LV2 Inertial Navigation Numerical Integration Methods

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Numerical integration methods are used to approximate the definite integral of a function. Various methods exist such as the Midpoint, Trapezoidal and Simpson Rules. The different methods that exist tend to offer a tradeoff between simplicity and accuracy. Simpson's method is the most accurate of the aforementioned methods and is used in the LV2 INS calculations.

Simpson's method is one that assumes that the samples used in the calculation are evenly spaced over the desired interval. More complicated methods exist, that offer greater accuracy I might add, that allow for unequally spaced samples over the desired interval. Using A/D converters for sampling continuous signals *usually* implies evenly spaced samples and so most numerical integration methods used in an engineering application, such as inertial navigation, will employ a fixed step size as opposed to a variable step size.

The methods used approximate the values of velocity and position by integrating over sets of intervals. For example, the present time velocity can be approximated using the summation of the results of integration.

First Order Integration

In order to accomplish first order integration, we will use a closed, extended version of Simpson's Rule. The method is actually given by Press, Teukolsky et. al in their book: Numerical Recipes in C and is a hybridization of Simpson's extended rule and the trapezoidal rule. The method uses six samples and achieves an error of $O\left(\frac{1}{N^4}\right)$. This method is used for calculating the velocity from a set of six acceleration values.

$$\boxed{\int_{x_1}^{x_N} f(x)dx = \Delta x\left[\frac{3}{8}f_1 + \frac{7}{6}f_2 + \frac{23}{24}f_3 + \frac{23}{24}f_4 + \frac{7}{6}f_5 + \frac{3}{8}f_6\right] + O\left(\frac{1}{N^4}\right)}$$

Second Order Integration

The second order integration method is used to calculate position from a set of six accelerometer samples. The difficulty is that there is no way to numerically integrate twice using the first order integration method. So, in order to calculate the position from acceleration values, we have to have a numerical, i.e formala format, method of calculating the integral of the integral.

This method was developed and is in a way based on Simpson's method because it fits a polynomial to the data points, and takes the definite integral of the indefinite integral over the interval. The formula is based on a 5th order polynomial and therefore achieves a very high accuracy as long as the function being approximated is not of higher order than five. The method is shown below and it's proof follows.

$$\int_{x_1}^{x_N}\int f(x)dxdx \approx \Delta x^2\left[\frac{1525}{1008}f_1+\frac{11875}{2016}f_2+\frac{625}{504}f_3+\frac{3125}{1008}f_4+\frac{625}{1008}f_5+\frac{275}{2016}f_6\right]+5Cx$$

where C is the initial condition of the first order integral at x_1, i.e. the initial velocity

Proof (Mathematica was used to ease the calculation of the linear system):

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Peq = {n^5, n^4, n^3, n^2, n, 1}
{n^5, n^4, n^3, n^2, n, 1}

m = {{0, 0, 0, 0, 0, 1},
  {x^5, x^4, x^3, x^2, x^1, 1},
  {32 x^5, 16 x^4, 8 x^3, 4 x^2, 2 x, 1},
  {243 x^5, 81 x^4, 27 x^3, 9 x^2, 3 x, 1},
  {1024 x^5, 256 x^4, 64 x^3, 16 x^2, 4 x, 1},
  {3125 x^5, 625 x^4, 125 x^3, 25 x^2, 5 x, 1}}
{{0, 0, 0, 0, 0, 1}, {x^5, x^4, x^3, x^2, x, 1},
 {32 x^5, 16 x^4, 8 x^3, 4 x^2, 2 x, 1},
 {243 x^5, 81 x^4, 27 x^3, 9 x^2, 3 x, 1},
 {1024 x^5, 256 x^4, 64 x^3, 16 x^2, 4 x, 1},
 {3125 x^5, 625 x^4, 125 x^3, 25 x^2, 5 x, 1}}

Res = LinearSolve[m, {p1, p2, p3, p4, p5, p6}]
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$$\left\{\frac{-p1+5\,p2-10\,p3+10\,p4-5\,p5+p6}{120\,x^5},\ \frac{3\,p1-14\,p2+26\,p3-24\,p4+11\,p5-2\,p6}{24\,x^4},\ \frac{-17\,p1+71\,p2-118\,p3+98\,p4-41\,p5+7\,p6}{24\,x^3},\ \frac{45\,p1-154\,p2+214\,p3-156\,p4+61\,p5-10\,p6}{24\,x^2},\ \frac{-137\,p1+300\,p2-300\,p3+200\,p4-75\,p5+12\,p6}{60\,x},\ p1\right\}$$

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MFunc = Peq.Res
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$$\mathrm{p1}+\frac{n^5(-\mathrm{p1}+5\,\mathrm{p2}-10\,\mathrm{p3}+10\,\mathrm{p4}-5\,\mathrm{p5}+\mathrm{p6})}{120\,x^5}+\frac{n^4(3\,\mathrm{p1}-14\,\mathrm{p2}+26\,\mathrm{p3}-24\,\mathrm{p4}+11\,\mathrm{p5}-2\,\mathrm{p6})}{24\,x^4}+\frac{n^3(-17\,\mathrm{p1}+71\,\mathrm{p2}-118\,\mathrm{p3}+98\,\mathrm{p4}-41\,\mathrm{p5}+7\,\mathrm{p6})}{24\,x^3}+\frac{1}{24\,x^2}(n^2(45\,\mathrm{p1}-154\,\mathrm{p2}+214\,\mathrm{p3}-156\,\mathrm{p4}+61\,\mathrm{p5}-10\,\mathrm{p6}))+\frac{1}{60\,x}(n(-137\,\mathrm{p1}+300\,\mathrm{p2}-300\,\mathrm{p3}+200\,\mathrm{p4}-75\,\mathrm{p5}+12\,\mathrm{p6}))$$

Vel = Integrate[MFunc, {n, 0, 5 x}]

$$\frac{95\,\mathrm{p1}\,x}{288}+\frac{125\,\mathrm{p2}\,x}{96}+\frac{125\,\mathrm{p3}\,x}{144}+\frac{125\,\mathrm{p4}\,x}{144}+\frac{125\,\mathrm{p5}\,x}{96}+\frac{95\,\mathrm{p6}\,x}{288}$$

Half = Integrate[MFunc, n]

$$n\,\mathrm{p1}+\frac{n^6(-\mathrm{p1}+5\,\mathrm{p2}-10\,\mathrm{p3}+10\,\mathrm{p4}-5\,\mathrm{p5}+\mathrm{p6})}{720\,x^5}+\frac{n^5(3\,\mathrm{p1}-14\,\mathrm{p2}+26\,\mathrm{p3}-24\,\mathrm{p4}+11\,\mathrm{p5}-2\,\mathrm{p6})}{120\,x^4}+\frac{n^4(-17\,\mathrm{p1}+71\,\mathrm{p2}-118\,\mathrm{p3}+98\,\mathrm{p4}-41\,\mathrm{p5}+7\,\mathrm{p6})}{96\,x^3}+\frac{1}{72\,x^2}(n^3(45\,\mathrm{p1}-154\,\mathrm{p2}+214\,\mathrm{p3}-156\,\mathrm{p4}+61\,\mathrm{p5}-10\,\mathrm{p6}))+\frac{1}{120\,x}(n^2(-137\,\mathrm{p1}+300\,\mathrm{p2}-300\,\mathrm{p3}+200\,\mathrm{p4}-75\,\mathrm{p5}+12\,\mathrm{p6}))$$

Dis = Integrate[Half + C, {n, 0, 5 x}]

$$5\,C\,x+\frac{1525\,\mathrm{p1}\,x^2}{1008}+\frac{11875\,\mathrm{p2}\,x^2}{2016}+\frac{625\,\mathrm{p3}\,x^2}{504}+\frac{3125\,\mathrm{p4}\,x^2}{1008}+\frac{625\,\mathrm{p5}\,x^2}{1008}+\frac{275\,\mathrm{p6}\,x^2}{2016}$$