bodies passes through the equatorial plane, is zero.

2.2.2 Historical Background

Precession was discovered by Hipparchus in 125 B.C. by observing an increase in the longitudes of stars with no perceptible change in latitudes. In his Principia, Sir Issac Newton gave the first explanation of precession in terms of dynamical theory.

Newton was also the first to note the nutation of the earth's axis due to the influence of the sun. In 1748 Bradley discovered nutations that were due to the movement of the moon's nodes. Shortly after, a theoretical explanation of nutation was presented by D'Alembert. The efforts of Euler and Laplace helped to improve his explanation. A large portion of the more recent investigations of nutation try to correlate theoretical and observed results by introducing earth models that account for such characteristics as elasticity, fluidity, inner core movement, and other physical properties. Analyses such as these were initiated when S. Newcomb pointed out that discrepancies between the theoretical 10-month period of Free Eulerian Nutation and the 14-month period measured by S. C. Chandler were due to the effects of the fluidity of the oceans and the elasticity of the earth. Since then many earth models have been invented. Increased geophysical data concerning the interior of the earth has helped solve many problems that were significant in determining a representative model of the earth.

The earth model that will be used in the following analysis is a rigid ellipsoid which is later simplified to an oblate spheroid. This model does not account for elasticity, fluidity and other physical properties of the earth; but it is sufficient to use for a fairly complete derivation of precession and nutation. It must be noted that the results of an analysis using such a simplified model will not be exactly correct, but are sufficient for most practical purposes. The complications that arise with a more complex earth model are quite extensive, and it is not thought that such considerations would significantly add to the discussion. However, empirical results from the American Ephemeris and Nautical Almanac are shown with the theoretically derived equations for precession, nutation, and coordinate correction. This presentation provides a set of best fit equations that correlate theory and observation.

The method of analysis used in the derivation and solution of the precession and nutation equations is basically the method presented by W. M. Smart in his Celestial Mechanics.

2.2.3 Equations of Motion of an Ellipsoid Earth Model in a Potential Field

2.2.3.1 Kinetic Energy of a Rotating Body

Consider an ellipsoid whose semimajor and semiminor axes, for purposes of generality, are all unequal. A body fixed coordinate system that is aligned with the principal axes of the ellipsoid and has its origin at the

center of the ellipsoid can be constructed. If the ellipsoid has some angular velocity, $\vec{\omega}$, it can be shown that the rotational kinetic energy, T , of this body is

$$T = \frac{1}{2} [\omega_x^2 I_{xx} + \omega_y^2 I_{yy} + \omega_z^2 I_{zz}] \quad (1)$$

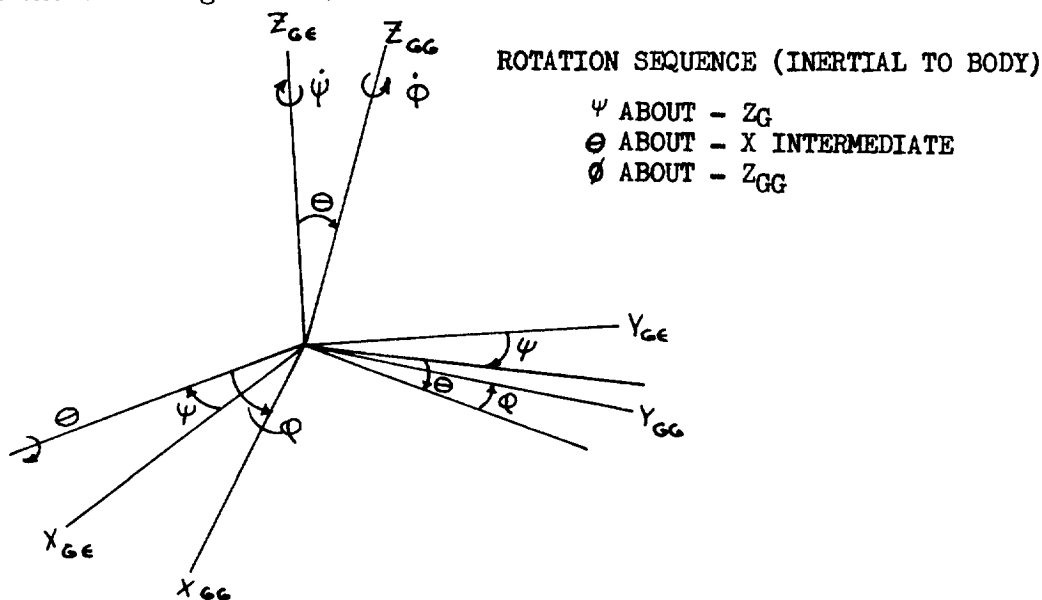
where

- T = rotational kinetic energy
- I_{xx} = moment of inertia about x axis
- I_{yy} = moment of inertia about y axis
- I_{zz} = moment of inertia about z axis
- ω_x = body fixed angular velocity - x direction
- ω_y = body fixed angular velocity - y direction
- ω_z = body fixed angular velocity - z direction

The total kinetic energy is required in Lagrange's Equation. However, the translation component is not a function of the variables of this analysis. For this reason it need not be considered.

2.2.3.2 Euler Transformation

The body fixed angular velocity can be related to an inertially fixed coordinate system by an Euler Transformation. This will enable the solution of angular rates that are equal to the precession and nutation angular rates. The axis system $X_{GE} Y_{GE} Z_{GE}$ represents the geocentric inertial axis system and $X_{GG} Y_{GG} Z_{GG}$ represents the body fixed system as illustrated in the following sketch.



This sketch illustrates the order of rotation from the inertial reference to body fixed axes. The corresponding relation between the body fixed angular velocities and the Euler angle rates is as follows:

$$\omega_x = \dot{\psi} \sin \theta \sin \phi - \dot{\theta} \cos \phi \quad (2)$$

$$\omega_y = \dot{\psi} \sin \theta \cos \phi + \dot{\theta} \sin \phi \quad (3)$$

$$\omega_z = \dot{\phi} - \dot{\psi} \cos \theta \quad (4)$$

The inertial axis system that is most useful in precession analyses is the geocliptic system. This system is defined for some epoch, t_0 , and can be considered fixed in space for all future derivations.

2.2.3.3 Application of Lagrange's Equation

The model of the solar system to be employed in the analyses of precession and nutation is conservative (i.e., the forces can be expressed as the gradient of a potential function). Thus, Lagrange's Equation can be written in generalized coordinates as:

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_r} \right) - \frac{\partial T}{\partial q_r} = \frac{\partial U}{\partial q_r} \quad (5)$$

where for this study T = total kinetic energy of rotation

U = total potential energy of earth-moon and earth-sun systems (U for this definition of potential is opposite the sense normally employed in potential theory. Thus, U is in reality the work function. This convention will be employed throughout this monograph.)

q_r = the r th generalized coordinate

Applying Lagrange's Equation to the Euler angle, ϕ , Equation (5) becomes

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\phi}} \right) - \frac{\partial T}{\partial \phi} = \frac{\partial U}{\partial \phi} \quad (6)$$

But

$$\frac{\partial T}{\partial \phi} = \frac{\partial T}{\partial \omega_x} \frac{\partial \omega_x}{\partial \phi} + \frac{\partial T}{\partial \omega_y} \frac{\partial \omega_y}{\partial \phi} + \frac{\partial T}{\partial \omega_z} + \frac{\partial T}{\partial \omega_z} \frac{\partial \omega_z}{\partial \phi} \quad (7)$$

so

$$\frac{\partial T}{\partial \dot{\phi}} = \frac{\partial T}{\partial \omega_z} = \omega_z I_{zz} \quad (8)$$

Similarly

$$\frac{\partial T}{\partial \dot{\phi}} = \frac{\partial T}{\partial \omega_x} \frac{\partial \omega_x}{\partial \dot{\phi}} + \frac{\partial T}{\partial \omega_y} \frac{\partial \omega_y}{\partial \dot{\phi}} + \frac{\partial T}{\partial \omega_z} \frac{\partial \omega_z}{\partial \dot{\phi}} \quad (9)$$

but

$$\frac{\partial T}{\partial \omega_x} = \omega_x I_{xx} \quad \text{and} \quad \frac{\partial T}{\partial \omega_y} = \omega_y I_{yy} \quad (10)$$

thus

$$\frac{\partial T}{\partial \dot{\phi}} = \omega_x \omega_y (I_{xx} - I_{yy}) \quad (11)$$

Substituting Lagrange's equation,

$$\frac{d}{dt} (\omega_z I_{zz}) - \omega_x \omega_y (I_{xx} - I_{yy}) = \frac{\partial U}{\partial \phi} \quad (12)$$

so

$$\dot{\omega}_z I_{zz} - \omega_x \omega_y (I_{xx} - I_{yy}) = \frac{\partial U}{\partial \phi} \quad (13)$$

This is Lagrange's equation for the angle ϕ . In a similar manner, the following equations for the Euler angles θ and ψ can be obtained as:

$$\begin{aligned} -I_{xx} \dot{\omega}_x \cos \phi + I_{yy} \dot{\omega}_y \sin \phi + (I_{xx} - I_{zz}) \omega_x \omega_z \sin \phi \\ + (I_{yy} - I_{zz}) \omega_y \omega_z \cos \phi = \frac{\partial U}{\partial \theta} \end{aligned} \quad (14)$$

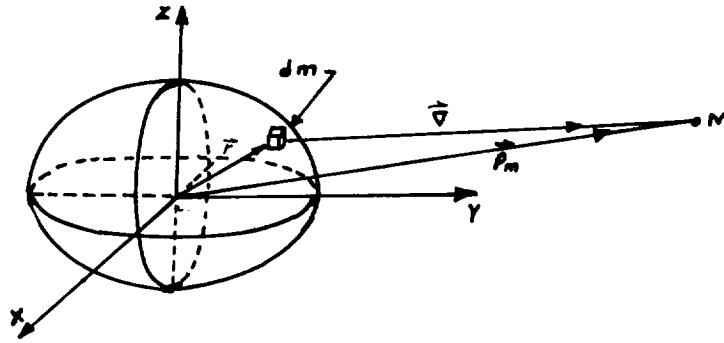
$$I_{xx} \dot{\omega}_x \sin \varphi + I_{yy} \dot{\omega}_y \cos \varphi - (I_{yy} - I_{zz}) \omega_y \omega_z \sin \varphi + (I_{xx} - I_{zz}) \omega_x \omega_z \cos \varphi = \cos \Theta \frac{\partial U}{\partial \varphi} + \frac{1}{\sin \Theta} \frac{\partial U}{\partial \psi} \quad (15)$$

2.2.3.4 Potential Energy due to Ellipsoid Earth Model

The potential of a point mass, M , at a distance ρ from the center of an ellipsoid earth model is

$$dU = GM \int \frac{1}{r} dm \quad (\text{See Page 58}) \quad (16)$$

where the geometry used for the potential energy formulation is defined in the following sketch:



It can be shown that the potential energy of the moon, assumed to be a point mass, with respect to the ellipsoid earth model is

$$U_m = GM_m \left[\frac{m_e}{\rho_m} + \frac{I_{yy} + I_{zz} - 2I_{xx}}{2\rho_m^3} + \frac{3}{2} \frac{(I_{xx} - I_{yy})^m y_{ce}^2 - (I_{zz} - I_{xx})^m z_{ce}^2}{\rho_m^5} \right] \quad (17)$$

where U_m = potential energy of moon-earth system
 G = gravitational constant
 m_m = mass of the moon
 m_e = mass of the earth
 m_s = mass of the sun
 ρ_m = distance between centers of gravity

See Appendix A for a detailed derivation of Equation (17). Similarly, the potential energy of the sun with respect to the earth is:

$$U_s = Gm_s \left[\frac{m_e}{\rho_s} + \frac{I_{yy} + I_{zz} - 2I_{xx}}{2\rho_s^3} + \frac{3}{2} \frac{(I_{xx} - I_{yy})(y_{ee}^2) - (I_{zz} - I_{xx})(z_{ee}^2)}{\rho_s^5} \right] \quad (18)$$

Since for the present purposes the potential energy expressions will be utilized in Lagrange's equation, the first two terms of each potential expression may be dropped since they are not functions of the Euler angles. The abbreviated potential expressions may now be written as:

$$U_m = \frac{3Gm_m}{2\rho_m^3} \left[(I_{xx} - I_{yy})(y_{ee}^2) - (I_{zz} - I_{xx})(z_{ee}^2) \right] \quad (19)$$

$$U_s = \frac{3Gm_s}{2\rho_s^3} \left[(I_{xx} - I_{yy})(y_{ee}^2) - (I_{zz} - I_{xx})(z_{ee}^2) \right] \quad (20)$$

2.2.3.5 The Oblate Spheroid Approximation

The earth can be considered to be an oblate spheroid. This approximation results in a simplification of the equations that have been derived thus far. The moments of inertia about the x and y axes are now equal, and Lagrange's equation reduces to the following:

a. Lagrange's equation for ϕ

$$I_{zz} \ddot{\omega}_z - (I_{xx} \rightarrow I_{yy}) \omega_x \omega_y = \frac{\partial U}{\partial \phi} \quad (21)$$

$$\dot{\omega}_z I_{zz} = \frac{\partial U}{\partial \phi} \quad (22)$$

Since ϕ is the Euler rotation about the z body axis, and the spheroid is symmetric about the z axis, there is no change in potential due to a change in ϕ only. The term $\partial U / \partial \phi$ is zero.

$$\text{Now } \dot{\omega}_z I_{zz} = 0 \quad (23)$$

$$\text{and } \omega_z = \text{constant} = \omega_{z_0} \quad (24)$$

It should be noted that this result is for the rigid body model and that the earth does experience very small changes in its angular velocity due to the effects of tidal motion, elasticity, and other physical properties.

b. Lagrange's equation for ψ (For simplicity, the energy equation will be used directly in Lagrange's equation)

$$\begin{aligned} T &= \frac{1}{2} [I_{xx} \omega_x^2 + I_{yy} \omega_y^2 + I_{zz} \omega_z^2] \\ &= \frac{1}{2} [I_{xx} (\omega_x^2 + \omega_y^2) + I_{zz} \omega_z^2] \end{aligned} \quad (25)$$

$$\text{Now } \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\psi}} \right) - \frac{\partial T}{\partial \psi} = \frac{\partial U}{\partial \psi} \quad (26)$$

$$\text{and } \frac{\partial T}{\partial \dot{\psi}} = I_{xx} \dot{\psi} \sin^2 \theta - I_{zz} \cos \theta (\dot{\phi} - \dot{\psi} \cos \theta) \quad (27)$$

$$\frac{\partial T}{\partial \psi} = \frac{\partial T}{\partial \omega_x} \left[\frac{\partial \omega_x}{\partial \psi} + \frac{\partial \omega_y}{\partial \psi} \right] + \frac{\partial T}{\partial \omega_z} \frac{\partial \omega_z}{\partial \psi} = 0 \quad (28)$$

so Lagrange's equation for ψ becomes

$$\begin{aligned} I_{xx} \ddot{\psi} \sin^2 \theta + 2 I_{xx} \dot{\theta} \dot{\psi} \sin \theta \cos \theta + I_{zz} \omega_{z_0} \dot{\theta} \sin \theta \\ - I_{zz} \cos \theta (\ddot{\phi} - \ddot{\psi} \cos \theta + \dot{\theta} \dot{\psi} \sin \theta) = \frac{\partial U}{\partial \psi} \end{aligned} \quad (29)$$

It can be shown that $\ddot{\psi}$, $\dot{\Theta}\dot{\psi}$, $\ddot{\varphi}$ are small when compared to $\omega_{z_0}\dot{\Theta}$.
In this light, the equation for ψ becomes

$$\dot{\Theta} = \frac{1}{I_{zz} \omega_{z_0} \sin \Theta} \frac{\partial U}{\partial \psi} \quad (30)$$

c. Lagrange's equation for Θ : direct application of Lagrange's equation yields

$$\begin{aligned} -I_{xx} \dot{\omega}_x \cos \varphi + I_{xx} \dot{\omega}_y \sin \varphi + (I_{xx} - I_{zz}) \omega_x \omega_z \sin \varphi \\ + (I_{xx} - I_{zz}) \omega_y \omega_z \cos \varphi = \frac{\partial U}{\partial \Theta} \end{aligned} \quad (31)$$

which can be simplified to

$$I_{xx} \ddot{\Theta} - I_{xx} \dot{\psi}^2 \sin \Theta \cos \Theta - I_{zz} \omega_{z_0} \dot{\psi} \sin \Theta = \frac{\partial U}{\partial \Theta} \quad (32)$$

Now assuming that $\ddot{\Theta}$ and $\dot{\psi}^2$ are small compared with $\omega_{z_0} \dot{\psi}$ the equation becomes

$$\dot{\psi} = \frac{1}{I_{zz} \omega_{z_0} \sin \Theta} \frac{\partial U}{\partial \Theta} \quad (33)$$

The abbreviated potential equations of the moon and sun can similarly be simplified via the oblate spheroid approximation.

$$U_m = \frac{-3 G m_m}{2 \rho_m^3} (I_{zz} - I_{xx}) Z_{GG}^2 \quad (34)$$

$$U_s = \frac{-3 G m_s}{2 \rho_s^3} (I_{zz} - I_{xx}) Z_{GG}^2 \quad (35)$$

so that the total potential is ($U = U_m + U_s$)

$$U = \frac{-3}{2} (I_{zz} - I_{xx}) \left[\frac{G m_m}{\rho_m^3} Z_{GG}^2 + \frac{G m_s}{\rho_s^3} Z_{GG}^2 \right] \quad (36)$$

or in a form to be used later

$$\frac{-U}{I_{zz} \omega_{z_0}} = K \left[L \left(\frac{a_m}{\rho_m} \right)^3 \left(\frac{z_0}{\rho_m} \right)^2 + \left(\frac{a_s}{\rho_s} \right)^3 \left(\frac{z_0}{\rho_s} \right)^2 \right] \quad (37)$$

where

$$K = \frac{3}{2} (I_{zz} - I_{xx}) G S / (I_{zz} a_s^3 \omega_{z_0})$$

a_s = semimajor axis of earth's orbit

a_m = semimajor axis of moon's orbit

$$L = \frac{m_m}{m_s} \left(\frac{a_s}{a_m} \right)^3$$

2.2.4 Solution of Equations

2.2.4.1 Free Eulerian Nutation

If an approximate evaluation of the potential is performed, it can be shown that the ratio of the potential energy to the kinetic energy is very small

$$\frac{|U|}{T} < 2 \times 10^{-7}$$

and, if use is made of the fact that $I_{xx} - I_{yy} \neq 0$, Equations (14) and (15) reduce to

$$I_{xx} \dot{\omega}_x - (I_{xx} - I_{zz}) \omega_{z_0} \omega_y = 0 \quad (38)$$

$$I_{yy} \dot{\omega}_y - (I_{zz} - I_{xx}) \omega_{z_0} \omega_x = 0 \quad (39)$$

or

$$\dot{\omega}_x + \left[\frac{(I_{zz} - I_{xx})}{I_{xx}} \omega_{z_0} \right] \omega_y = 0 \quad (40)$$

$$\dot{\omega}_y - \left[\frac{(I_{zz} - I_{xx})}{I_{xx}} \omega_{z_0} \right] \omega_x = 0 \quad (41)$$

The solution to these differential equations is:

$$\omega_x = B \cos (p t + \gamma) \quad (42)$$

$$\omega_y = B \sin (p t + \gamma) \quad (43)$$

where $p = \omega_{z_0} (I_{zz} - I_{xx}) / I_{xx}$ (44)
 B and γ are integration constants

But, it is known that

$$(I_{zz} - I_{xx}) / I_{zz} = 1 / 304.2$$

and since $\omega_{z_0} = 2\pi$ radians/sidereal day,

$$\begin{aligned} p &= \omega_{z_0} \frac{I_{zz} - I_{xx}}{I_{xx}} \frac{I_{zz}}{I_{xx}} \\ &= 2\pi \left(\frac{1}{304.2} \right) \left(\frac{1}{.99671} \right) \quad \text{radians/sidereal day} \end{aligned}$$

This value of p corresponds to a period of

$$\gamma = \frac{2\pi}{p} = (304.2) (.99671) = 303.91 \text{ sidereal days}$$

which is approximately 302 mean solar days. Hence, the theoretical value for the Eulerian free period of nutation is about ten months.

From the above derivation, a variation in the instantaneous latitude would be expected equal to

$$\varphi = \varphi_0 + C \cos (p t + \gamma) \quad (45)$$

where

ϕ_0 = the mean latitude of some reference point

C = amplitude of the Eulerian Free Nutation

p = frequency of the Eulerian Free Nutation

Observations, however, reveal that the actual variation in latitude is given as the sum of two periodic terms:

$$\phi = \phi_0 + C_1 \cos(p_1 t + \gamma_1) + C_2 \cos(p_2 t + \gamma_2) \quad (46)$$

in which the period of the first periodic term is one year and that of the second about 14 months. Furthermore, the maximum deviation from ϕ_0 is 0".3. Since the period of the observed nutation differs from the theoretical value, some individuals have ignored the Eulerian Nutation and have attributed the observed nutations to meteorological causes producing periodical changes in the principal moments of inertia of the earth. Other individuals have theoretically shown that the elasticity and fluidity of the earth, when taken into account, can significantly extend the theoretical period of Eulerian Nutation. T. H. Sloudshy determined a period of 12 or 14 months for Eulerian Nutation using a simplified theory of rotation of the Earth with a fluid core.

The fact remains that the earth does experience a variation in latitude of the order of

$$\phi = \phi_0 + 0".09 \cos(p_1 t + \gamma_1) + 0".18 \cos(p_2 t + \gamma_2) \quad (47)$$

where

$p_1 = 2\pi$ radians/year

$p_2 = \frac{12}{7}\pi$ radians/year

γ_1 = phase constant

γ_2 = phase constant

2.2.4.2 Solution of Equations for Precession and Forced Nutation

Equations (30) and (33) describe precession and nutation; however, since these equations must be integrated, they will be repeated for convenience.

$$\dot{\psi} = \frac{-1}{I_{zz} \omega_z \sin \theta} \frac{\partial U}{\partial \psi} \quad (48)$$

$$\dot{\Theta} = \frac{1}{I_{22} \omega_{2_0} \sin \Theta} \frac{\partial U}{\partial \Psi} \quad (49)$$

The solution to these equations can be found when the potential can be expressed in terms of Θ and Ψ . This operation is very tedious, however, and does not supplement this discussion. Thus, the development has been placed in Appendix B. and the expression for the abbreviated form of potential energy in terms of the Euler angles is repeated below:

$$-\frac{U}{I_{22} \omega_{2_0}} = F \sin^2 \Theta + [G_1 (g_s \cos \Psi - g \sin \Psi) \sin \Theta \cos \Theta + H_1 \sin^2 \Theta] t + V \quad (50)$$

where

$$F = K[L(\frac{1}{2} + \frac{3}{4} e_n^2 - \frac{3}{4} \sin^2 i_0) + \frac{1}{2} + \frac{3}{4} e_0^2]$$

$$G_1 = K(L+1)$$

$$H_1 = \frac{3}{2} K e_0 e'$$

$$g_s = \text{secular constant } (\dot{\lambda}_r \cos \Omega = g_s t + h_s t^2)$$

$$g_n = \text{secular constant } (\dot{\lambda}_0 \sin \Omega = g_n t + h_n t^2)$$

$\dot{\lambda}_r$ = obliquity of true ecliptic with respect to the fixed ecliptic at some epoch

Ω = longitude of the earth's ascending node

i_0 = obliquity of moons orbital plane with respect to the true ecliptic

$$e_0 = e_s - \frac{de}{dt} / t$$

e_0 = adjusted eccentricity of the apparent orbit of the sun

e_s = eccentricity of earth's orbit at epoch

e_m = eccentricity of moon's orbit

$$h_m = e_m \sin \tilde{\omega}_m$$

$$h_s = e_s \sin \tilde{\omega}_s$$

$$\begin{aligned} V = & K \left\{ L \sin i_0 \left(1 - \frac{1}{2} \sin^2 i_0 + \frac{3}{2} e_s^2 \right) \sin \Theta \cos \Theta \cos(N + \Psi) \right. \\ & - \frac{1}{4} L \sin^2 i_0 \sin^2 \Theta \cos(2N + 2\Psi) - \frac{1}{2} [L \cos(2M_m + 2\tilde{\omega}_m + 2\Psi) \\ & \left. + \cos(2M_s + 2\tilde{\omega}_s + 2\Psi)] \sin^2 \Theta + \frac{3}{2} [L e_m \cos M_m + e_s \cos M_s] \sin^2 \Theta \right\} \end{aligned}$$

N = Ω + the angle between the earth's ascending node and the lunar node

M_m = mean anomaly of moon

$\tilde{\omega}_m$ = longitude of perihelion of moon's orbit

M_s = mean anomaly in earth's orbit

$\tilde{\omega}_s$ = longitude of perihelion of earth's orbit

Equation (50) contains two significant groups of terms. The expression for " V " is solely a collection of periodic terms that represent the fluctuations in the potential that are associated with nutations. The rest of the potential expression consists of terms that are associated with precession. The significance of the two-part potential expression is that it enables a two-part solution. The general solutions may be written as follows:

$$\Psi = \Psi_m + \Psi \quad (51)$$

$$\Theta = \Theta_m + \Theta \quad (52)$$

or

$$\dot{\Psi} = \dot{\Psi}_m + \dot{\Psi} \quad (53)$$

$$\dot{\Theta} = \dot{\Theta}_m + \dot{\Theta}' \quad (54)$$

where

Ψ_m = luni-solar precession

Θ_m = obliquity associated with the luni-solar precession
as contrasted with i_e which is the corresponding
obliquity for the true case

Ψ = principal terms for nutation in longitude

Θ' = obliquity associated with nutation

When the potential energy equation is combined with the precession and nutation expression, the result is:

$$\dot{\Psi} = 2F \cos \Theta + [G_1(g_s \cos \Psi - g_m \sin \Psi) \frac{\cos 2\Theta}{\sin \Theta} + 2H_1 \cos \Theta] t \quad (55)$$

$$+ \frac{1}{\sin \Theta} \frac{\partial V}{\partial \Theta}$$

$$\dot{\Theta} = G_1 \cos \Theta (g_s \sin \Psi + g_m \cos \Psi) t - \frac{1}{\sin \Theta} \frac{\partial V}{\partial \Psi} \quad (56)$$

The solution of Equations (55) and (56) can be simplified by choosing the epoch t_0 such that Ψ is a very small angle. This can be accomplished by choosing the X_{ce} axis to be very close to the lunar node at t_0 . Using the simplifications mentioned above, and neglecting V , the expressions for precession reduce to:

$$\dot{\Psi}_m = 2F \cos \Theta_0 + (G_1 g_s \frac{\cos 2\Theta_0}{\sin \Theta_0} + 2H_1 \cos \Theta_0) t \quad (57)$$

$$\dot{\Theta}_m = (G_1 g_m \cos \Theta_0) t \quad (58)$$

Integration of these equations yields

$$\Psi_m = \Psi_0 + (2F \cos \Theta_0) t + (G_1 g_s \frac{\cos 2\Theta_0}{\sin \Theta_0} + 2H_1 \cos \Theta_0) \frac{t^2}{2} \quad (59)$$

$$\Theta_m = \Theta_o + (G, g_m \cos \Theta_o) \frac{t^2}{2} \quad (60)$$

For an epoch $t_o = 1950$ January 0, Greenwich Mean Noon, and the unit of time taken as the Julian year of 365.25 mean solar days, the following values for the coefficients for the above expressions have been determined by Dr. J. G. Porter of the H. M. Nautical Almanac Office. For convenience, they are shown with their respective equations.

$$\Psi_m = 50''.3732 t - 0''.0001072 t^2 \quad (61)$$

$$\Theta_m = 23^\circ 26' 44''.84 + 5''.608 \times 10^{-6} t^2 \quad (62)$$

The equations for nutation are derived by utilizing the remaining periodic terms of Equations (55) and (56), i.e., those terms containing V .

$$\Psi = \frac{1}{\sin \Theta} \frac{\partial V}{\partial \Theta} \quad (63)$$

$$\Theta = \frac{-1}{\sin \Theta} \frac{\partial V}{\partial \Psi} \quad (64)$$

Now upon replacing Θ by Θ_o after differentiation, the nutation equations become

$$\begin{aligned} \dot{\Psi} = & KL \sin i_o \left(1 - \frac{1}{2} \sin^2 i_o + \frac{3}{2} e_n^2\right) \frac{\cos 2\Theta_o}{\sin \Theta_o} \cos(N + \Psi) \\ & - K \cos \Theta_o \left[\frac{1}{2} L \sin^2 i_o \cos(2N + 2\Psi) + L \cos(2M_m + 2\tilde{\omega}_m + 2\Psi) \right. \\ & \left. + \cos(2M_s + 2\tilde{\omega}_s + 2\Psi) - 3(L e_m \cos M_m + e_s \cos M_s) \right] \end{aligned} \quad (65)$$

$$\dot{\Theta} = KL \sin i_0 \left(1 - \frac{1}{2} \sin^2 i_0 + \frac{3}{2} e^2\right) \cos \Theta_0 \sin(N + \Psi) - K \sin \Theta_0 \left[\frac{1}{2} L \sin^2 i_0 \sin(2N + 2\Psi) + L \sin(2M_m + 2\tilde{\omega}_m + 2\Psi) + \sin(2M_s + 2\tilde{\omega}_s + 2\Psi) \right] \quad (66)$$

But, the lunar node retrogrades on the ecliptic in a period of about 18.6 years, so N can be written in the form

$$N = N_0 - N' t \quad (67)$$

And for present purposes, it is sufficient to write the precession angle as

$$\Psi = \Psi_0 + K' t \quad (68)$$

Thus, by adding Equations (67) and (68)

$$N + \Psi = \nu - \xi t \quad (69)$$

where $\nu = N_0 + \Psi_0$

and $\xi = N' - K'$

In order to simplify the algebra in dealing with Equations (65) and (66), the following substitutions are introduced:

$$\xi D \equiv KL \sin i_0 \left(1 - \frac{1}{2} \sin^2 i_0 + \frac{3}{2} e^2\right) \frac{\cos 2\Theta_0}{\sin \Theta_0} \quad (70)$$

$$\xi E \equiv KL \sin i_0 \left(1 - \frac{1}{2} \sin^2 i_0 + \frac{3}{2} e^2\right) \cos \Theta_0 \quad (71)$$

Integrating Equations (65) and (66) now yields the expressions for the forced nutation in longitude and the associated obliquity.

$$\begin{aligned}\Psi = & -D \sin(N+\psi) + K \cos \Theta_0 \left[\frac{L \sin^2 i_0}{4 \alpha} \sin(2N+2\psi) \right. \\ & - \frac{L}{2\eta_m} \sin(2M_m + 2\tilde{\omega}_m + 2\psi) - \frac{1}{2\eta_s} \sin(2M_s + 2\tilde{\omega}_s + 2\psi) \\ & \left. + 3 \left(\frac{L e_m}{\eta_m} \sin M_m + \frac{e_s}{\eta_s} \sin M_s \right) \right]\end{aligned}\quad (72)$$

$$\begin{aligned}\Theta = & E \cos(N+\psi) - K \sin \Theta_0 \left[\frac{L \sin^2 i_0}{4 \alpha} \cos(2N+2\psi) \right. \\ & \left. - \frac{L}{2\eta_m} \cos(2M_m + 2\tilde{\omega}_m + 2\psi) - \frac{1}{2\eta_s} \cos(2M_s + 2\tilde{\omega}_s + 2\psi) \right]\end{aligned}\quad (73)$$

where η_m = the mean angular velocity of the moon about its orbit

η_s = the mean angular velocity of the earth about its orbit

The Nautical Almanac lists the calculated values for the nutation equations to be:

$$\begin{aligned}\Psi = & -17''.23 \sin(N+\psi) + 0''.21 \sin(2N+2\psi) - 1''.27 \sin(2M_s + 2\tilde{\omega}_s) \\ & - 0''.21 \sin(2M_m + 2\tilde{\omega}_m) + 0''.07 \sin M_m + 0''.13 \sin M_s\end{aligned}\quad (74)$$

$$\begin{aligned}\Theta = & 9''.21 \cos(N+\psi) - 0''.09 \cos(2N+2\psi) + 0''.55 \cos(2M_s + 2\tilde{\omega}_s) \\ & + 0''.09 \cos(2M_m + 2\tilde{\omega}_m)\end{aligned}\quad (75)$$

All terms of amplitudes less than $0''.05$ have been neglected.

2.2.5 Coordinate Correction

Consider a point in space the position of which is measured with respect to an earth fixed coordinate system by the angles α_0 and δ_0 , the Right

$$Z_{GE} = |\vec{r}| \sin \delta_0 \quad (78)$$

where $|\vec{r}|$ is the magnitude of the vector discussed.

The components of this vector may now be expressed in terms of inertial components as follows:

$$\begin{bmatrix} X_{GE80} \\ Y_{GE80} \\ Z_{GE80} \end{bmatrix} = \begin{bmatrix} \cos \psi_m & \cos \theta_m \sin \psi_m & \sin \theta_m \sin \psi_m \\ -\sin \psi_m & \cos \theta_m \cos \psi_m & \sin \theta_m \cos \psi_m \\ 0 & -\sin \theta_m & \cos \theta_m \end{bmatrix} \begin{bmatrix} |\vec{r}| \cos \delta_0 \cos \alpha_0 \\ |\vec{r}| \cos \delta_0 \sin \alpha_0 \\ |\vec{r}| \sin \delta_0 \end{bmatrix}$$

so

$$X_{GE80} = |\vec{r}| \left[\cos \delta_0 \cos \alpha_0 \cos \psi_m + \cos \delta_0 \sin \alpha_0 \cos \theta_m \sin \psi_m + \sin \delta_0 \sin \theta_m \sin \psi_m \right] \quad (79)$$

$$Y_{GE80} = |\vec{r}| \left[-\cos \delta_0 \cos \alpha_0 \sin \psi_m + \cos \delta_0 \sin \alpha_0 \cos \theta_m \cos \psi_m + \sin \delta_0 \sin \theta_m \cos \psi_m \right] \quad (80)$$

$$Z_{GE80} = |\vec{r}| \left[-\cos \delta_0 \sin \alpha_0 \sin \theta_m + \sin \delta_0 \cos \theta_m \right] \quad (81)$$

But, the coordinates of the vector with respect to the inertial reference can be expressed by the Right Ascension and Declination coordinates α , δ , or

$$X_{GE80} = |\vec{r}| \cos \delta \cos \alpha \quad (82)$$

Ascension and Declination, respectively. If it is desired to find the corresponding position of this point with respect to some inertially fixed coordinate system X_{GE50} Y_{GE50} Z_{GE50} , the angular corrections for precession and nutation must be taken into account. The following is a derivation of the correction terms for precession. Correction terms for nutation will be presented following the precession discussion.

2.2.5.1 Precession Correction

The previously defined Euler angle rotations must be used to relate the inertially fixed reference to the precessing coordinate system of date. This can be accomplished by using the known precession angle and its associated obliquity, ψ_m and θ_m , respectively. The inertial reference in this case would be the one from which the precession is referenced, i.e., the one determined by the mean equinox at epoch, the mean equatorial pole at epoch, and a perpendicular to the previous two. The coordinate system of date is the one determined by the mean equinox of date, the mean equatorial pole of date, and a perpendicular to the previous two.

A vector, \vec{r} , in the equatorial coordinate system defined by the mean equinox of date can be transformed to the inertial reference by the following operator:

$$\begin{bmatrix} X_{GE50} \\ Y_{GE50} \\ Z_{GE50} \end{bmatrix} = \begin{bmatrix} \cos \psi_m & \cos \theta_m \sin \psi_m & \sin \theta_m \sin \psi_m \\ -\sin \psi_m & \cos \theta_m \cos \psi_m & \sin \theta_m \cos \psi_m \\ 0 & -\sin \theta_m & \cos \theta_m \end{bmatrix} \begin{bmatrix} X_{GE} \\ Y_{GE} \\ Z_{GE} \end{bmatrix}_{MEAN}$$

or $\vec{r}_{GE50} = [P] \vec{r}_{GEmean}$

The above operator transforms any vector measured in the coordinate system of date to an inertial system in which the precession angle ψ_m and the obliquity angle θ_m define the transformation as a function of time (equation 61, 62).

The quantity that is measured in a sighting is in the true reference frame of date. If Right Ascension and Declination Coordinates are used, the vector can be expressed in rectangular components which are also in a true frame of date.

$$X_{GE} = |\vec{r}| \cos \delta_o \cos \alpha_o \quad (76)$$

$$Y_{GE} = |\vec{r}| \cos \delta_o \sin \alpha_o \quad (77)$$

$$Y_{GES0} = |\vec{r}| \cos \delta \sin \alpha \quad (83)$$

$$Z_{GES0} = |\vec{r}| \sin \delta \quad (84)$$

so that equating the two sets of inertial components for the point, a relationship between the measured and inertial Right Ascension and Declination coordinates is found.

$$\begin{aligned} \cos \delta \cos \alpha = & \cos \delta_0 \cos \alpha_0 \cos \psi_m + \cos \delta_0 \sin \alpha_0 \cos \theta_m \sin \psi_m \\ & + \sin \delta_0 \sin \theta_m \sin \psi_m \end{aligned} \quad (85)$$

$$\begin{aligned} \cos \delta \sin \alpha = & -\cos \delta_0 \cos \alpha_0 \sin \psi_m + \cos \delta_0 \sin \alpha_0 \cos \theta_m \cos \psi_m \\ & + \sin \delta_0 \sin \theta_m \cos \psi_m \end{aligned} \quad (86)$$

$$\sin \delta = -\cos \delta_0 \sin \alpha_0 \sin \theta_m + \sin \delta_0 \cos \theta_m \quad (87)$$

In order to solve **these** equations for the correction term, $\alpha - \alpha_0$ the first equation is multiplied by $\sin \alpha_0$ and the second equation by $\cos \alpha_0$. The first is then subtracted from the second. The results on the left side of the equation are as follows:

$$\begin{aligned}
& \cos \delta \sin \alpha \cos \alpha_0 - \cos \delta \cos \alpha \sin \alpha_0 \\
&= \cos \delta (\sin \alpha \cos \alpha_0 - \cos \alpha \sin \alpha_0) \\
&= \cos \delta \sin(\alpha - \alpha_0)
\end{aligned} \tag{88}$$

Now this equation can be solved for $\alpha - \alpha_0$ which is the correction term for the Right Ascension term for precession. The detailed equation follows:

$$\begin{aligned}
(\alpha - \alpha_0) = \text{ARCSIN} & \left\{ \frac{-\cos \delta_0 \cos \alpha_0 (\sin \psi_m \cos \alpha_0 + \cos \psi_m \sin \alpha_0)}{\sqrt{1 - \sin^2 \delta}} \right. \\
& + \frac{\cos \delta_0 \sin \alpha_0 \cos \theta_m (\cos \psi_m \cos \alpha_0 - \sin \psi_m \sin \alpha_0)}{\sqrt{1 - \sin^2 \delta}} \\
& \left. + \frac{\sin \delta_0 \sin \theta_m (\cos \psi_m \cos \alpha_0 - \sin \psi_m \sin \alpha_0)}{\sqrt{1 - \sin^2 \delta}} \right\} \tag{89}
\end{aligned}$$

where $\sin \delta$ is defined in Equation (87). Since θ_m and ψ_m are explicit functions of time, the correction term, $\alpha - \alpha_0$, is a function of δ_0 , α_0 and t . The correction term for the Declination, $\delta - \delta_0$, coordinate may be derived by expressing $\sin \delta$ as follows:

$$\begin{aligned}
\sin \delta &= \sin[\delta_0 + (\delta - \delta_0)] = \sin \delta_0 \cos(\delta - \delta_0) \\
&+ \cos \delta_0 \sin(\delta - \delta_0)
\end{aligned} \tag{90}$$

Now since $\delta - \delta_0$ is very small, the following approximations are valid:

$$\cos(\delta - \delta_0) \approx 1$$

$$\sin(\delta - \delta_0) \cong \delta - \delta_0 \text{ (radians)}$$

Equation (87) thus reduces to

$$\sin \delta_0 (1) + (\cos \delta_0)(\delta - \delta_0) = -\cos \delta_0 \sin \alpha_0 \sin \theta_m + \sin \delta_0 \cos \theta_m \quad (91)$$

$$\text{and } (\delta - \delta_0) = \frac{-\cos \delta_0 \sin \alpha_0 \sin \theta_m + \sin \delta_0 \cos \theta_m - \sin \delta_0}{\cos \delta_0} \quad (92)$$

The declination correction term is seen to be a function of δ_0 , α_0 and θ ; this is analogous to the Right Ascension correction term.

2.2.5.2 Nutation Correction

The correction matrix for nutation is obtained by taking rotations of Θ about the true $+X_{GE}$ axis and Ψ about $+Z_{GE}$. Thus, the nutation correction matrix is:

$$\begin{bmatrix} X_{GE} \\ Y_{GE} \\ Z_{GE} \end{bmatrix}_{MEAN} = \begin{bmatrix} 1 & \Psi(\cos \theta_m - \Theta \sin \theta_m) & \Psi(\Theta \cos \theta_m + \sin \theta_m) \\ -\Psi \cos \theta_m & \cos \theta_m (\cos \theta_m - \Theta \sin \theta_m) + \sin \theta_m (\sin \theta_m + \Theta \cos \theta_m) & \cos \theta_m (\Theta \cos \theta_m + \sin \theta_m) - \sin \theta_m (-\Theta \sin \theta_m + \cos \theta_m) \\ -\Psi \sin \theta_m & \sin \theta_m (\cos \theta_m - \Theta \sin \theta_m) + \cos \theta_m (-\sin \theta_m - \Theta \cos \theta_m) & \sin \theta_m (\Theta \cos \theta_m + \sin \theta_m) + \cos \theta_m (-\Theta \sin \theta_m + \cos \theta_m) \end{bmatrix} \begin{bmatrix} X_{GE} \\ Y_{GE} \\ Z_{GE} \end{bmatrix}_{TRUE}$$

or $\hat{r}_{GE,MEAN} = [N] \hat{r}_{GE,TRUE}$ (93)

Equations (74) and (75) give the detailed expressions for Θ and Ψ . When these values are used with θ_m and ψ_m in N , the appropriate nutation correction matrix is obtained for an orthogonal transformation from the true geographic frame of date to the mean geographic frame of date. Once the nutation correction is performed, the correction for precession may be made with the matrix $[P]$.

The Right Ascension and Declination correction terms for nutation can be derived in a similar manner to that of the precession correction terms.

$$(\alpha - \alpha'_0) = \text{ARCSIN} \left\{ \frac{\cos \alpha'_0 [\eta_{21} \cos \delta'_0 \cos \alpha'_0 + \eta_{22} \cos \delta'_0 \sin \alpha'_0 + \eta_{23} \sin \delta'_0]}{\sqrt{1 - \sin^2 \delta'_0}} - \frac{\sin \alpha'_0 [\eta_{11} \cos \delta'_0 \cos \alpha'_0 + \eta_{12} \cos \delta'_0 \sin \alpha'_0 + \eta_{13} \sin \delta'_0]}{\sqrt{1 - \sin^2 \delta'_0}} \right\}$$

$$(\delta_o - \delta_o') = \frac{\eta_{31} \cos \delta_o' \cos \alpha_o' + \eta_{32} \cos \delta_o' \sin \alpha_o' + \eta_{33} \sin \delta_o' - \sin \delta_o'}{\cos \delta_o'}$$

where

η_{ij} = the elements of $[N]$

α_o' = Right Ascension value uncorrected for nutation

δ_o' = Declination value uncorrected for nutation

α_o = Right Ascension value corrected for nutation but not corrected for precession

δ_o = Declination value corrected for nutation but not corrected for precession.

If it is desired, a direct transformation may be performed from the true reference to the inertial reference in one transformation.

$$\vec{r}_{GE50} = [P] [N] \vec{r}_{GEtrue}$$

$$[P] [N] = \begin{bmatrix} \cos \psi_m - \psi \sin \psi_m & \left[(\psi \cos \psi_m + \sin \psi_m) \cdot (\cos \theta_m - \Theta \sin \theta_m) \right] & \left[(\psi \cos \psi_m + \sin \psi_m) \cdot (\Theta \cos \theta_m + \sin \theta_m) \right] \\ -\sin \psi_m - \psi \cos \psi_m & \left[(-\psi \sin \psi_m + \cos \psi_m) \cdot (\cos \theta_m - \Theta \sin \theta_m) \right] & \left[(-\psi \sin \psi_m + \cos \psi_m) \cdot (\Theta \cos \theta_m + \sin \theta_m) \right] \\ 0 & -\sin \theta_m - \Theta \cos \theta_m & -\Theta \sin \theta_m + \cos \theta_m \end{bmatrix}$$

This matrix accounts for both precession and nutation.

Due to the orthogonality of all the transformations used above, it is possible to determine the inverse relationships easily by employing the transpose of the matrices.

$$\vec{r}_{GEmean} = [P]^{-1} \vec{r}_{GE50}$$

$$\vec{r}_{GEtrue} = [N]^{-1} \vec{r}_{GEmean}$$

$$\vec{r}_{GEtrue} = [N]^{-1} [P]^{-1} \vec{r}_{GE50}$$

2.2.5.3 Empirical Correction

The Ephemeris contains the empirical equations for total general precession. It must be noted that the previous analysis developed all equations with the assumption that planetary precession was negligible. Therefore, extremely small deviations would be expected in certain elements of the transformation matrices (e.g., terms that were theoretically zero will be extremely small numbers).

The values for the elements, p_{ij} , of the precession correction matrix, $[P]$, as listed in the Ephemeris are as follows (epoch 1950.0)

$$P_{11} = 1 - (29696 T^2 + 13 T^3) \times 10^{-8}$$

$$P_{12} = (2234941 T + 676 T^2 - 221 T^3) \times 10^{-8}$$

$$P_{13} = (971690 T - 207 T^2 - 96 T^3) \times 10^{-8}$$

$$P_{21} = - P_{12}$$

$$P_{22} = 1 - (24975 T^2 + 15 T^3) \times 10^{-8}$$

$$P_{23} = - (10858 T^2) \times 10^{-8}$$

$$P_{31} = - P_{13}$$

$$P_{32} = P_{23}$$

$$P_{33} = 1 - (4721 T^2) \times 10^{-8}$$

where T is measured in tropical centuries from 1950.0.

The empirical results for nutation correction are slightly more complicated but can be expressed as functions of time. The Nautical Almanac uses the following parameters to describe nutation:

Γ' = mean longitude of the lunar perigee, measured in the ecliptic from the mean equinox of date to the mean ascending node of the lunar orbit and then along the orbit.

L = geometric mean longitude, mean equinox of date.

Γ = mean longitude of perigee, mean equinox of date.

\mathcal{Q} = mean longitude of Moon, measured in ecliptic from mean equinox of date to mean ascending node of lunar orbit and then along the orbit.

Ω = longitude of the mean ascending node of the lunar orbit on the ecliptic measured from the mean equinox of date

where

$$\begin{aligned}
 \Omega &= 12^{\circ}.112790 - .05253922D + .0020795T + .002081T^2 + .000002T^3 \\
 \mathcal{C} &= 64^{\circ}.375452 + 13.176397D - .001131575T - .00113015T^2 - .0000019T^3 \\
 \Gamma' &= 208^{\circ}.84399 + .11140408D - .010334T - .010343T^2 - .000012T^3 \\
 \Gamma &= 282^{\circ}.08053 + .000047068D + .0004553T + .0004575T^2 + .000003T^3 \\
 L &= 280^{\circ}.08121 + .98564734D + .000303(T+T^2)
 \end{aligned}$$

and

D = Days since reference epoch (1950.0, J.D. 2433282.423)

T = Julian centuries past reference epoch

The Nautical Almanac also separates the short period nutations and the long period nutations as follows:

$$\Psi = \Delta \psi + d\Psi$$

$$\Theta = \Delta \epsilon + d\epsilon$$

where $\Delta \epsilon$ = long period obliquity of nutation

$\Delta \psi$ = long period nutation in longitude

$d\epsilon$ = short period obliquity of nutation

$d\psi$ = short period nutation in longitude

and gives the values for these parameters as:

$$\begin{aligned}
 \Delta \psi \times 10^4 &= -(47^{\circ}.8927 + .0482T) \sin \Omega \\
 &+ .5800 \sin 2\Omega - 3.5361 \sin 2L - .1378 \sin (3L - \Gamma) \\
 &+ .0594 \sin (L + \Gamma) + .0344 \sin (2L - \Omega) + .0125 \sin (2\Gamma' - \Omega) \\
 &+ .3500 \sin (L - \Gamma) + .0125 \sin (2L - 2\Gamma') \\
 d\psi \times 10^4 &= -.5658 \sin 2\mathcal{C} - .0950 \sin (2\mathcal{C} - \Omega) \\
 &- .0725 \sin (3\mathcal{C} - \Gamma') + .0317 \sin (\mathcal{C} + \Gamma') \\
 &+ .0161 \sin (\mathcal{C} - \Gamma' + \Omega) + .0158 \sin (\mathcal{C} - \Gamma' - \Omega) \\
 &- .0144 \sin (3\mathcal{C} + \Gamma' - 2L) - .0122 \sin (3\mathcal{C} - \Gamma' - \Omega)
 \end{aligned}$$

$$\begin{aligned}
& + .1875 \sin (\ell - \Gamma') + .0078 \sin (2\ell - 2\Gamma') \\
& + .414 \sin (\ell + \Gamma' - 2L) + .0167 \sin (2\ell - 2L) \\
& - .0089 \sin (4\ell - 2L).
\end{aligned}$$

$$\begin{aligned}
\Delta \epsilon \times 10^4 &= 25.5844 \cos \Omega - .2511 \cos 2\Omega \\
&+ 1.5336 \cos 2L + .0666 \cos (3L - \Gamma) \\
&- .0258 \cos (L + \Gamma) - .0183 \cos (2L - \Omega) \\
&- .0067 \cos (2\Gamma' - \Omega) \\
d\epsilon \times 10^4 &= .2456 \cos 2\ell + .0508 \cos (2\ell - \Omega) \\
&+ .0369 \cos (3\ell - \Gamma') - .0139 \cos (\ell + \Gamma') \\
&- .0086 \cos (\ell - \Gamma' + \Omega) + .0083 \cos (\ell - \Gamma' - \Omega) \\
&+ .0061 \cos (3\ell + \Gamma' - 2L) + .0064 \cos (3\ell - \Gamma' - \Omega)
\end{aligned}$$

The nutation correction matrix can be approximated as:

$$N = \begin{bmatrix} 1 & \psi \cos i_e & \psi \sin i_e \\ -\psi \cos i_e & 1 & \psi \\ -\psi \sin i_e & -\psi & 1 \end{bmatrix}$$

where i_e is utilized rather than Θ_m due to the fact that it represents general precession better. This quantity is numerically equal to

$$\begin{aligned}
i_e &= 23.4457587^\circ - .01309404T \\
&- .00000088T^2 + .00000050T^3
\end{aligned}$$

2.3 TIME STANDARDS, MEASURE AND CONVERSION

2.3.1 Astronomical Time Standards

Time in the most basic sense is a measure of the elapsed interval between two observed events relative to the period of a stable oscillation referred to as the time reckoner. Thus, time measurement may be based on any of a number of observed uniform and periodic phenomenon (for example, the decay of radioactive isotopes, the observed motions of some man-made periodic system, the observed motions of celestial bodies, etc.). However, until the advent of the atomic clock, no system was devised which was capable of approaching the accuracy obtained by employing the astronomical time reckoner. Since this system of measure is still employed as the time standard, the paragraphs which follow are considered essential for the successful interpretation of material presented in all of the remaining monographs of this series.

2.3.1.1 Sidereal Time

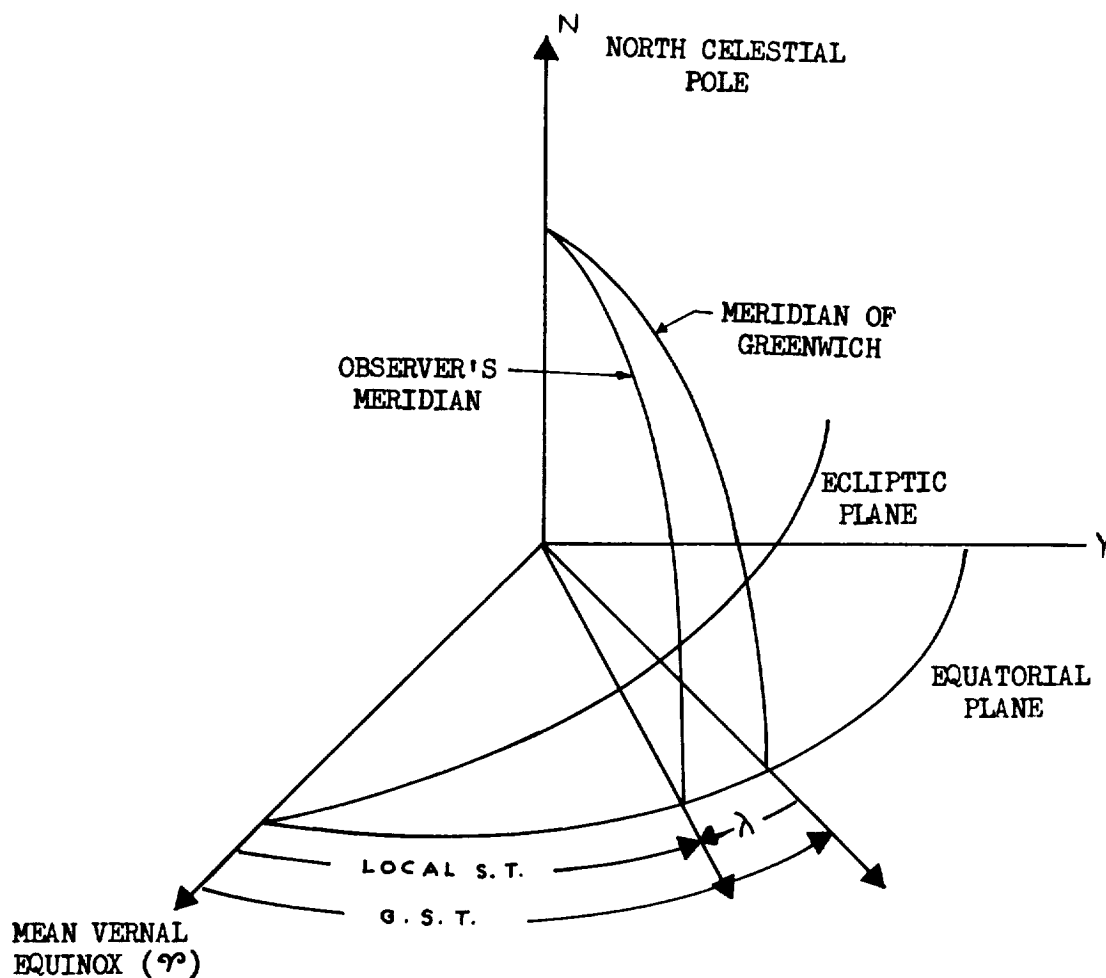
This measure of time is based on the apparent motion of the stars relative to an observer on the earth and uses for the fundamental period the interval between two successive transits through the observer's meridian of any selected star (1 sidereal day = 86164.09054 mean solar seconds = 360 degrees). This interval (or the corresponding spin rate of the earth) thus defines the spin vector for, or angular momentum corresponding to, the earth's rotation at the epoch in question. Discussions elsewhere in this monograph pertaining to the rate of change of this angular momentum shows that the vector is not constant but rather nutates and precesses. These discussions do not indicate, however, that the magnitude of the spin is variable; References 2, 3, 4, 5, and 6 indicate that the mathematical model utilized to derive these results was slightly in error; and that tidal friction and other forces of similar nature are producing secular and periodic changes in the rotational rate of the earth. The observed effects on a day-to-day basis are, however, almost completely inappreciable. For this reason, the sidereal day mentioned previously or a uniform measure of the perturbed day will be considered to be known; and the effects of the earth's spin variations will be enumerated when they are significant.

While the star which is selected for the definition of this period is seen to be unimportant, the base of measure for each of several observations must be known in order to incorporate a data acquired by others in any given analysis. Thus, to avoid numerous problems, the vernal equinox or first point of Aries (though the vernal equinox no longer lies in the constellation Aries) has been selected as the reference for all sidereal times. In the most general sense then, sidereal time is the hour angle (angle between the observers meridian and the meridian of the object being observed measured positive toward west) of the vernal equinox. But since two observers at

different longitudes could measure the same sidereal times (at different times, of course), it is also necessary to differentiate between the various local sidereal times. This step is accomplished by establishing a reference meridian (that of Greenwich) and correcting all times in such a fashion that the instantaneous position of the reference meridian is computed

$$\text{Greenwich Sidereal Time} = \text{local S.T.} + \text{longitude } (\lambda)$$

where: longitude is measured in hours (1 hour = 15 degrees) in the astronomical convention, i.e., positive to the west,



and by adopting the convention that zero hours Greenwich sidereal time corresponds to the upper transit of the vernal equinox through the meridian of Greenwich.

There is, however, a problem arising from the referencing of sidereal times to the vernal equinox since a unit vector in this direction moves due to the motion of the ecliptic plane and the change in orientation of the spin vector of the earth. This fact has led to the definition of two sidereal times: (1) apparent sidereal time and (2) mean sidereal time. Apparent sidereal time is referenced to the true equator of date and, therefore, includes adjustments for both nutation and precession. However, since the nutation correction is itself a variable, this definition of sidereal time is nonuniform and is not generally employed. Rather, a correction tabulated in the ephemeris and known as the equation of the equinox is applied to the apparent sidereal time to reduce it to mean sidereal time (referenced to the mean equator of date which precesses at a uniform rate).

2.3.1.2 Universal Time

From the standpoint of the sequencing of civil functions, the most obvious time standard is the apparent motion of the sun about the earth. Unfortunately, due to eccentricity of the earth's orbit, such a time standard would be nonuniform. This situation leads to the definition of a fictitious sun moving in a circular orbit in the plane of the earth's equator with a period exactly equal to that of the true sun (the equatorial year or the interval between upper transits of the sun through the mean equator is 365.242,198,79 - .000,006,14T, where T denotes Julian centuries since 1900.0, zero hours January Zero on the Besselian Calendar). Successive upper transits of this fictitious sun (referred to most frequently as the mean sun) through the meridian of an observer then defined the mean solar day and its divisions (1 mean solar sec = 1/86400 mean solar day).

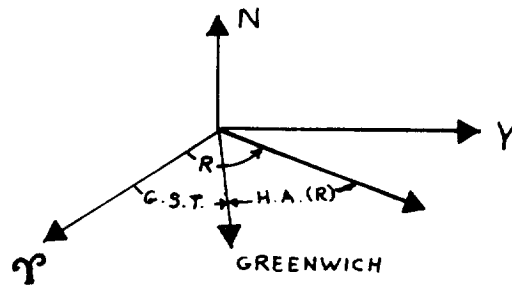
With the introduction of Newcomb's Tables of the Sun, terminology changed slightly and a standard solar time referred to as Universal Time was defined. In his table, the Right Ascension of the mean sun was given by (Newcomb's Notation)

$$R = 18^h 38^m 45^s.836 + 8640184^s.542T + 0^s.0919 T^2$$

where T is centuries from Greenwich mean noon January 0, 1900. Corresponding to this expression for R , Universal Time (or Greenwich mean time, GMT) referenced to mean midnight rather than noon was defined to be

$$\begin{aligned} \text{U.T.} &= 12^h + \text{Greenwich hour angle of } R \\ &= 12^h + \text{H.A. } (R) \end{aligned}$$

Now expanding the hour angle (by referring to the following sketch), U.T. can be related directly to Greenwich sidereal time and R



$$U.T. = 12^h + G.S.T. - R$$

This equation, thus, shows that universal time and sidereal time are equivalent measures.

Unfortunately, Newcomb considered T to be a measure of mean solar time (though he did not specify the manner in which T was to be measured) and did not have information available to him pertaining to the variable rate of rotation of the earth. Therefore, the value of R predicted utilizing these assumptions does not correspond to the true position of the mean sun (it is noted that the errors are very small compared to the hour angle of the true sun with respect to the mean sun). However, since the mean sun itself was an artificial means of defining a uniform time, Newcomb's equation for R has been retained in the definition of U.T. with T now defined to be Julian centuries (36525 days of U.T.) elapsed since 12 hours E. T. on January 0 1900.

Thus, Universal Time is a precise measure of time, by definition. Further, since it is precise, and since the predicted Right Ascensions of the time reckoner correspond so closely to those of the mean sun (and thus to the true sun), U.T. is the most logical standard to be utilized as the reference for all civil times.

Now since there is a uniform time defined for points along the Greenwich meridian, a local mean time for other meridians (not to be confused with zonal time which is the local mean time for a 15° interval of longitudes equal to time along the bisecting meridian) can be defined by subtracting the longitude of the local station from U.T., i.e.,

$$L.M.T. = U.T. - \lambda$$

where λ is measured in hours, etc., positive to the west of Greenwich to be consistent with astronomical convention.

2.3.1.3 Ephemeris Time

Based on the observations of the introductory sentences to this section of the monograph, it can be argued that there is no absolute time scale. Thus, in periodic dynamical systems, the problem becomes one of measuring time on

some arbitrary scale convenient for the problem at hand. The dynamical systems of astronomical interest are no exception since here a uniform time depending on the equations of motion for determination can be adopted and time defined by matching the predicted and observed behavior of the system. This step has, in fact, been accomplished in the definition of Ephemeris Time by requiring that the motion of the sun relative to the earth be observed and by selecting a time scale which agreed as well as possible with the actual solar motions (any other body could have been employed without loss of generality) and universal time during the 19th century.

The fundamental epoch from which Ephemeris Time is reckoned was adopted by Newcomb as 12 E.T. January 0 1900 when the mean longitude (= mean anomaly plus argument of periaipse plus longitude of the ascending node) of the sun was $279^{\circ} 41' 48'' .04$. At the same time Newcomb defined the fundamental period as the tropical year (i.e., the interval required for the sun's mean longitude to increase by 360 degrees) and expressed the mean longitude in the following equation

$$L = 279^{\circ} 41' 48'' .04 + 129602768'' .13 T + 1'' .089 T^2$$

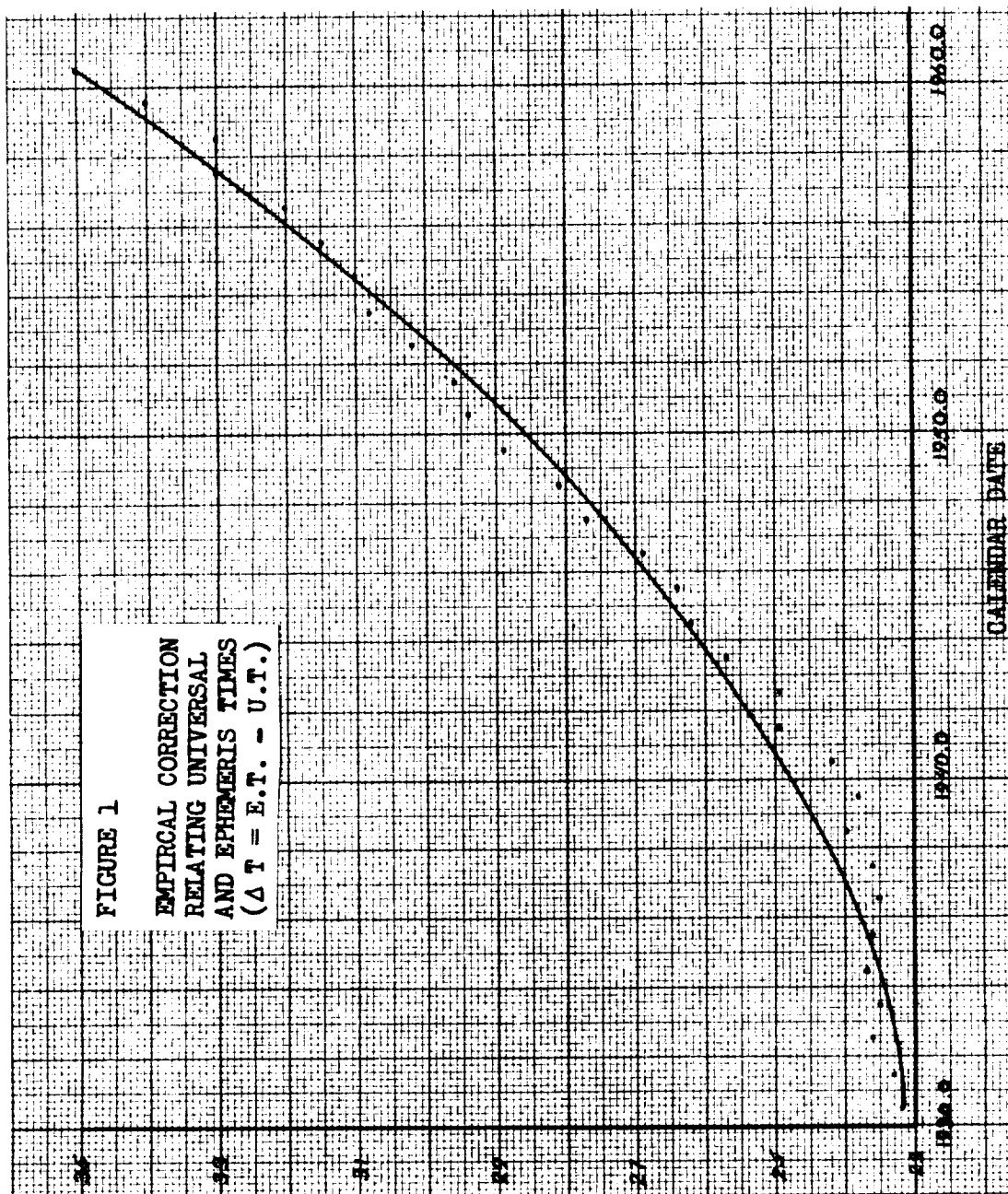
where T is centuries of 36525 ephemeris days elapsed since the fundamental epoch. This equation exhibits the fact that the tropical year is not uniform. Thus, to conserve the system of time used by Newcomb in his theories of motion and assure its uniformity, the Comite Internationale des Poids et Mesures in 1957 adopted the ephemeris second of mean noon E.T. January 0 1900 as the fundamental invariable unit of ephemeris time (this unit is $1/31556925.9747$ tropical year of 1900).

Ephemeris time at any given instant is obtained empirically by comparing the observed and gravitationally predicted positions of some body. When, after interpolation, the argument in the tabulated ephemeris is found, Ephemeris time is known and the empirical correction to universal times can be computed.

$$\Delta T = E.T. - U.T.$$

Raw (that is, unsmoothed) data for ΔT can be found on Page 89 of the "Explanatory Supplement to the Ephemeris." These data and the least squares parabolic which fits them are presented in Figure 1 for the time period of 1930.5 to 1960.5. Therefore, an approximate value of ΔT can be computed for any time in the vicinity of the interval for which there was data and universal time corrected to yield the desired ephemeris epoch.

As may be seen, ephemeris time is not connected to the rotational rate of the earth and is thus not suitable for the computation of quantities which depend on this rotational rate. However, a meridian on an imaginary ellipsoid enclosing the earth's surface and rotating at a uniform rather than a variable rate has been introduced and defined as the ephemeris meridian. This ephemeris meridian was selected in such a manner that had the earth rotated uniformly since mean noon January 0 1900, the Greenwich and ephemeris meridians would coincide. Actually,



they differ and the ephemeris meridian is 1.002738 Δ T sec east of the Greenwich meridian. Thus, while E.T. is not generally employed in such problems, it can be adopted and the results related to the true earth. This approach is not recommended for most problems; however, for those jobs in which a great deal of ephemeris lookup is encountered, it may be convenient to operate with a single time standard.

2.3.1.4 Julian Date

All of the previously defined times have been described in such a manner that no readily available chronological time record exists. This deficiency requires cumbersome conversion of dates and greatly magnifies problems of analysis, since in order to relate any two measurements in time, they must first be referred to the same epoch. This step, in turn, requires only that some epoch pre-dating the time period in question be selected and that universal time be recorded on a continuous basis from this epoch. But, rather than selecting a new epoch for each problem attacked, an arbitrary date sufficiently far in the past to pre-date recorded history was adopted (Greenwich mean noon 1 January 4713 BC on the Julian proleptic calendar) and given the name Julian Day (J.D.) zero.

It is noted before passing that since ephemeris time and universal time differ only by the small empirical correction, the Julian Day numbers can also be interpreted as Julian Ephemeris Dates (J.E.D.) if the initial epoch is reckoned to be 12 E.T. 1 January 4713 BC.

2.3.2 Determination and Conversion of Astronomical Times

As was apparent in the discussions which preceded, the various time scales were equivalent (or were relatable if observed empirical corrections could be estimated). Thus, in order to define any of the three, it is necessary to determine at least one. While this determination itself may not be of great interest in most problems connected with this study, the reduction of the observed data and the interrelation of these time scales is of direct interest. For this reason, a modified numerical example based on the example presented in the "Supplement to the Ephemeris" (Page 84) will be presented. This numerical example assumes that local apparent sidereal time is available and progresses through mean sidereal time to universal time to ephemeris time. However, it must be noted that the process can be inverted at any point in the sample should one of the other times be given.

It is assumed that an identifiable portion of the celestial sphere has been observed from a known and calibrated station and that the observed position data have been correlated with an ephemeris of the apparent positions of the observed stars for the purpose of computing the position of the vernal equinox of date, the corresponding local apparent sidereal time, and the values of universal and ephemeris time. Data for the sample are as follows:

Date: 7 March 1960 (J.D. 2437000.5 = 0^h U.T.)

Longitude: 5^h 08^m 15^s.75 W

Local Apparent S.T.: 13^h 05^m 37^s.249

Approximately 2^h local mean time (= 7^h U.T.)

and reduction proceeds as:

observed local apparent sidereal time	13 ^h	05 ^m	37 ^s .249
equation of the equinoxes (interpolated to 7 ^h U.T. obtained from the American Ephemeris and Nautical Almanac (AENA))	-		.046
local mean sidereal time	13	05	37.295
longitude (add if W)	5	05	15.750
Greenwich mean sidereal time	18	13	53.045
minus right ascension of time reckoner +12 ^h J.D. 0 ^h U.T. of epoch date 243 7000.5 J.D. 1900.5 241 5020.0 elapsed time T. 21980.5 Julian centuries T=060179329227 -R +12 ^h = -6 ^h 38 ^m 45 ^s .836 -8640184.542 T -0.0929 T ²	-10	58	50.971
universal time of observation, sidereal units (1 sidereal day = 86164.09054 mean solar sec)	7	15	2.074
conversion to mean solar time units (1 day = 86400 MSS) direct conversion or by Table VIII AENA (reduction to mean solar time)	-	1	11.269
universal time of observation (solar units)	7	13	50.805
Δ T (the empirical correction to convert U.T. to Ephemeris Time) from Fig. 1			35.
Ephemeris Time	7	14	26.

2.3.3 Dynamical Time

As was pointed out in the discussion of Ephemeris time, the time scale is not absolute and may be selected arbitrarily to simplify the solution. Thus, in problems concerning the motion of a small satellite in the vicinity of a large mass, it is sometimes desirable to adopt a measure of time other than one of those discussed (for internal computations) and defer relation to these standards until a later point in the computation. In these cases it is convenient to consider the equations of motion for a particle whose position vector is \vec{r} in a central force field of a mass whose gravitational constant is μ .

$$\ddot{\vec{r}} = -\mu \frac{\vec{r}}{r^3}$$

and note that if τ is defined to be $\tau = \sqrt{\mu} t$ then

$$\frac{d^2 \vec{r}}{d\tau^2} = -\frac{\vec{r}}{r^3}$$

and

$$\frac{d\vec{r}}{d\tau} = \vec{v}' = \frac{\vec{v}}{\sqrt{\mu}}$$

The gravitational constant has, thus, been absorbed (note that the new time variable τ has the units of $L^{1/2}$). This absorption process appears, on its face, to accomplish little other than the removal of scaling operation at various points in the numerical solution of the equations of motion by substituting a scaling at the two extremes of the calculation procedure. However, it is important to note that the trajectory in terms of the variable τ can be computed to any precision desired and that the results become more accurate as the uncertainty in μ is reduced (References 7, 8, 9, 10 11). In this sense, then, the trajectory is universal, and the process is akin to that employed in the preparation of the ephemeris where again the accuracy improvement is not obtainable without the absorption process.

Since time now has the general units of length, it is important to specify this measure. Various standards of length have been employed; however, the most desirable standard for studies of motion in the vicinity of the earth is the earth's equatorial radius (R_e). The advisability of this selection is apparent when it is realized that R_e like μ is uncertain (thus subject to improvement) and that all vector components (cartesian) both relative to points on earth and to some base frame can, thus, be rescaled as R_e becomes better known. Best values of these constants in conventional units are (Reference 11)

$$\begin{aligned}\mu &= (398,601.5 \pm 1.) \text{ Km}^3/(\text{mean solar sec})^2 \\ R_e &= (6378.163 \pm .02) \text{ Km}\end{aligned}$$

3. RECOMMENDED PROCEDURES

The material presented in this monograph pertains primarily to definition and standardization of terminology to be employed in the remaining monographs of the series. Where this material duplicates information in the literature and where previously accepted standards exist, no alteration has been proposed. However, there are several areas (principally within the discussion of coordinate systems) in which it has been necessary to adopt a notation, for example, the manner of identifying coordinate frames. These adoptions are the result of careful attention to matters pertaining to the entire series of monographs and with minor exception are recommended for these applications.

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Potential Energy of an Ellipsoid Earth Model

Referring to the sketch on Page 61 for geometrical definitions and to W. M. Smart (Celestial Mechanics) for the basic theory leads to the development of the potential energy of an ellipsoidal earth. This development will be presented below.

From the sketch it is seen that

$$\bar{\Delta} = \bar{P} - \bar{r} \quad (A1)$$

If the dot product of each side is taken with itself equation (A1) becomes

$$|\bar{\Delta}|^2 = |\bar{P}|^2 - 2\bar{P} \cdot \bar{r} + |\bar{r}|^2 \quad (A2)$$

A new right handed orthogonal coordinate system is now chosen with its principal direction in the \bar{P} direction. The \bar{r} vector is expressed as (ξ, η, ζ) in this new system. In this light

$$-2\bar{P} \cdot \bar{r} = -2[P_x \xi + P_y \eta + P_z \zeta] = -2|\bar{P}| \xi \quad (A3)$$

$$\text{Now } |\bar{\Delta}|^2 = |\bar{P}|^2 - 2|\bar{P}| \xi + |\bar{r}|^2 \quad (A4)$$

$$\text{or } \Delta^2 = P^2 - 2P\xi + r^2 \quad (A5)$$

The vector notation has been omitted for simplicity. It is understood that the symbols stand for the magnitude of their respective vectors. Equation (A4) can be used in Equation 16 if it is expressed as

$$\Delta = P \left(1 - \frac{2\xi}{P} + \frac{r^2}{P^2} \right)^{1/2}$$

$$\text{or } \frac{1}{\Delta} = \frac{1}{P} \left(1 - \frac{2\xi}{P} + \frac{r^2}{P^2} \right)^{-1/2} \quad (A6)$$

It is possible to expand equation (A6) in a binomial series and get a close approximation for $\frac{1}{\Delta}$ because ξ/P and r/P are very small (the distance to dm as compared to distance to the Sun or Moon). Carrying terms to the order of $\left(\frac{\xi}{P}\right)^4$, equation (16) becomes

$$U = \frac{Gm_i}{P} \int_{dm} \left(1 + \frac{\xi}{P} + \frac{3\xi^2 - r^2}{2P^2} + \frac{5\xi^3 - 3\xi r^2}{2P^3} + \frac{35\xi^4 - 30\xi^2 r^2 + 3r^4}{8P^4} \right) \quad (A7)$$

where $m_i = m_s$ or m_m

Let (l, m, n) be the direction cosines for \bar{e} with respect to the body fixed axes. Since the ξ axis was chosen to be in this direction, a unit vector, $\hat{\xi}$, may be defined to be

$$\hat{\xi} = (l, m, n) \quad (A8)$$

In terms of the body fixed axis system, the \bar{r} vector can be expressed as

$$\bar{r} = (x, y, z) \quad (A9)$$

Since ξ is the component of \bar{r} in the \bar{e} direction, it can be found as follows:

$$\xi = \hat{\xi} \cdot \bar{r} = lx + my + nz \quad (A10)$$

Now the various terms of equation (A7) can be evaluated.

$$(1) \quad \int_V dm = m_e \quad (A11)$$

$$(2) \quad \int_V \xi dm = (lx + my + nz) dm = 0 \quad (A12)$$

(due to symmetry)

$$(3) \quad \int_V (3\xi^2 - r^2) dm = \int_V \{2r^2 - 3(\eta^2 + \zeta^2)\} dm \quad (A13)$$

Now from the definition of moment of inertia, the following is true:

$$I_{xx} + I_{yy} + I_{zz} = \int_V 2(x^2 + y^2 + z^2) dm = \int_V 2r^2 dm \quad (A14)$$

$$\text{and} \quad \int_V (\eta^2 + \zeta^2) dm = I' \quad (A15)$$

where I' is the moment of inertia about the ξ axis.

$$\text{Hence} \quad \int_V (3\xi^2 - r^2) dm = I_{xx} + I_{yy} + I_{zz} - 3I' \quad (A16)$$

$$(4) \quad \int_V (5\xi^3 - 3\xi r^2) dm = \int_V [5(lx + my + nz)^3 - 3(lx + my + nz)(\cancel{\eta^2 + \zeta^2} + n^2)(x^2 + y^2 + z^2)] dm \quad (A17)$$

All terms of this expression integrate to zero when taken over the entire volume because of symmetry.

$$\text{So} \quad \int_V (5\xi^3 - 3\xi r^2) dm = 0 \quad (A18)$$

$$(5) \quad \int_V (35\xi^4 - 30\xi^2 r^2 + 3r^4) dm \text{ is of the order of } m_e a^4, \text{ where}$$

a is a semi axis of the ellipsoid earth model.

Similarly,

$$\int_V (3\xi^2 - r^2) dm \text{ is of the order of } m_e a^2.$$

Comparing the contribution of the third and fifth terms of equation A7, it is seen from the above expressions that the order of magnitude of the fifth term is $\frac{a^2}{r^2}$ times as small as the third. Since this corresponds to $\left(\frac{1}{60}\right)^2$, the fifth term may be neglected.

The expression for potential correct to order $\left(\frac{a}{r}\right)^3$ may now be written as

$$U = Gm_i \left[\frac{m_e}{r} + \frac{I_{xx} + I_{yy} + I_{zz} - 3I'}{2r^3} \right] \quad (A19)$$

It is well known that the moment of inertia about any axis in terms of the principal moments of inertia is:

$$I' = I_{xx} l^2 + I_{yy} m^2 + I_{zz} n^2 \quad (A20)$$

$$\text{also } l^2 + m^2 + n^2 = 1 \quad (A21)$$

$$\text{so } I' = I_{xx} - (I_{xx} - I_{yy}) m^2 + (I_{zz} - I_{xx}) n^2 \quad (A22)$$

The position in space of the body m_i is related to the earth fixed axis system by the direction cosines.

$$m = \frac{y}{r} \quad n = \frac{z}{r}$$

Finally, (A19) becomes:

$$U = Gm_i \left[\frac{m_e}{r} + \frac{I_{xx} + I_{yy} + I_{zz} - 2I_{xx}}{2r^3} + \frac{3(I_{xx} - I_{yy}) y^2 - 3(I_{zz} - I_{xx}) z^2}{2r^5} \right] \quad (A23)$$

APPENDIX B

Expansion of U in Terms of Θ and Ψ

If reference is made to the figure on page 58 for the definition of Euler angles and to W. M. Smart (Celestial Mechanics) for the basic method of analysis, the expansion of the potential energy of the ellipsoidal earth model can be expressed in terms of the angles Θ and Ψ . This derivation is shown below.

From a cartesian coordinate transformation it is seen that

$$m_{Z_{GG}} = m_{X_{G\epsilon}} \sin \Theta \sin \Psi + m_{Y_{G\epsilon}} \sin \Theta \cos \Psi + m_{Z_{G\epsilon}} \cos \Theta \quad (B1)$$

$$s_{Z_{GG}} = s_{X_{G\epsilon}} \sin \Theta \sin \Psi + s_{Y_{G\epsilon}} \sin \Theta \cos \Psi + m_{Z_{G\epsilon}} \cos \Theta \quad (B2)$$

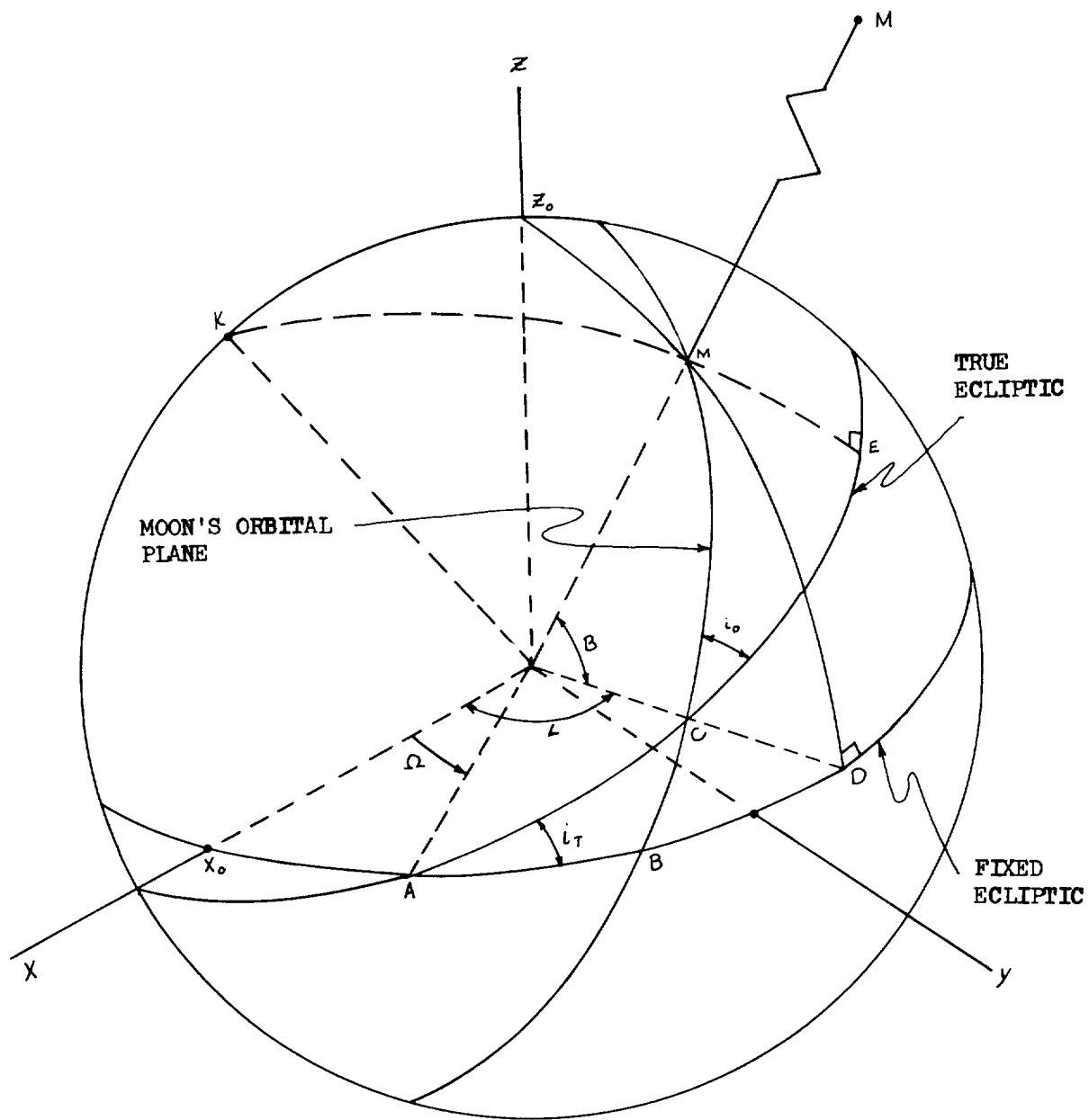
Also, the standard polar spherical to rectangular transformation can be used to find $X_{G\epsilon}$, $Y_{G\epsilon}$ and $Z_{G\epsilon}$.

$$\left. \begin{aligned} X_{G\epsilon} &= \rho \cos L \cos B \\ X_{G\epsilon} &= \rho \sin L \cos B \\ Z_{G\epsilon} &= \rho \sin B \end{aligned} \right\} \quad (B3)$$

Hence,

$$\frac{Z_{GG}}{\rho} = \cos B \sin \Theta \sin (L + \Psi) + \sin B \cos \Theta \quad (B4)$$

The following sketch defines the geometry to be used in the subsequent analysis.



If equation (B4) is expressed in terms of the variables used in equation (50), then the expression for \mathcal{U} can be found in terms of the Euler angles Θ and Ψ by using equation (37). The following angles are now defined:

$$\begin{aligned} KZ_O &= i_T \\ Z_O M &= \pi/2 - B \\ KM &= \pi/2 - b \\ KZ_O M &= \pi/2 + L - \Omega \\ Z_O KM &= \pi/2 - (\ell - \Omega) \end{aligned}$$

where

$$\begin{aligned} \ell &= X_O A + AE \\ b &= EM \end{aligned}$$

Also, the following identities are true:

$$\sin B = \sin b \cos i_T + \cos b \sin i_T \sin (\ell - \Omega) \quad (B5)$$

$$\cos B \sin (L - \Omega) = -\sin b \sin i_T + \cos b \cos i_T \sin (\ell - \Omega) \quad (B6)$$

$$\cos B \cos (L - \Omega) = \cos b \cos (\ell - \Omega) \quad (B7)$$

Define $X_O A + AC = N$

$$N + CM = \mathcal{V}$$

so $MC = \mathcal{V} - N$

$$ME = b$$

$$CE = \ell - N$$

$$MCE = i_O$$

$$MEC = \pi/2$$

Now $\cos(\mathcal{V} - N) = \cos b \cos (\ell - N) \quad (B8)$

$$\sin(\mathcal{V} - N) \cos i_O = \cos b \sin (\ell - N) \quad (B9)$$

$$\sin(\mathcal{V} - N) \sin i_O = \sin b \quad (B10)$$

Writing $(L - \Omega) + (\psi + \Omega)$ for $(L + \psi)$, $\frac{Z_{GG}}{\rho}$
becomes $\frac{Z_{GG}}{\rho} = \sin \theta \cos(\Omega + \psi) [\cos B \sin(L - \Omega)]$
 $+ \sin \theta \sin(\Omega + \psi) [\cos B \cos(L - \Omega)] + \cos \theta \sin B$ (B11)

Since i_T is a very small angle, it is sufficient to write:

$$\sin i_T = i_T$$

$$\cos i_T = 1$$

Now equation (B6) becomes

$$\cos B \sin(L - \Omega) = -i_T \sin b + \cos b \sin(\ell - \Omega) \quad (B12)$$

Writing $(\ell - N) + (N - \Omega)$ for $(\ell - \Omega)$ and using equations (B8), (B9), and (B10), equation (B12) becomes

$$\cos B = -i_T \sin i_0 \sin(\nu - N) + \cos i_0 \sin(\nu - N) \cos(N - \Omega) \\ + \cos(\nu - N) \sin(N - \Omega) \quad (B13)$$

Since the angle i_0 is approximately $5^\circ 2'$, $\cos i_0$ can be written as

$$\cos i_0 = 1 - 1/2 \sin^2 i_0$$

with sufficient accuracy (using a binomial expansion and neglecting terms of higher order than $\sin^3 i_0$).

Now

$$\cos B \sin(L - \Omega) = (1 - 1/4 \sin^2 i_0) \sin(\nu - \Omega) \quad (B14)$$

$$- 1/4 \sin^2 i_0 \sin(\nu - 2N + \Omega)$$

$$\cos B \cos(L - \Omega) = (1 - 1/4 \sin^2 i_0) \cos(\nu - \Omega) \\ + 1/4 \sin^2 i_0 \cos(\nu - 2N + \Omega) \quad (B15)$$

$$\sin B = \sin i_0 \sin(\nu - N) + i_r \sin(\nu - \Omega) \quad (B16)$$

The expression for $\frac{Z_{GG}}{r}$ now becomes

$$\begin{aligned} \frac{Z_{GG}}{r} = & (1 - 1/4 \sin^2 i_0) \sin \Theta \sin(\nu + \psi) + \sin i_0 \cos \Theta \sin(\nu - N) \\ & + i_T \cos \Theta \sin(\nu - \Omega) - 1/4 \sin^2 i_0 \sin \Theta \sin(\nu - 2N + \psi) \end{aligned} \quad (B17)$$

Equation (37) requires an expression for $(\frac{Z_{GG}}{r})^2$ in order for it to be useful. If equation (B17) is squared a very complex expression results. However, all periodic terms whose coefficients are of greater order than i_T or $\sin^2 i_0$ do not contribute a significant amount in the end result when an integration of Lagrange's equation is performed.

Hence,

$$\begin{aligned} \left(\frac{Z_{GG}}{r}\right)^2 = & 1/2 \sin^2 i_0 + (1/2 - 3/4 \sin^2 i_0) \sin^2 \Theta \\ & - (1/2 - 1/4 \sin^2 i_0) \sin^2 \Theta \cos(2\nu + 2\psi) \\ & - 1/4 \sin^2 i_0 \sin^2 \Theta \cos(2N + 2\psi) \\ & + \sin^2 i_0 (1 - 1/2 \sin^2 i_0) \sin \Theta \cos \Theta \cos(N + \psi) \\ & + i_T \sin \Theta \cos \Theta \cos(\Omega + \psi) \end{aligned} \quad (B18)$$

The other term needed in order to express equation (37) as desired is $(\frac{a}{r})^3$. Expressed in terms of the mean anomaly ($M_m = n_m t + \epsilon - \tilde{\omega}$),

$$\left(\frac{a}{r}\right)^3 = 1 + 3/2 e_m^2 + 3e_m \cos M + 9/2 e_m^2 \cos 2M_m \quad (B19)$$

Now

$$\begin{aligned} \left(\frac{a}{r}\right)^3 \left(\frac{Z_{GG}}{r}\right)^2 = & (1/2 + 3/4 e_m^2 - 3/4 \sin^2 i_0) \sin^2 \Theta \\ & + \sin i_0 (1 - 1/2 \sin^2 i_0 + 3/2 e_m^2) \sin \Theta \cos \Theta \cos(N + \psi) \\ & - 1/4 \sin^2 i_0 \sin^2 \Theta \cos(2N + 2\psi) + i_T \sin \Theta \cos \Theta \cos(N + \psi) + (continued) \end{aligned}$$

$$+ 3/2 e_m \sin^2 \theta \cos M_m - 1/2 \sin^2 \theta \cos(2M_m + 2\tilde{\omega} + 2\psi) \quad (B20)$$

All terms which do not contribute to $\frac{\partial U}{\partial \theta}$ or $\frac{\partial U}{\partial \phi}$ have been neglected.

Employing a subscript m for Moon and s for Sun and recalling that i_0 is zero for the Sun, the following expression is written:

$$\begin{aligned} - \frac{U}{K \omega_{z0} I_{zz}} &= \left[L(1/2 + 3/4 e_m^2 - 3/4 \sin^2 i_0) + 1/2 + 3/4 e_s^2 \right] \sin^2 \theta \\ &+ \left[L \sin i_0 (1 - 1/2 \sin^2 i_0 + 3/4 e_m^2) \right] \sin \theta \cos \theta \cos(N + \psi) \\ &- 1/4 \sin^2 i_0 \sin^2 \theta \cos(2N + 2\psi) L \\ &- 1/2 \left[L \cos(2M_m + 2\tilde{\omega}_m + 2\psi) + \cos(2M_s + 2\tilde{\omega}_s + 2\psi) \right] \sin^2 \theta \\ &+ i_T(L + 1) \cos(\psi + \Omega) \sin \theta \cos \theta \\ &+ 3/2(L e_m \cos M_m + e_s \cos M_s) \sin^2 \theta \end{aligned} \quad (B21)$$

where K and L are defined on Page 59.

Using the substitutions defined on Page 62 and 63, equation (50) may now be written.

$$\begin{aligned} - \frac{U}{I_{zz} \omega_{z0}} &= F \sin^2 \theta + \left[G_1(g_1 \cos \psi - g \sin \psi) \sin \theta \cos \theta + H, \sin^2 \theta \right] t \\ &+ V \end{aligned} \quad (50)$$

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