Conversion of Geodetic coordinates to the Local Tangent Plane

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In this paper the equations are presented to convert WGS−84 Latitude, Longitude and Height measurements to Local Tangent Plane coordinates. The method used passes through the Earth-Centered, Earth-Fixed rectangular coordinate system on the way to the Local Tangent Plane. The basic reference for this paper is J.Farrell & M.Barth "The Global Positioning System & Inertial Navigation" © 1999 by McGraw Hill.

The customary reference frame for terrestrial navigation is known as the Earth centered-Earth fixed geodetic (ECEF-g) coordinate system. The coordinates for this frame are usually written <λ, ϕ, h> for latitude, longitude, and altitude (height), respectively. Note that altitude is defined as the perpendicular distance above an assumed earth shape known as the geoid which is mathematically an ellipsoid of revolution meant to approximate the true shape of the earth. ECEF-g is the coordinate system most commonly associated with the GPS. Since longitude is measured relative to a fixed point on the surface of the earth, ECEF-g rotates at the earth rate, i.e. it is a non-inertial frame.

As an alternative reference frame it is sometimes convenient to define an ECEF rectangular (ECEF-r) system. In this scheme the origin is at the earth center, the z-axis is directed up through the north pole, the x-axis is perpendicular passing through the prime meridian and the y-axis is chosen to complete an <x, y, z> orthogonal right handed coordinate system. The ECEF-r system is fairly easy to relate to local rectangular frames

To precisely apply the ECEF-g coordinate system the exact shape of the Earth's ellipsoid must be given. This shape is derived from measurement, and has been defined differently by different groups. The assumed shape (sometimes known as the "datum") used in the GPS is known as WGS−84 and contains the following parameters:

Constant	Value	Unit	What?
π	3.1415926535898		Pi
a	6378137.0	m	WGS-84 Earth semimajor axis
b	6356752.3142	m	WGS-84 Earth semiminor axis
ω_{ie}	$7.2921151467 \times 10^{-5}$	r / s	WGS-84 value for Earth rotation rate
μ	3.986005×10^{14}	m^3 / s^2	WGS-84 value for Earth's gravitational constant

Only the first three values in this table are required here.

The transformation from ECEF-g to ECEF-r can be found in our reference.

First define the ellipsoid flatness

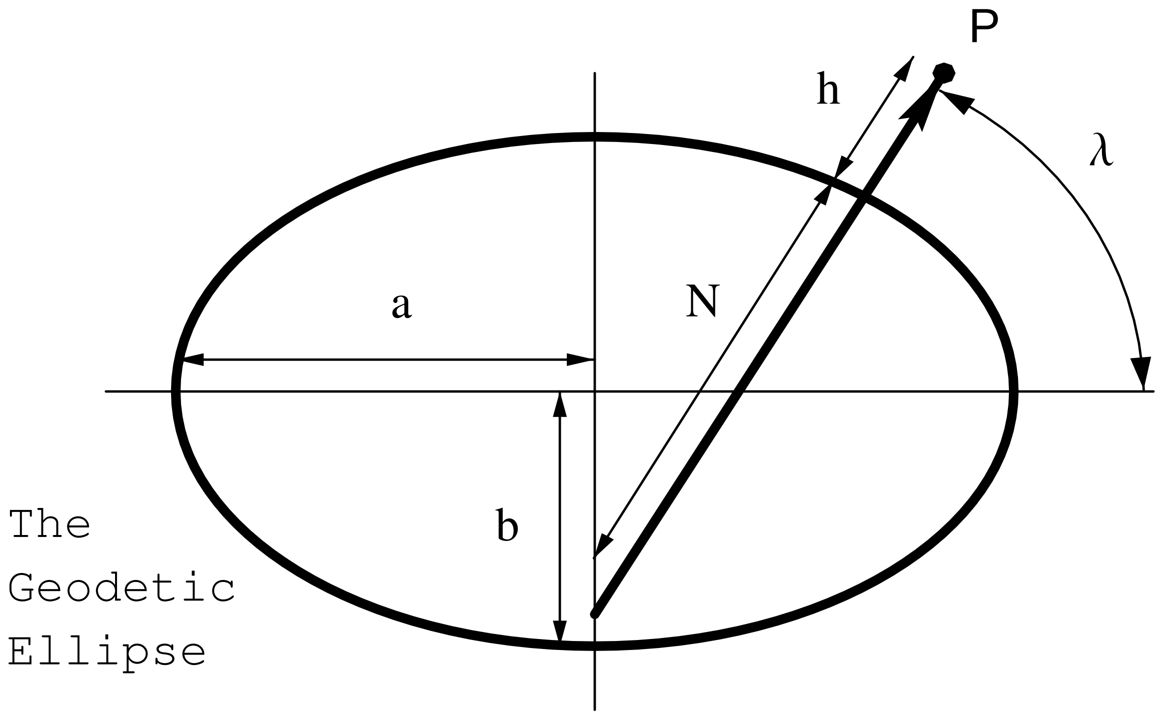
$$f = \frac{a - b}{a} = 3.3528107 \times 10^{-3}$$

The eccentricity may then be defined as

$$e = \sqrt{f\,(2 - f)} = 8.1819191 \times 10^{-2}$$

The distance from the surface to the z-axis along the ellipsoid normal (the plumb line) is then found as a function of λ as

$$N(\lambda) = \frac{a}{\sqrt{1 - e^2 \sin^2(\lambda)}}$$



The diagram shows the relationship of the normal distance *(N)*, the height *(h)* and the latitude (λ) for a point (**P**). Note that the latitude is defined by the Earth normal, not the line connecting the point to the Earth's center.

The ECEF-r coordinates can be calculated from the geodetic as

$$x = (h + N) \cos\lambda \cos\phi$$
$$y = (h + N) \cos\lambda \sin\phi$$
$$z = (h + (1 - e^2)\, N) \sin\lambda$$

The inverse transformation is somewhat more involved, so it is not quoted here.

It is desired to further transform the ECEF-r coordinates into what will be termed here the Local Tangent Plane (LTP). This is an orthogonal, rectangular, reference system defined with its origin at an arbitrary point on (or possibly near) the Earth's surface. The vertical axis is then taken to be straight up, opposite the plumb line. The other axes are perpendicular to the vertical axis aligned with local geographic east and north. This system is written as <e, n, u> and forms a right handed coordinate system with a strong analogy to the usual <x, y, z> coordinates. Note that this is not the only possible arrangement of axes for the local tangent plane, but the author feels that this definition has some intuitive advantages. The great advantage of the Local Tangent system is that its axes coincide with the expectation of people on the ground concerning such ingrained things as up, and north, which something like ECEF-r totally does not.

The transformation from ECEF-r to LTP can be built up from a sequence of simple translation and rotations.

Let ($\mathbf{x}_0$) be the origin of the LTP expressed in ECEF-r coordinates. Transforming the origin from the center of the Earth to the Tangent Plane origin is accomplished by subtracting ($\mathbf{x}_0$) from the ECEF-r coordinate. Call the ECEF−r coordinate ($\mathbf{x}$), and the new translated coordinate system ($\mathbf{x}'$)

$$\mathbf{x}' = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

Note that the <x', y', z'> axes are parallel to the ECEF−r <x, y, z> axes.

Transforming (**x**') coordinates to LTP requires only rotation to align the axes.

Let $<\lambda, \phi>$ be the latitude and longitude respectively of the Local Tangent Plane origin in ECEF−g coordinates. Define an intermediate coordinate system (**x**'') by rotating (**x**') about the z−axis so that the x'−axis passes through the projection of the LTP origin onto the <xy> plane. Since the LTP origin has longitude (ϕ), this requires a positive rotation of magnitude (ϕ). After rotation, the y'−axis will be parallel to the LTP east direction. With this in mind, define the new double−prime coordinate system like this

$$\mathbf{x}'' = \begin{pmatrix} e \\ u'' \\ z \end{pmatrix} = \mathbf{R}_1[\phi] \cdot \mathbf{x}' = \begin{pmatrix} -\sin(\phi) & \cos(\phi) & 0 \\ \cos(\phi) & \sin(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

Looking at the above transformation, several slightly tricky things are going on. The first component of (**x**'') is called (e) not (e''), this makes sense because (e) is truly the LTP east coordinate, requiring no further transformation. The second component (u'') by contrast is not the same as the LTP up−coordinate, though the u''−axis can be made parallel to the up−axis by a simple rotation about the east−axis.

The transformation to (**x**'') raises a natural question. Is the matrix ($\mathbf{R}_1$) defined by the equation correct? Any 2D plane rotation matrix will have row and column parings of two sine and two cosine functions, but at a given matrix position how can we check for sine versus cosine, or plus versus minus? One technique which works fairly well is to let the rotation angle (ϕ) be a very small positive number, thus ($\cos(\phi) \lesssim 1$), and ($\sin(\phi) \gtrsim 0$), then either as a purely mental exercise, or by drawing a diagram, check in succession the effect upon the output vector of a vector parallel to each of the input axes. For example in this case a small (ϕ) means that the LTP origin is very close to the prime meridian. Thus increasing the x'−coordinate causes an increasingly *negative* east−coordinate, but the closer the rotation angle gets to zero, the smaller the contribution the x'−coordinate makes in the east direction, until at zero rotation, the x'−coordinate has no influence on the east−coordinate at all. Of the 4 possibilities, only negative−sine has the correct behavior. Similarly, at zero longitude, the y'−axis is parallel to the LTP east direction, so the only suitable function is positive−cosine.

To complete the transformation to LTP coordinates, the u''−axis must be rotated about the east−axis so that it points straight up in the Local Tangent Plane. Fortunately, the latitude (λ) is defined precisely as the angle between the equator and the local normal vector. (Note that a positive rotation about the east−axis rotates the u''−axis toward the south pole, while latitude is defined to be positive north of the equator.)

In any case, the final rotation may be expressed like this

$$\mathbf{x}_t = \begin{pmatrix} e \\ n \\ u \end{pmatrix} = \mathbf{R}_2[\lambda] \cdot \mathbf{x}'' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\sin(\lambda) & \cos(\lambda) \\ 0 & \cos(\lambda) & \sin(\lambda) \end{pmatrix} \cdot \begin{pmatrix} e \\ u'' \\ z \end{pmatrix}$$

Where ($\mathbf{x}_t$) refers to coordinates expressed in the Local Tangent Plane.

Writing the whole transformation out gives

$$\mathbf{x}_t = \begin{pmatrix} e \\ n \\ u \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\sin(\lambda) & \cos(\lambda) \\ 0 & \cos(\lambda) & \sin(\lambda) \end{pmatrix} \cdot \begin{pmatrix} -\sin(\phi) & \cos(\phi) & 0 \\ \cos(\phi) & \sin(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \right)$$

$$= \begin{pmatrix} -\sin(\phi) & \cos(\phi) & 0 \\ -\cos(\phi)\sin(\lambda) & -\sin(\lambda)\sin(\phi) & \cos(\lambda) \\ \cos(\lambda)\cos(\phi) & \cos(\lambda)\sin(\phi) & \sin(\lambda) \end{pmatrix} \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} = \mathbf{R}_e^t\,(\mathbf{x}_e - \mathbf{x}_0)$$

As an example, taken from our reference, a point near Los Angeles has geodetic coordinates

$$\lambda = 34\,°0'\ 0.00174''\ \text{North} = 0.59341195\ \text{radians}$$
$$\phi = 117\,°20'\ 0.84965''\ \text{West} = -2.0478571\ \text{radians}$$
$$\text{h} = 251.702\ \text{m}$$

The normal distance is found as

$$N(\lambda) = \frac{a}{\sqrt{1 - e^2\ \sin^2(\lambda)}} = \frac{6378137.0}{\sqrt{1 - \left(8.1819191 \times 10^{-2}\right)^2\ \sin^2(0.59341195)}} = 6.3848232 \times 10^6\ \text{m}$$

Therefore the ECEF-r coordinates of the Tangent Plane origin are

$$\mathbf{x}_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} -2{,}430{,}601.8 \\ -4{,}702{,}442.7 \\ 3{,}546{,}587.4 \end{pmatrix} \text{m}$$

Using the formula given above, the rotation matrix which transforms ECEF-r coordinates to the example Tangent Plane coordinates is

$$\mathbf{R}_e^t = \begin{pmatrix} 0.88834836 & -0.45917011 & 0.00000000 \\ 0.25676467 & 0.49675810 & 0.82903757 \\ -0.38066927 & -0.73647416 & 0.55919291 \end{pmatrix}$$

The complete transformation from ECEF-r to Tangent Plane for our example is then

$$\mathbf{x}_t = \begin{pmatrix} e \\ n \\ u \end{pmatrix} = \mathbf{R}_e^t\ (\mathbf{x}_e - \mathbf{x}_0)$$

And the inverse transform is

$$\mathbf{x}_e = \mathbf{x}_0 + \mathbf{R}_t^e\ \mathbf{x}_t = \mathbf{x}_0 + (\mathbf{R}_e^t)^T\ \mathbf{x}_t$$

Where the superscript T represents the matrix transpose

As a final check, the unit gravity direction vector in the Local Tangent Plane is given by $\langle 0, 0, -1 \rangle$, using the inverse transformation the unit vector in the ECEF-r frame is

$$\mathbf{R}_t^e \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0.380669 \\ 0.736474 \\ -0.559193 \end{pmatrix}$$

The origin vector ($\mathbf{x}_0$) has not been added here because only direction, not offset, is being considered.

This result agrees with Farrell & Barth, which serves as a double check, so all this may even be right.